EXAM

Practice Final

Math 157

December 17, 2013

ANSWERS

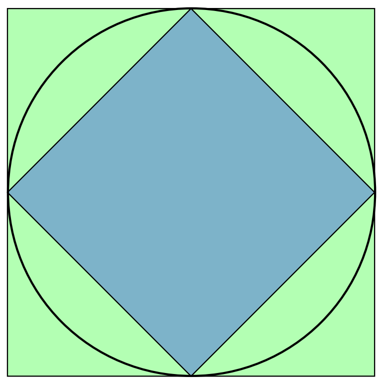
Problem 1. Everyone ought to try to approximate π.

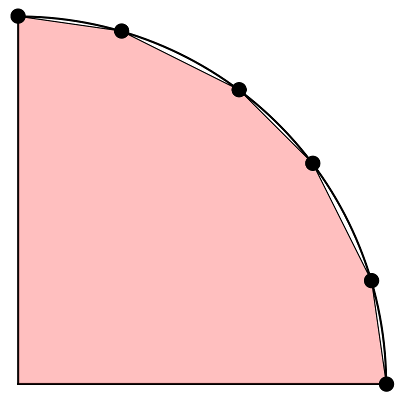
(a) Define the number π.

Answer:

π is defined to be the area of the unit circle.

(b) The picture on the left makes it easy to see that $2 < \pi < 4$:





It takes more work to see that $3 < \pi$. Here's one way: Use the following points

$$(0,1), \quad \left(\frac{7}{25}, \frac{24}{25}\right), \quad \left(\frac{3}{5}, \frac{4}{5}\right), \quad \left(\frac{4}{5}, \frac{3}{5}\right), \quad \left(\frac{24}{25}, \frac{7}{25}\right), \quad (1,0)$$

as the vertices of a polygon (as pictured on the right) to approximate the area of the quarter unit circle. Use this to (under) approximate π.

Answer:

The region under the polygon from $x = 0$ to $x = \frac{7}{25}$ consists of a rectange of width $\frac{7}{25}$ and height $\frac{24}{25}$ and a triangle of height $\frac{1}{25}$ and width $\frac{7}{25}$ giving an area of

$$\left(\frac{7}{25}\right)\left(\frac{24}{25}\right) + \frac{1}{2}\left(\frac{7}{25}\right)\left(\frac{1}{25}\right) = \frac{343}{1250}.$$

Similarly, the area of the polygon between $x = \frac{7}{25}$ and $x = \frac{3}{5}$ is given by

$$\left(\frac{7}{25} - \frac{3}{5}\right)\left(\frac{4}{5}\right) + \frac{1}{2}\left(\frac{7}{25} - \frac{3}{5}\right)\left(\frac{24}{25} - \frac{4}{25}\right) = \frac{176}{625}.$$

Continuing in this way we obtain the area of the polygon is

$$\frac{343}{1250} + \frac{176}{625} + \frac{7}{50} + \frac{44}{625} + \frac{7}{1250} = \frac{193}{250}.$$

So, $\pi \approx 4\left(\frac{193}{250}\right) = \frac{386}{125} > \frac{375}{125} = 3.$

Problem 1.

(c) Here's another way to approximate π. Let $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ be a partition of $[0,1]$ into n subintervals of equal length. Define step a function $s : [0,1] \to \mathbb{R}$ by

$$s(x) = \sqrt{1 - x_{k+1}^2} \text{ if } x_k \leq x < x_{k+1}$$

Note that $\displaystyle\int_0^1 s < \frac{\pi}{4}$. How large must n be in order for $\displaystyle 3.14 < 4\int_0^1 s$?

Answer:

By defining a step function $t : [0,1] \to \mathbb{R}$ by $t(x) = \sqrt{1 - x_k^2}$ if $x_k \leq x < x_{k+1}$, we have

$$\int_0^1 s \leq \int_0^1 \sqrt{1-x^2}dx \leq \int_0^1 t.$$

Since

$$\int_0^1 t - \int_0^1 s = \frac{1}{n}(1-0) = \frac{1}{n}$$

we have

$$\int_0^1 \sqrt{1-x^2}dx - \int_0^1 s < \frac{1}{n} \Rightarrow \frac{\pi}{4} - \int_0^1 s < \frac{1}{n} \Rightarrow \pi - \frac{4}{n} < 4\int_0^1 s.$$

Since $3.14 < \pi - \frac{1}{1000}$, if we choose n large enough for $\frac{4}{n} < \frac{1}{1000}$, we will have $\displaystyle 3.14 < 4\int_0^1 s$. So if $n > 4000$ we have $\displaystyle 3.14 < 4\int_0^1 s$.

