**EXAM**

Practice Exam 1

Math 157

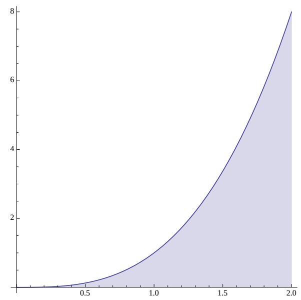
October 3, 2013

# ANSWERS

**Problem 1**. Use mathematical induction to prove that for all $n \in \mathbb{N}$,

$$0^3 + 1^3 + 2^3 + \cdots + (n-1)^3 < \frac{n^4}{4} \text{ and } \frac{n^4}{4} < 1^3 + 2^3 + \cdots + (n-1)^3 + n^3.$$

Use this result to compute the area of the region pictured below (the vertical distance between the point $b$ units from 0 is $b^3$).




***Answer***:
We proved in class that $0^3 + 1^3 + 2^3 + \cdots + (n-1)^3 < \frac{n^4}{4}$ for all $n \in \mathbb{N}$ and that $\frac{n^4}{4} < 1^3 + 2^3 + \cdots + (n-1)^3 + n^3$ for all $n \in \mathbb{N}$. Here, I'll write the details for the second one.
To prove that $\frac{n^4}{4} < 1^3 + 2^3 + \cdots + (n-1)^3 + n^3$ for all $n \in \mathbb{N}$ by induction, consider the base case: since $\frac{1^4}{4} < 1$, the inequality is holds when $n = 1$.
Now, assume the inequality holds for $n = k$. That is, suppose that for some natural number $k$, we have

$$\frac{k^4}{4} < 1^3 + 2^3 + \cdots + k^3.$$

After adding $(k+1)^3$ to both sides we have

$$\frac{k^4}{4} + (k+1)^3 < 1^3 + 2^3 + \cdots + k^3 + (k+1)^3$$

and doing a little arithmetic yields

$$\frac{k^4}{4} + \frac{4k^3}{4} + \frac{12k^2}{4} + \frac{12k}{4} + \frac{4}{4} < 1^3 + 2^3 + \cdots + k^3 + (k+1)^3. \tag{1}$$

Also,

$$\frac{k^4}{4} + \frac{4k^3}{4} + \frac{6k^2}{4} + \frac{4k}{4} + \frac{1}{4} < \frac{k^4}{4} + \frac{4k^3}{4} + \frac{12k^2}{4} + \frac{12k}{4} + \frac{4}{4}. \tag{2}$$

Combining (1) and (2) gives

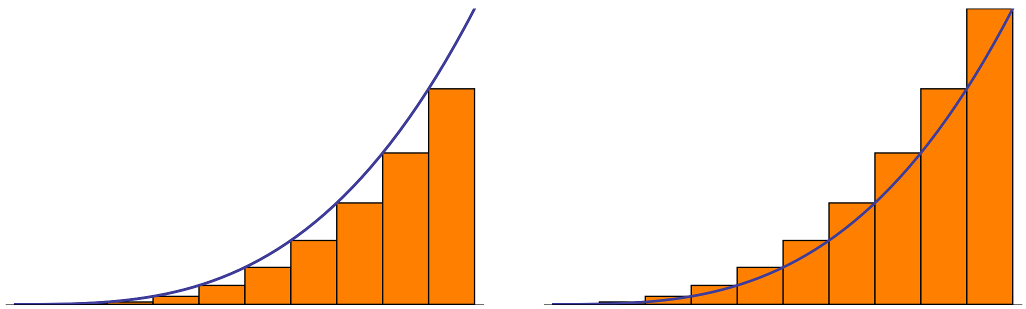
$$\frac{k^4}{4} + \frac{4k^3}{4} + \frac{6k^2}{4} + \frac{4k}{4} + \frac{1}{4} < 1^3 + 2^3 + \cdots + k^3 + (k+1)^3.$$

That is,

$$\frac{(k+1)^4}{4} < 1^3 + 2^3 + \cdots + k^3 + (k+1)^3.$$

This proves that if the inequality holds for $k$ then it holds for $k+1$.
So, by the principle of mathematical induction, the inequality holds for all $n \in \mathbb{N}$.

