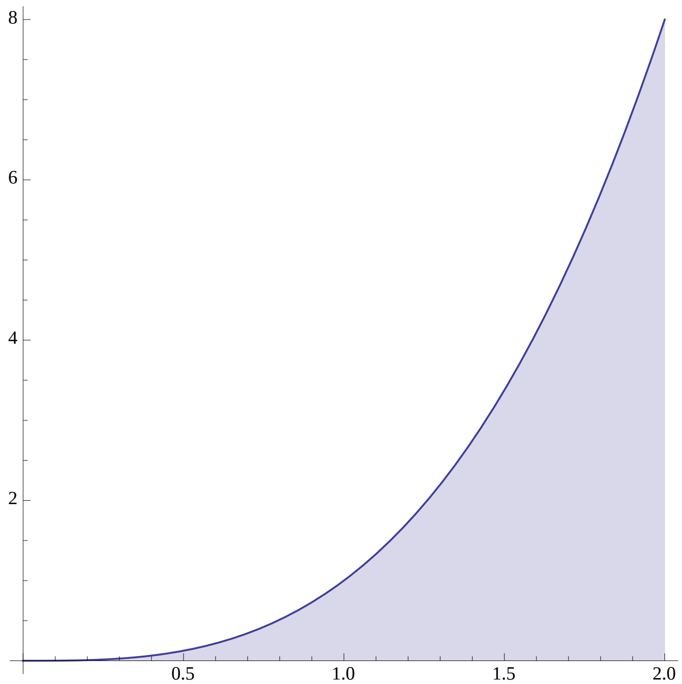
Problem 1. Use mathematical induction to prove that for all $n \in \mathbb{N}$,

$$0^3 + 1^3 + 2^3 + \cdots + (n-1)^3 < \frac{n^4}{4} \text{ and } \frac{n^4}{4} < 1^3 + 2^3 + \cdots + (n-1)^3 + n^3.$$

Use this result to compute the area of the region pictured below (the vertical distance between the point b units from 0 is b^3).



Problem 2. Prove that $|x+y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.

Problem 3. Let $a, b \in \mathbb{R}$ with $a \neq 0$. Use the field axioms of $\mathbb{R}$ to carefully prove that if $ax + b = 0$ then $x = \frac{-b}{a}$. Justify all your steps.

Problem 4. True or False: If S is a nonempty subset of rational numbers that is bounded above, then the least upper bound of S is rational.

Problem 5. Prove or disprove: For all propositions p, q, r, s, we have

$$((p \Rightarrow q) \Rightarrow r) \Rightarrow s \equiv ((p \wedge q) \wedge r) \Rightarrow s.$$

Problem 6. Let X and Y be sets. Recall, for sets X and Y, we define the set $X \setminus Y$ to be

$$X \setminus Y = \{x \in X \text{ satisfying } x \notin Y\}.$$

Prove or disprove:

(a) For all sets A, B, C
$$(A \setminus B) \cup C = (A \cup C) \setminus (B \cup C).$$

(b) For all sets A, B, C
$$A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C).$$

(c) For all sets A, B, C
$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

Problem 7. Let X and Y be subsets of real numbers. Define the set $X - Y$ to be

$$X - Y = \{x - y \text{ where } x \in X \text{ and } y \in Y\}.$$

(a) Suppose that $X, Y \subset \mathbb{R}$ and that L is the least upper bound of X and M is the least upper bound of Y. Prove or disprove: the least upper bound of $X - Y$ is $L - M$.

(b) Prove or disprove: For any sets $X, Y \subset \mathbb{R}$, if $X \subset X - Y$ then $0 \in Y$.

Problem 8. Use the order axioms of $\mathbb{R}$ to prove that if x, y are positive real numbers with $x < y$ then $x^2 < y^2$.

Problem 9. More induction: use mathematical induction to prove that

(a) $\displaystyle\sum_{k=1}^{n} = \frac{(n)(n+1)}{2}$ for all $n \in \mathbb{N}$.

(b) $n! > 2^n$ for all natural numbers $n \geq 4$.

Problem 10. Negate the following propositions. Decide whether the proposition or its negation are true.

(a) $\forall x \in \mathbb{R} \exists n \in \mathbb{N} (x < n)$

(b) $\exists x \in \mathbb{R} \forall n \in \mathbb{N} (x > n)$

(c) $\forall x > 0 \exists n \in \mathbb{N} \left(\frac{1}{n} < x\right)$

(d) $\forall x \in \mathbb{R} \forall y \in \mathbb{R} \left(x < y \Rightarrow \exists z \in \mathbb{R}(x < z < y)\right)$

EXAM

Practice Exam 1

Math 157

September 27, 2013

- These problems are *practice* for the first exam, which will be in class on October 3. While these problems involve the right material and are about the right difficulty, this practice is exam is quite a bit longer than the in-class exam will be.
- This practice exam can be a good diagnostic tool and a good way to study for the in class exam. You might want to give yourself a few hours, say 3 hours, to take the exam as if it were an in class exam. Then, you'll have a good idea of what you know well and what you need to study. Go back over the problems later with the book, notes, and classmates to figure out any problems you might have missed.
- I'll also post solutions. I'll also answer questions in class on Tuesday, October 1.

Success!