For fun, I computed this and got $\int_0^1 s = 3.14108800514810316 71\ldots$ *when* $n = 4000$. *But note that smaller* n *work–the smallest integer* n *for which* $3.14 < \int_0^1 s$ *is* $n = 1277$

Problem 2. Expressions like $\int_0^1 \frac{1}{\sqrt{x}}dx$ and $\int_1^\infty \frac{1}{x^2}dx$ are undefined because (so far!) we have defined $\int_a^b f$ only for finite intervals $[a,b]$ and for bounded functions f. Nonetheless, by using limits we can make sense of these "improper integrals." We define

$$\int_0^1 \frac{1}{\sqrt{x}}dx := \lim_{B\to 0^+}\int_B^1 \frac{1}{\sqrt{x}}dx \qquad \text{and} \qquad \int_1^\infty \frac{1}{x^2}dx := \lim_{B\to\infty}\int_1^B \frac{1}{x^2}dx.$$

(a) Compute $\int_0^1 \frac{1}{\sqrt{x}}dx$

Answer:

$$\int_0^1 \frac{1}{\sqrt{x}}dx := \lim_{B\to 0^+}\int_B^1 \frac{1}{\sqrt{x}}dx = \lim_{B\to 0^+} 2\sqrt{1} - 2\sqrt{B} = 2.$$

(b) Compute $\int_1^\infty \frac{1}{x^2}dx$.

Answer:

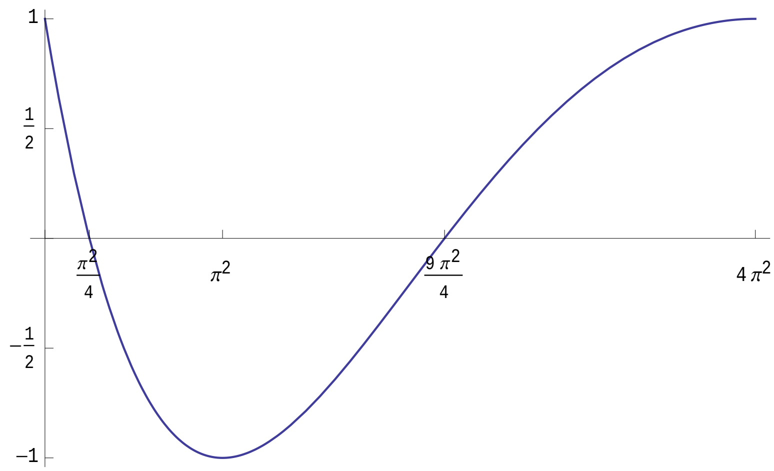
$$\int_1^\infty \frac{1}{x^2}dx := \lim_{B\to\infty}\int_1^B \frac{1}{x^2}dx = \lim_{B\to\infty}\left(-\frac{1}{B}\right) - \left(-\frac{1}{1}\right) = 1.$$

Problem 3. Let $f : [0, \infty) \to \mathbb{R}$ by $f(t) = \cos(\sqrt{t})$ and define $A : [0, \infty) \to \mathbb{R}$ by

$$A(x) = \int_0^x f(t)dt.$$

(a) Sketch the graph of f.

Answer:



(b) Approximate $A(x)$ for $x = 0, \pi^2, \frac{9\pi^2}{4}$, and $4\pi^2$.

Answer:

Looking at the graph of f and using $\pi^2 \approx 10$, $\frac{9}{4}\pi^2 \approx 22$ and $4\pi^2 \approx 40$, I estimate

$$A(\pi^2) \approx -4, \quad A\left(\frac{9}{4}\pi^2\right) \approx -10, \quad A\left(4\pi^2\right) \approx 0.$$

(c) On what intervals is A increasing? decreasing?