***Answer***:
To compute the area of the region in question, we bound it by two rectangular regions obtained by inscribing $n$ rectangles of equal width and circumscribing $n$ rectangles of equal width, as pictured below:



Let $s_n$ be the area of the inscribed rectangular region, let $A$ be the area of the curved region, and $S_n$ be the area of the circumscribed rectangular region. We have

$$s_n < A < S_n.$$

$$\begin{aligned}
s_n &= \frac{2}{n}((0)^3) + \frac{2}{n}\left(\left(\frac{2}{n}\right)^3\right) + \frac{2}{n}\left(\left(\frac{4}{n}\right)^3\right) + \cdots + \frac{2}{n}\left(\left(\frac{(n-1)2}{n}\right)^3\right) \\
&= \frac{2}{n}\left(\left(\frac{2}{n}\right)^3 + \left(\frac{4}{n}\right)^3 + \cdots + \left(\frac{(n-1)2}{n}\right)^3\right) \\
&= \frac{2^4}{n^4}\left(1^3 + 2^3 + \cdots (n-1)^3\right)
\end{aligned}$$

Similarly,

$$S_n = \frac{2^4}{n^4}\left(1^3 + 2^3 + \cdots (n)^3\right)$$

By the inequalities proved above, we have

$$s_n < \left(\frac{2^4}{n^4}\right)\left(\frac{n^4}{4}\right) = 4 \text{ and } 4 = \left(\frac{2^4}{n^4}\right)\left(\frac{n^4}{4}\right) < S_n.$$

So, for every $n \in \mathbb{N}$, we have

$$s_n < 4 < S_n \text{ and } s_n < A < S_n.$$

Since $S_n - s_n = \frac{16}{n}$, we conclude that there is only one real number that lies between $s_n$ and $S_n$ for every $n \in \mathbb{N}$. Therefore, $A = 4$.

**Problem 2**. Prove that $|x+y| \leq |x| + |y|$ for all $x, y \in \mathbb{R}$.

***Answer***:
First, we prove a lemma. For all $z \in \mathbb{R}$ and all $a \geq 0$,

$$|z| \leq a \Leftrightarrow -a \leq z \leq a.$$

To prove that $|z| \leq a \Rightarrow -a \leq z \leq a$, suppose that $|z| \leq a$. If $z \geq 0$, then we automatically get $-a \leq z$ and since $|z| \leq a$ means $z \leq a$ we get $-a \leq z \leq a$ also. If $z \leq 0$, then we automatically have $z \leq a$ and since $|z| \leq a$ means $-z \leq a$ we get $-a \leq z$, and so $-a \leq z \leq a$.
To prove that if $-a \leq z \leq a \Rightarrow |z| \leq a$, suppose that $-a \leq z \leq a$. If $z \geq 0$, then $|z| = z \leq a$. If $z \leq 0$, then $|z| = -z$ and $-a \leq z \Rightarrow -z \leq a \Leftrightarrow |z| \leq a$.
Now, since $|x| \leq |x|$ the Lemma (by substituting $z = x$ and $a = |x|$) implies

$$-|x| \leq x \leq |x|.$$

Similarly, we have

$$-|y| < y < |y|.$$

Adding gives

$$-(|x| + |y|) \leq x + y \leq |x| + |y|$$

which the Lemma (by substituting $z = x + y$ and $a = |x| + |y|$) implies that $|x + y| \leq |x| + |y|$.

**Problem 3**. Let $a, b \in \mathbb{R}$ with $a \neq 0$. Use the field axioms of $\mathbb{R}$ to carefully prove that if $ax + b = 0$ then $x = -b\left(\frac{1}{a}\right)$. Justify all your steps.

***Answer***:

$$
\begin{aligned}
ax + b = 0 &\Rightarrow (ax+b) - b = 0 - b && \text{existence of negatives} \\
&\Rightarrow ax + (b + -b) = -b && \text{associativity of } + \text{ and def of } 0 \\
&\Rightarrow ax = -b && \text{since } b + -b = 0 \\
&\Rightarrow \left(\frac{1}{a}\right)(ax) = \left(\frac{1}{a}\right)(-b) && \text{existence of recipricals} \\
&\Rightarrow \left(\left(\frac{1}{a}\right)a\right)(x) = \left(\frac{1}{a}\right)(-b) && \text{associativity of } \times \\
&\Rightarrow 1x = \left(\frac{1}{a}\right)(-b) && \text{since } \left(\frac{1}{a}\right)a = 1 \\
&\Rightarrow x = \left(\frac{1}{a}\right)(-b) && \text{by definition of } 1
\end{aligned}
$$

**Problem 4**. True or False: If $S$ is a nonempty subset of rational numbers that is bounded above, then the least upper bound of $S$ is rational.

