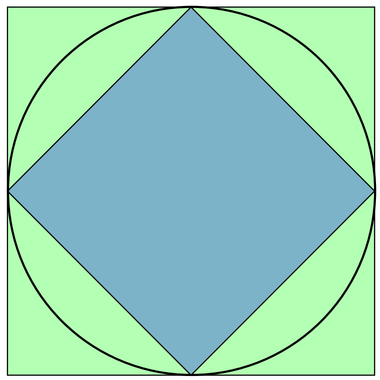
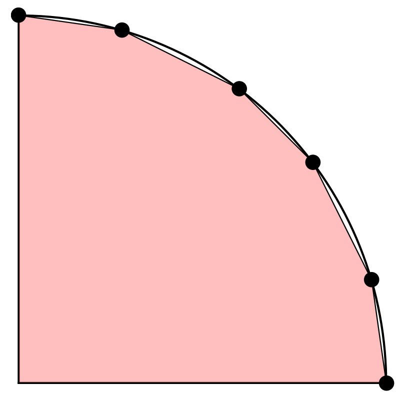
Problem 1. Everyone ought to try to approximate π.

(a) Define the number π.

(b) The picture on the left makes it easy to see that $2 < \pi < 4$:





It takes more work to see that $3 < \pi$. Here's one way: Use the following points

$$(0,1), \quad \left(\frac{7}{25}, \frac{24}{25}\right), \quad \left(\frac{3}{5}, \frac{4}{5}\right), \quad \left(\frac{4}{5}, \frac{3}{5}\right), \quad \left(\frac{24}{25}, \frac{7}{25}\right), \quad (1,0)$$

as the vertices of a polygon (as pictured on the right) to approximate the area of the quarter unit circle. Use this to (under) approximate π.

(c) Here's another way to approximate π. Let $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[0,1]$ into n subintervals of equal length. Define step a function $s : [0,1] \to \mathbb{R}$ by

$$s(x) = \sqrt{1 - x_{k+1}^2} \text{ if } x_k \leq x < x_{k+1}$$

Note that $\int_0^1 s < \frac{\pi}{4}$. How large must n be in order for $3.14 < 4 \int_0^1 s$?

Problem 2. Expressions like $\int_0^1 \frac{1}{\sqrt{x}}dx$ and $\int_1^\infty \frac{1}{x^2}dx$ are undefined because (so far!) we have defined $\int_a^b f$ only for finite intervals $[a, b]$ and for bounded functions f. Nonetheless, by using limits we can make sense of these "improper integrals." We define

$$\int_0^1 \frac{1}{\sqrt{x}}dx := \lim_{B\to 0^+} \int_B^1 \frac{1}{\sqrt{x}}dx \qquad \text{and} \qquad \int_1^\infty \frac{1}{x^2}dx := \lim_{B\to\infty} \int_1^B \frac{1}{x^2}dx.$$

(a) Compute $\int_0^1 \frac{1}{\sqrt{x}}dx$

(b) Compute $\int_1^\infty \frac{1}{x^2}dx$.

Problem 3. Let $f : [0, \infty) \to \mathbb{R}$ by $f(t) = \cos(\sqrt{t})$ and define $A : [0, \infty) \to \mathbb{R}$ by

$$A(x) = \int_0^x f(t)dt.$$

(a) Sketch the graph of f.

(b) Approximate $A(x)$ for $x = 0, \pi^2, \frac{9\pi^2}{4}$, and $4\pi^2$.

(c) On what intervals is A increasing? decreasing?

(d) On what intervals is A concave up? concave down?

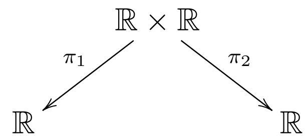
(e) Sketch a good picture of the graph of A.

Problem 4. The k-th Fibonacci number F_k is defined inductively for all k by $F_1 = 1$, $F_2 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for all $k > 2$. The first few Fibonacci numbers are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

Prove that for every $n \in \mathbb{N}$, $F(4n)$ is divisible by 3.

Problem 5. Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ and $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be the canonical projections



For each of the following sets $A \subset \mathbb{R}^2$ describe $\pi_1(A)$ and $\pi_2(A)$.

