Problem 1. [2 points] Use the four fundamental properties of sine and cosine on page 95 of the Apostol's book to prove that

$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}.$$

Then, use the other properties of sine and cosine listed in sections 2.5, 2.6, and 2.7 to compute

$$\int_0^\pi \left|\frac{1}{2} + \cos(t)\right| dt$$

Answer. The values we know from the fundamental properties are

$$\cos(0) = \sin\left(\frac{\pi}{2}\right) = 1 \text{ and } \cos(\pi) = -1 \tag{1}$$

and the relation we know is

$$\cos(y - x) = \cos(y)\cos(x) + \sin(y)\sin(x). \tag{2}$$

Now, specializing Equation (2) using the values we know from (1) yields

$$\sin(0) = 0 \text{ (by setting } y = x = 0) \tag{3}$$
$$\sin(\pi) = 0 \text{ (by setting } y = x = \pi) \tag{4}$$
$$\cos\left(\frac{\pi}{2}\right) = 0 \text{ (by setting } y = \pi \text{ and } x = \frac{\pi}{2}). \tag{5}$$

Now, setting $y = \frac{\pi}{2}$ in Equation (2) yields

$$\cos\left(\frac{\pi}{2} - x\right) = \sin(x). \tag{6}$$

Using $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$ in Equation (6) yields the two equations

$$\cos\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) \text{ and } \cos\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{6}\right). \tag{7}$$

Then, setting $y = \frac{\pi}{3}$ and $x = \frac{\pi}{6}$ into Equation (2) yields

$$\cos\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right). \tag{8}$$

Using (7) to replace the parts of (8) with $\frac{\pi}{6}$'s gives

$$\sin\left(\frac{\pi}{3}\right) = \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right). \tag{9}$$

Cancelling the $\sin\left(\frac{\pi}{3}\right)$ in Equation (9) and solving gives

$$1 = \cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right) \Rightarrow \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}. \tag{10}$$

Finally, using $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and setting $y = \pi$ and $x = \frac{\pi}{3}$ into Equation (2) gives the result

$$\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}.$$

Now, we show that $\displaystyle\int_0^\pi \left|\frac{1}{2} + \cos(t)\right| = \frac{\pi}{6} + \sqrt{3}$. Since cosine is strictly decreasing on $[0, \pi]$ and $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$, we have

$$0 < t < \frac{2\pi}{3} \Rightarrow \cos(t) > -\frac{1}{2} \Rightarrow \frac{1}{2} + \cos(t) > 0$$
$$\frac{2\pi}{3} < t < \pi \Rightarrow \cos(t) < -\frac{1}{2} \Rightarrow \frac{1}{2} + \cos(t) < 0$$

So

$$\left|\frac{1}{2} + \cos(t)\right| = \begin{cases} \frac{1}{2} + \cos(t) & \text{if } 0 \le t \le \frac{2\pi}{3}, \\ -\frac{1}{2} - \cos(t) & \text{if } \frac{2\pi}{3} < t \le \pi. \end{cases}$$

Now we compute

$$\begin{aligned} \int_0^\pi \left|\frac{1}{2} + \cos(t)\right| &= \int_0^{\frac{2\pi}{3}} \frac{1}{2} + \cos(t) + \int_{\frac{2\pi}{3}}^\pi -\frac{1}{2} - \cos(t) \\ &= \int_0^{\frac{2\pi}{3}} \frac{1}{2} + \int_0^{\frac{2\pi}{3}} \cos(t) - \int_{\frac{2\pi}{3}}^\pi \frac{1}{2} - \int_{\frac{2\pi}{3}}^\pi \cos(t) \\ &= \left(\frac{\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right) - \left(\frac{\pi}{6}\right) - \sin(\pi) + \sin\left(\frac{2\pi}{3}\right) \\ &= \frac{\pi}{6} + 2\sin\left(\frac{2\pi}{3}\right) \\ &= \frac{\pi}{6} + \sqrt{3}. \end{aligned}$$

The last equation follows from the fact that $\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$, which we deduce by using $\sin^2(x) + \cos^2(x) = 1$ for all x, $\cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$, and $\sin(x) > 0$ for $0 < x < \pi$.

Problem 2. [1 point each] True or False. Completely justify your answers.

(a) $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}}{2\sqrt{2}}$.

Answer. False. Sine is increasing on $[0, \pi]$. We know $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, so $\sin\left(\frac{\pi}{12}\right) < \frac{1}{2}$. But $\frac{\sqrt{3}}{2\sqrt{2}} > \frac{1}{2}$. That's the end of my answer. For another way to see that $\sin\left(\frac{\pi}{12}\right) \neq \frac{\sqrt{3}}{2\sqrt{2}}$, just compute $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{3}-1}{2\sqrt{2}} \neq \frac{\sqrt{3}}{2\sqrt{2}}$ using the fact that $\frac{\pi}{12} = \frac{\pi}{4} - \frac{\pi}{6}$ and the formula $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$.