***Answer***:
False. The set

$$S = \{r \in \mathbb{Q} : r < \sqrt{3}\}$$

is bounded above and the least upper bound is $\sqrt{3}$ which is irrational.
We proved that $\sqrt{3}$ isn't rational in class. It's clear that $\sqrt{3}$ is an upper bound for $S$. To see that $\sqrt{3}$ is the least upper bound for $S$, suppose $B < \sqrt{3}$. Since there is a rational number between any two distinct real numbers, we know there exists a rational number $r$ with $B < r < \sqrt{3}$, and that says that $B$ is not an upper bound for $S$.

**Problem 5**. Prove or disprove: For all propositions $p, q, r, s$, we have

$$((p \Rightarrow q) \Rightarrow r) \Rightarrow s \equiv ((p \wedge q) \wedge r) \Rightarrow s.$$

***Answer***:
If $p, r$ are true and $q, s$ are false, then $p \Rightarrow q$ is false, $(p \Rightarrow q) \Rightarrow r$ is true, hence $((p \Rightarrow q) \Rightarrow r) \Rightarrow s$ is false. On the other hand $((p \wedge q) \wedge r)$ is false, so $((p \wedge q) \wedge r) \Rightarrow s$ is true.

**Problem 6**. Let $X$ and $Y$ be sets. Recall, for sets $X$ and $Y$, we define the set $X \setminus Y$ to be

$$X \setminus Y = \{x \in X \text{ satisfying } x \notin Y\}.$$

Prove or disprove:

(a) For all sets $A, B, C$

$$(A \setminus B) \cup C = (A \cup C) \setminus (B \cup C).$$

***Answer***:

False. For example, let $A = \{1, 2, 3\}$, $B = \{1, 4\}$ and $C = \{3, 4, 5\}$. Then

$(A \setminus B) \cup C = \{2, 3\} \cup C = \{2, 3, 4, 5\}$ and $(A \cup C) \setminus (B \cup C) = \{1, 2, 3, 4, 5\} \setminus \{1, 3, 4, 5\} = \{2\}$.

(b) For all sets $A, B, C$

***Answer***:

False. For example, let $A = \{1, 2, 3\}$, $B = \{1, 4\}$ and $C = \{3, 4, 5\}$. Then

$A \setminus (B \cup C) = A \setminus \{1, 3, 4, 5\} = \{2\}$ and $(A \setminus B) \cup (A \setminus C) = \{2, 3\} \cup \{1, 2\} = \{1, 2, 3\}$

(c) For all sets $A, B, C$

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C).$$

***Answer***:

This is true. To prove that $A \setminus (B \cap C) \subseteq (A \setminus B) \cup (A \setminus C)$, let $a \in A \setminus (B \cap C)$. This means $a \in A$ and $a\ notin (B \cap C)$. Since $a \notin B \cap C$, we know $a \notin B$ or $a \notin C$. If $a \notin B$, then $a \in A \setminus B$ so $a \in A(\setminus B) \cup (A \setminus C)$. If $a \notin C$, then $a \in A \setminus C$ so $a \in A(\setminus B) \cup (A \setminus C)$.

To prove that $(A \setminus B) \cup (A \setminus C) \subseteq A \setminus (B \cap C) \subseteq (A \setminus B)$, let $a \in (A \setminus B) \cup (A \setminus C)$. This means $a \in A \setminus B$ or $a \in A \setminus C$. In the first case, $a \in A \setminus B$ means $a \in A$ and $a \notin B$. Since $a \notin B$, $a \notin B \cap C$, so we have $a \in A$ and $a \notin B \cap C$, so $a \in A \setminus (B \cap C)$. In the second case, $a \in A \setminus C$ means $a \in A$ and $a \notin C$. Since $a \notin C$, $a \notin B \cap C$, so we have $a \in A$ and $a \notin B \cap C$, so $a \in A \setminus (B \cap C)$.

**Problem 7**. Let $X$ and $Y$ be subsets of real numbers. Define the set $X - Y$ to be

$$X - Y = \{x - y \text{ where } x \in X \text{ and } y \in Y\}.$$

(a) Suppose that $X, Y \subset \mathbb{R}$ and that $L$ is the least upper bound of $X$ and $M$ is the least upper bound of $Y$. Prove or disprove: the least upper bound of $X - Y$ is $L - M$.