(a) $A =$ the graph of g where $g : \mathbb{R} \to \mathbb{R}$ is given by $g(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$ and $\frac{1}{r}$ if $x = \frac{p}{q} \in \mathbb{Q}$ is in lowest terms.

(b) A is the ordinate set of g where $g : \mathbb{R} \to \mathbb{R}$ is given by $g(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$ and $\frac{1}{r}$ if $x = \frac{p}{q} \in \mathbb{Q}$ is in lowest terms.

(c) A is the unit circle.

(d) $A = \{(x, y) : 1 < x \leq 2 \text{ and } 5 \leq x < 7\}$.

Problem 6. Give an example, or prove that no such example exists.

(a) Two functions f, g satisfying $\lim_{x \to 0} f(x) = 0$, $\lim_{x \to 0} g(x) = 0$ and $\lim_{x \to 0} \dfrac{f(x)}{g(x)} = 3$

(b) A continuous bijection $f : (0, 1) \to \mathbb{R}$

(c) A bijection $f : [0, 2\pi) \to C$ where $C \subset \mathbb{R}^2$ is the unit circle

(d) A function $f : [1, 3] \to \mathbb{R}$ satisfying $f(1) = -1$, $f(3) = 1$, and $f(x) \neq 0$ for any number x

(e) A continuous function f whose domain is all real numbers satisfying $f(n) = n!$ for all $n \in \mathbb{N}$

(f) A function with domain $[0, 2]$ and range $[0, 2] \cup [3, 4]$

(g) A function with domain $[0, 2]$ and range $(0, 2)$

(h) A continuous function with domain $[0, 2]$ and range $[0, 2) \cup (2, 3]$

(i) A function whose domain is $\mathbb{R}$ and whose range is $(0, 1)$

Problem 7. Compute.

(a) $\lim_{h\to 0} \left(\frac{1}{h}\right)\left(\frac{3}{2+h} - \frac{3}{2}\right)$

(b) $\int_1^5 \left[x^2 - 3x + 2\right] dx$

(c) $\lim_{x\to\infty} \sqrt{x^2+x} - x$

(d) $\int_0^{\pi} \cos\left(\frac{x}{2}\right) dx$

(e) $\int_0^{\pi} |\sin(x) + \cos(x)| dx$

(f) $\int_0^2 \frac{x + \sqrt{4-x^2}}{2} dx$

(g) $\int_0^1 x^{\frac{1}{5}} dx$

(h) $\lim_{h\to 0} \frac{\cos\left(\frac{\pi}{3} + h\right) - \frac{1}{2}}{h}$

(i) $\lim_{x\to 0} \frac{\sin(x)}{x}$

(j) $\lim_{x\to\infty} \frac{\sin(x)}{x}$

Problem 8. True or false.

(a) If $f : X \to Y$ is injective, then for all $A \subset X$,

$$f(X \setminus A) = Y \setminus f(A).$$

(b) For any $a, b, c, d \in \mathbb{R}$, $(ac + bd)^2 \leq (a^2 + c^2)(b^2 + d^2)$.

(c) If A and B satisfy

$$\frac{17 - 6n}{3} \leq A \leq B \leq \frac{17 + 6n}{3}$$

for every $n \in \mathbb{N}$, then $A = B$.

(d) If $f : [a, b] \to \mathbb{R}$ is monotonic then f is integrable.

(e) If $f : [a, b] \to \mathbb{R}$ is bounded then f is integrable.

(f) If $f : [a, b] \to \mathbb{R}$ is bounded then f is continuous.

(g) If $f : [a, b] \to \mathbb{R}$ is integrable then f is monotonic.

(h) If $f : [a, b] \to \mathbb{R}$ is integrable then f is continuous.

(i) If $f : [a, b] \to \mathbb{R}$ is monotonic then f is invertible.

(j) If $f : [a, b] \to \mathbb{R}$ is invertible then f is monotonic.

(k) If $f : [a, b] \to \mathbb{R}$ is integrable then f is bounded.

(l) If $f : [a, b] \to \mathbb{R}$ is integrable then f is invertible.

(m) If $f : [a, b] \to \mathbb{R}$ is continuous then f is bounded.

(n) For every real number x there exists a real number y with $y^2 = x$.

EXAM

Practice Final

Math 157

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