(b) If f is increasing, then $f(b)(b-a) \geq \int_a^b f$.

Answer. True. First, we note that if f is increasing, then f is integrable hence $\int_a^b f$ exists. Since f is increasing, $f(x) \leq f(b)$ for all $x \leq b$. Thus, $\int_a^b f(x) \leq \int_a^b f(b) = f(b)(b-a)$.

(c) If f is integrable and satisfies $f(t+1) = f(t)$ for all t, then $A(x) = A(x+1)$ where A is defined by $A(x) = \int_a^x f(t)$.

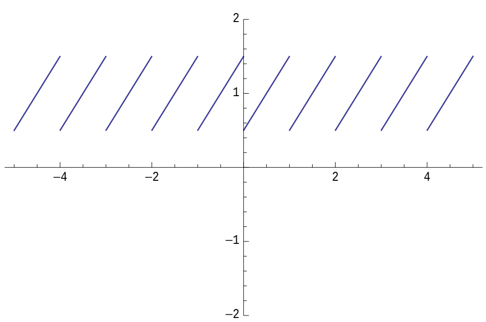
Answer. False. The constant function$f(t) = 1$ satisfies $f(t+1) = f(t)$ for all t. But $A(x) = \int_0^x f(t)dt = x$ does not satisfy $A(x) = A(x+1)$ for any x.

(d) If f is increasing, then the function A defined by $A(x) = \int_a^x f(t)$ is also increasing.

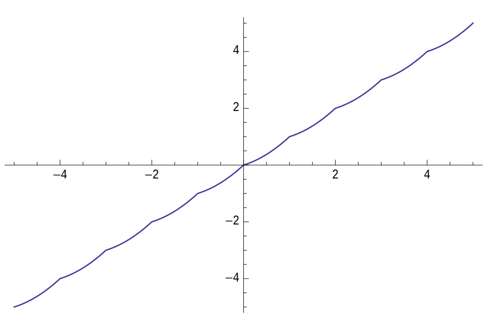
Answer. False. Let $f(t) = t$. Then $A(x) = \int_0^x f(t)dt = \frac{1}{2}x^2$ which is not increasing when $x \leq 0$.

Problem 3. [2 points] Let $f(t) = t - [t] + \frac{1}{2}$ and $A(x) = \int_0^x f(t)dt$. Sketch the graph of A on the interval $[-10, 10]$.

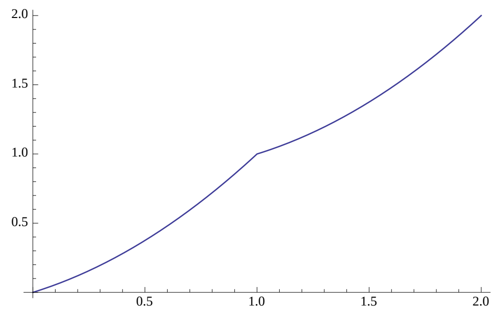
Answer. Here's a sketch of the graph of f



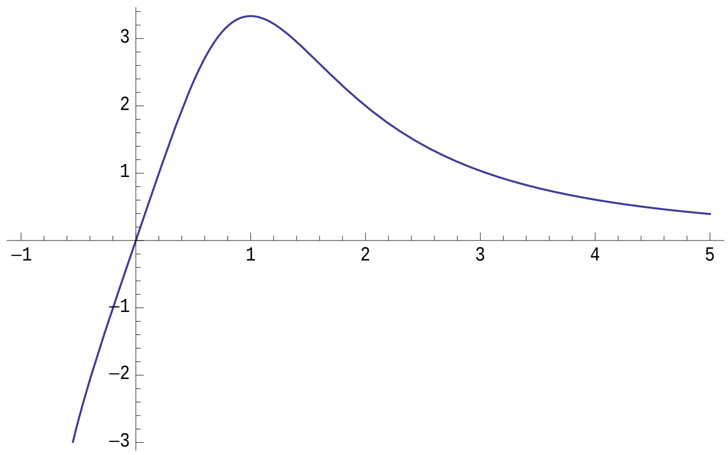
Here's a sketch of the graph of A



Here's a closeup of the graph of A showing that it's a union of parabolic segments. Over $[0, 1]$, the graph of A is the same as the curve $y = x^2$. Over $[1, 2]$, the graph of A is congruent to the graph over $[0, 1]$, translated over one unit and up one unit. And so on...



Problem 4. [3 points] Let $A(x) = \displaystyle\int_1^x \frac{10t\,dt}{2+t^3}$ for $x \geq -\sqrt[3]{2}$. Here's a sketch of $y = \dfrac{10t}{2+t^3}$.



(a) Determine $A(x)$ for a few values of x, say $x = -1, 0, 1, 5$, Just eyeball it, or use a couple of rectangles to approximate.