***Answer***:

Let $X = \{1, 2, 3\}$ and $Y = \{-1, 1\}$. Then $X - Y = \{2, 3, 4, 0, 1, 2\} = \{0, 1, 2, 3, 4\}$. Not that $X \subset X - Y$ and $0 \notin Y$.

(b) Prove or disprove: For any sets $X, Y \subset \mathbb{R}$, if $X \subset X - Y$ then $0 \in Y$.

***Answer***:

The same example above disproves this statement for $\sup X = 3$, $\sup Y = 1$ but $\sup X - Y = 3 \neq 3 - 1$.

**Problem 8**. Use the order axioms of $\mathbb{R}$ to prove that if $x, y$ are positive real numbers with $x < y$ then $x^2 < y^2$.

***Answer***:
Suppose $x, y > 0$ and $x < y$. Since $x, y > 0$, the sum $y + x > 0$. Also, $x < y$ implies $y - x > 0$. Therefore the product $(y + x)(y - x) > 0$. This says that $y^2 - x^2 > 0$, which means that $x^2 < y^2$.

**Problem 9**. More induction: use mathematical induction to prove that

(a) $\displaystyle\sum_{k=1}^{n} k = \frac{(n)(n+1)}{2}$ for all $n \in \mathbb{N}$.

***Answer***:

When $n = 1$, the statement is that $1 = \frac{(1)(2)}{2}$ which is true.

Assume that for some $m \in \mathbb{N}$ we have $\displaystyle\sum_{k=1}^{m} k = \frac{(m)(m+1)}{2}$. Now consider

$$\begin{aligned}
\sum_{k=1}^{m+1} k &= \left(\sum_{k=1}^{m} k\right) + m + 1 \\
&= \frac{(m)(m+1)}{2} + m + 1 \\
&= \frac{(m)(m+1) + 2(m+1)}{2} \\
&= \frac{(m+1)(m+2)}{2}
\end{aligned}$$

This shows that if the statement is true for $m \in \mathbb{N}$, it's true for $m + 1 \in \mathbb{N}$, so by PMI, the statement is true for all $n \in \mathbb{N}$.

(b) $n! > 2^n$ for all natural numbers $n \geq 4$.

***Answer***:

Here we do a modified version of mathematical induction where the base step verifies that it's true for $n = 4$. For $n = 4$, the statement is $24 > 16$, which is true.

Now assume that for some $k \in \mathbb{N}$, $k! > 2^k$. Now consider

$$(k+1)! = (k+1)k! > (k+1)2^k > (2)(2^k) = 2^{k+1}.$$

The first inequality in the line above follows from the inductive hypothesis and the second follows from the fact that $k + 1 > 2$.

That completes the proof.

**Problem 10**. Negate the following propositions. Decide whether the proposition or its negation are true.

(a) $\forall x \in \mathbb{R} \exists n \in \mathbb{N}(x < n)$

***Answer***:

$\exists x \in \mathbb{R} \forall n \in \mathbb{N}(n \leq x)$. The original statement is true—it's the Archimedean property of $\mathbb{R}$. The negation is false, it says that $\mathbb{N}$ is bounded above by some real number.

(b) $\exists x \in \mathbb{R} \forall n \in \mathbb{N}(x > n)$

***Answer***:

$\forall x \in \mathbb{R} \exists n \in N(x \leq n)$. Here the original statement is false—it says that $\mathbb{N}$ is bounded above by some real number.

(c) $\forall x > 0 \exists n \in \mathbb{N} \left(\frac{1}{n} < x\right)$

***Answer***:

$\exists x > 0 \forall n \in N \left(\frac{1}{n} \geq x\right)$ The original statement is true and the negation is false. The negation says that there's a real number less than $\frac{1}{2}$ less than $\frac{1}{3}$, and less than $\frac{1}{n}$ for every $n \in \mathbb{N}$.

(d) $\forall x \in \mathbb{R} \forall y \in \mathbb{R} \left(x < y \Rightarrow \exists z \in \mathbb{R}(x < z < y)\right)$

***Answer***:

$\exists x \in \mathbb{R} \exists y \in \mathbb{R} \left(x < y \text{ and } \forall z \in \mathbb{R}(z \leq x \text{ or } y \leq z\right)$. The original statement is true. There exists a real number between any two distinct real numbers.