Answer:

A is increasing when f is positive. That is $\left(0, \frac{\pi^2}{4}\right) \cup \left(\frac{9}{4}\pi^2, 4\pi^2\right)$.

A is decreasing when f is negative. That is $\left(\frac{\pi^2}{4}, \frac{9}{4}\pi^2\right)$.

(d) On what intervals is A concave up? concave down?

Answer:

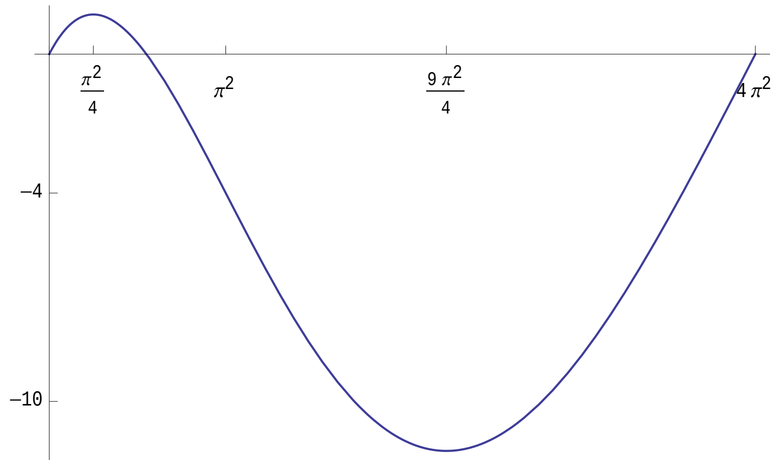
A is concave up when f is increasing. That is $(\pi^2, 4\pi^2)$.

A is concave down when f is decreasing. That is $(0, \pi^2)$.

Problem 3.

(e) Sketch a good picture of the graph of A.

Answer:



Problem 4. The k-th Fibonacci number F_k is defined inductively for all k by $F_1 = 1$, $F_2 = 1$, and $F_k = F_{k-1} + F_{k-2}$ for all $k > 2$. The first few Fibonacci numbers are

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

Prove that for every $n \in \mathbb{N}$, $F(4n)$ is divisible by 3.

Answer:
Base step. Note that $F_4 = 3$ is divisible by 3.
Inductive step. Suppose that F_{4k} is divisible by 3 for some natural number k. Now look at $F_{4(k+1)}$:

$$\begin{aligned} F_{4(k+1)} &= F_{4k+4} \\ &= F_{4k+3} + F_{4k+2} \\ &= F_{4k+2} + F_{4k+1} + F_{4k+1} + F_{4k} \\ &= F_{4k+1} + F_{4k} + F_{4k+1} + F_{4k+1} + F_{4k} \\ &= 3F_{4k+1} + F_{4k} \end{aligned}$$

Since F_{4k} is divisible by 3 and $3F_{4k+1}$ is divisible by 3, the sum is divisible by 3. That is, $F_{4(k+1)}$ is divisible by 3.
Therefore, by the principle of mathematical induction, F_{4n} is divisible by 3 for all $n \in \mathbb{N}$.

Problem 5. Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ and $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be the canonical projections

$$\begin{tikzcd} & \mathbb{R} \times \mathbb{R} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{R} & & \mathbb{R} \end{tikzcd}$$

For each of the following sets $A \subset \mathbb{R}^2$ describe $\pi_1(A)$ and $\pi_2(A)$.

(a) $A =$ the graph of g where $g : \mathbb{R} \to \mathbb{R}$ is given by $g(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$ and $\frac{1}{r}$ if $x = \frac{p}{q} \in \mathbb{Q}$ is in lowest terms.

Answer:

$\pi_1(A) = \mathbb{R}$ and $\pi_2(A) = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$.

(b) A is the ordinate set of g where $g : \mathbb{R} \to \mathbb{R}$ is given by $g(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$ and $\frac{1}{r}$ if $x = \frac{p}{q} \in \mathbb{Q}$ is in lowest terms.