Answer. Let

$$f(t) = \frac{10t}{2+t^3}.$$

For $A(5) = \int_1^5 f(t)dt$, taking a guess at the area, I'd say $A(5) \approx 5$. I know $A(1) = 0$ exactly. Note that $A(0) = \int_1^0 f(t)dt$ is negative the area under the curve $y = f(t)$ from $t = 0$ to $t = 1$, which I approximate as $A(0) \approx -2$—it looks like it's more than half of the rectangle of width 1 and height $\frac{10}{3}$. To approximate $A(-1) = \int_1^{-1} f(t)dt$, we look at the area trapped between the curve $y = f(t)$ over the interval $[-1, 1]$ with $A(-1) = \int_1^{-1} f(t)dt$ being the area below minus the area above. I guess that the area below is greater than the area above (note that $f(-1) = -10$), so $A(-1)$ will be positive again. I estimate $A(-1) \approx 1$. I summarize

$$A(-1) \approx 1 \qquad A(0) \approx -2 \qquad A(1) = 0 \qquad A(5) \approx 5.$$

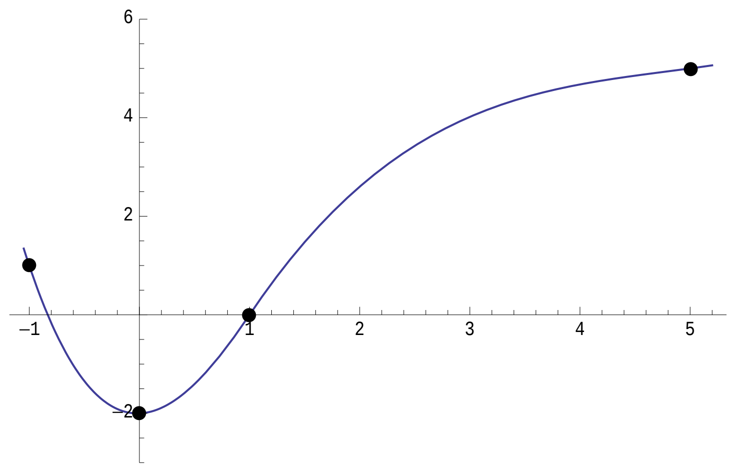
(b) A has a minimum on $(-\sqrt[3]{2}, \infty)$. What is it?

Answer. Since $f(t) < 0$ for $t < 0$, A is decreasing when $t < 0$. Then, for $t > 0$, $f(t) > 0$, so A is increasing when $t > 0$. Therefore A has a minimum when $x = 0$.

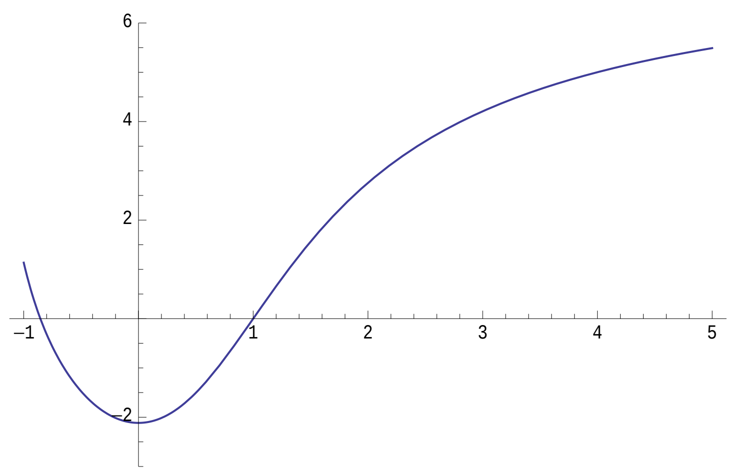
(c) Sketch the graph of A.

Answer. Here's a sketch using

- My estimates: $A(-1) \approx 1 \qquad A(0) \approx -2 \qquad A(1) = 0 \qquad A(5) \approx 5.$
- A is decreasing when $t < 0$ and A is increasing when $t > 0$.
- A is convex for $t < 1$ and concave when $t > 1$.



Remark. Here's a sketch of the graph of A produced by a computer using some very good estimates:



For example, the computer estimated that

$$\begin{aligned} &A(-1) \\ &= -\frac{5\left(\frac{\pi}{\sqrt{3}} + \log(2) + \log\left(2 + \sqrt[3]{2} - 2^{2/3}\right) - 2\log\left(2 + 2^{2/3}\right) + 2\sqrt{3}\tan^{-1}\left(\frac{2^{2/3}-1}{\sqrt{3}}\right)\right)}{3\sqrt[3]{2}} \\ &\approx -2.11528920395134306608622449326591198141 \end{aligned}$$

Problem 5. **[Bonus. 2 points]** Find a function f so that

$$\int_1^x f(t)dt = x^2 + 2x + 5$$

or prove that no such function exists.

Answer. No such function exists. The left hand side $\int_1^x f(t)dt = 0$ when $x = 1$ but the righthand side $x^2 + 2x + 5 = 8$ when $x = 1$.