Answer:

$\pi_1(A) = \mathbb{R}$ and $\pi_2(A) = [0, 1]$

(c) A is the unit circle.

Answer:

$\pi_1(A) = [-1, 1]$ and $\pi_2(A) = [-1, 1]$

(d) $A = \{(x, y) : 1 < x \leq 2 \text{ and } 5 \leq y < 7\}$.

Answer:

$\pi_1(A) = (1, 2]$ and $\pi_2(A) = [5, 7)$

Problem 6. Give an example, or prove that no such example exists.

(a) Two functions f, g satisfying $\lim_{x\to 0} f(x) = 0$, $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} \dfrac{f(x)}{g(x)} = 3$

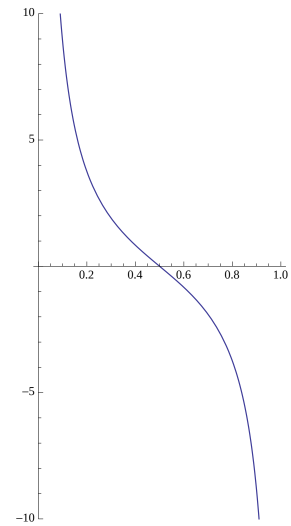
Answer:

$f(x) = 3x$, $g(x) = x$.

(b) A continuous bijection $f : (0, 1) \to \mathbb{R}$

Answer:

Here's a sketch of the graph of one such continuous bijection



(c) A bijection $f : [0, 2\pi) \to C$ where $C \subset \mathbb{R}^2$ is the unit circle

Answer:

$f(t) = (\cos(t), \sin(t))$

(d) A function $f : [1, 3] \to \mathbb{R}$ satisfying $f(1) = -1$, $f(3) = 1$, and $f(x) \neq 0$ for any number x

Answer:

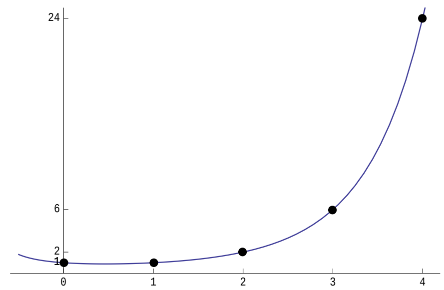
$f(x) = -1$ for $x = 1$ and $f(x) = 1$ for all $1 < x \leq 3$.

Problem 6.

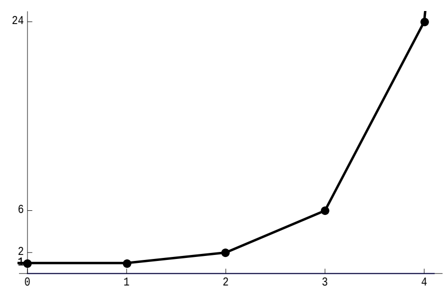
(e) A continuous function f whose domain is all real numbers satisfying $f(n) = n!$ for all $n \in \mathbb{N}$

Answer:

Here's a sketch of one such function



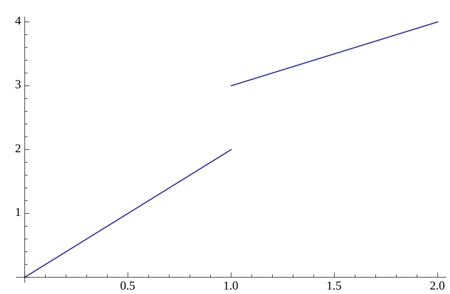
But there are many. Here's another:



(f) A function with domain $[0, 2]$ and range $[0, 2] \cup [3, 4]$

Answer:

Here's a sketch of the graph of one possible function:



(g) A function with domain $[0, 2]$ and range $(0, 2)$

Answer:

Let $f(0) = 1$, $f(2) = 1$, and $f(x) = x$ for all $0 < x < 2$.

(h) A continuous function with domain $[0, 2]$ and range $[0, 2) \cup (2, 3]$

Answer:

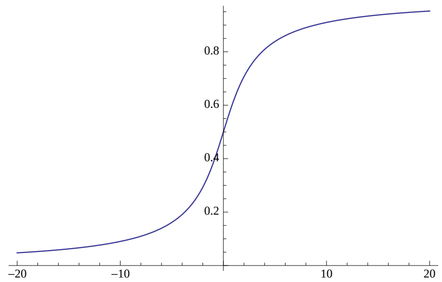
Not possible. By the intermediate value theorem, if 0 and 3 are in the range, then so must 2 be.

Problem 6.

(i) A function whose domain is $\mathbb{R}$ and whose range is $(0, 1)$

Answer:

The inverse of the function from part b works. So does the function whose graph is pictured here:



Problem 7. Compute.

Answer:
I'm giving the answer so you can check your work and hint in case you were stuck.

(a) $\lim_{h\to 0}\left(\frac{1}{h}\right)\left(\frac{3}{2+h}-\frac{3}{2}\right)=-\frac{3}{4}$ algebra, cancel the h.

(b) $\int_1^5 \left|x^2-3x+2\right|dx=\frac{41}{3}$ factor and find where $x^2-3x+2=0$

(c) $\lim_{x\to\infty}\sqrt{x^2+x}-x=\frac{1}{2}$ multiply and divide by $\sqrt{x^2+x}+x$.

(d) $\int_0^{\pi}\cos\left(\frac{x}{2}\right)dx=2$ use thm 1.19 on page 81 about expansion or contraction.

(e) $\int_0^{\pi}|\sin(x)+\cos(x)|dx=2\sqrt{2}$ you know the graphs of sine and cosine.

(f) $\int_0^2 \frac{x+\sqrt{4-x^2}}{2}dx=1+\frac{\pi}{2}$ you know the area of a circle of radius 2...

(g) $\int_0^1 x^{\frac{1}{5}}dx=\frac{5}{6}$.

(h) $\lim_{h\to 0}\frac{\cos\left(\frac{\pi}{3}+h\right)-\frac{1}{2}}{h}=-\frac{\sqrt{3}}{2}$ use $\cos(x+y)=\cdots$

(i) $\lim_{x\to 0}\frac{\sin(x)}{x}=1$ use the fundamental inequality and the squeezing principle

(j) $\lim_{x\to\infty}\frac{\sin(x)}{x}=0$ use the fact that $-1\leq\sin(x)\leq 1$ and squeeze

Problem 8. True or false.

Answer:
If true, you should have a good idea as to why. If false, you should be able to give a good simple example. Here are the answers.

(a) If $f : X \to Y$ is injective, then for all $A \subset X$,

$$f(X \setminus A) = Y \setminus f(A).$$

Answer:

False.

(b) For any $a, b, c, d \in \mathbb{R}$, $(ac + bd)^2 \leq (a^2 + c^2)(b^2 + d^2)$.

Answer:

True.

(c) If A and B satisfy

$$\frac{17 - 6n}{3} \leq A \leq B \leq \frac{17 + 6n}{3}$$

for every $n \in \mathbb{N}$, then $A = B$.

Answer:

True.

(d) If $f : [a, b] \to \mathbb{R}$ is monotonic then f is integrable.

Answer:

True.

(e) If $f : [a, b] \to \mathbb{R}$ is bounded then f is integrable.

Answer:

False.

(f) If $f : [a, b] \to \mathbb{R}$ is bounded then f is continuous.

Answer:

False.

Problem 8.

(g) If $f : [a, b] \to \mathbb{R}$ is integrable then f is monotonic.

Answer:

False.

(h) If $f : [a, b] \to \mathbb{R}$ is integrable then f is continuous.

Answer:

False.

(i) If $f : [a, b] \to \mathbb{R}$ is monotonic then f is invertible.

Answer:

False. If f is strictly monotonic, then f is invertible.

(j) If $f : [a, b] \to \mathbb{R}$ is invertible then f is monotonic.

Answer:

False.

(k) If $f : [a, b] \to \mathbb{R}$ is integrable then f is bounded.

Answer:

True.

(l) If $f : [a, b] \to \mathbb{R}$ is integrable then f is invertible.

Answer:

False.

(m) If $f : [a, b] \to \mathbb{R}$ is continuous then f is bounded.

Answer:

True.

(n) For every real number x there exists a real number y with $y^2 = x$.

Answer:

False.