Problem 1. **[1 point each]** Compute:

(a) $\lim_{x\to 2} \dfrac{3x^2-5x-2}{2x^2-9x+10}$.

Answer.

$$\begin{aligned}\lim_{x\to 2} \frac{3x^2-5x-2}{2x^2-9x+10} &= \lim_{x\to 2} \frac{(x-2)(3x+1)}{(x-2)(2x-5)} \\ &= \lim_{x\to 2} \frac{3x+1}{2x-5} \\ &= -7.\end{aligned}$$

(b) $\lim_{x\to 0} x^2 \sin\left(\dfrac{1}{x}\right)$.

Answer. Since $-1 \le \sin\left(\frac{1}{x}\right) \le 1$ for all x for all $x \ne 0$, we have

$$-x^2 \le \sin\left(\frac{1}{x}\right) \le x^2$$

for all $x \ne 0$. Since $\lim_{x\to 0} -x^2 = \lim_{x\to 0} x^2 = 0$, the Squeezing Principle (Theorem 3.3 in the book) says $\lim_{x\to 0} x^2 \sin\left(\dfrac{1}{x}\right) = 0$ also.

(c) $\lim_{x\to 0} \dfrac{\sqrt{1-x}-1}{x}$.

Answer.

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{1-x}-1}{x} &= \lim_{x\to 0} \frac{\sqrt{1-x}-1}{x}\left(\frac{\sqrt{1-x}+1}{\sqrt{1-x}+1}\right) \\ &= \lim_{x\to 0} \frac{1-x-1}{x\left(\sqrt{1-x}+1\right)} \\ &= \lim_{x\to 0} \frac{-x}{x\left(\sqrt{1-x}+1\right)} \\ &= \lim_{x\to 0} \frac{-1}{\sqrt{1-x}+1} \\ &= -\frac{1}{2}.\end{aligned}$$

(d) $\lim_{x\to 0} \dfrac{\tan(2x)}{\sin 3x}$.

Answer. We use the fact that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1.$$

This limit follows from the fundamental inequality

$$0 < \cos(x) < \frac{\sin(x)}{x} < \frac{1}{\cos(x)} \text{ for } 0 < x < \frac{\pi}{2}$$

and the squeeze theorem. See Example 4 on page 134 for all the details. It follows that

$$\lim_{x\to 0} \frac{\sin(2x)}{2x} = 1 \text{ and } \lim_{x\to 0} \frac{3x}{\sin(3x)} = 1.$$

Now,

$$\begin{aligned}\lim_{x\to 0} \frac{\tan(2x)}{\sin(3x)} &= \lim_{x\to 0} \frac{\sin(2x)}{\cos(2x)} \frac{1}{\sin(3x)} \\ &= \lim_{x\to 0} \frac{1}{\cos(2x)} \left(\frac{2}{3}\right) \left(\frac{\sin(2x)}{2x}\right) \left(\frac{3x}{\sin(3x)}\right) \\ &= \frac{2}{3}.\end{aligned}$$

Problem 2. [1 point each] True or False. Give proofs or counterexamples.

(a) If $\lim_{x\to a} f(x)$ does not exist and $\lim_{x\to a} g(x) = L$, then $\lim_{x\to a} f(x) + g(x)$ does not exist.

Answer. True. Suppose both limits $\lim_{x\to a} g(x)$ and $\lim_{x\to a} f(x) + g(x)$ exist. Say, $\lim_{x\to a} g(x) = L$ and $\lim_{x\to a} f(x) + g(x) = M$. Then, by Theorem 3.1 part (ii), we know the limit of the difference

$$\lim_{x\to a} (f(x) + g(x)) - f(x)$$

exists and equals $M - L$. That is, if $\lim_{x\to a} g(x)$ exists and $\lim_{x\to a} f(x) + g(x)$ exists, then $\lim_{x\to a} f(x)$ exists also. Therefore, if $\lim_{x\to a} f(x)$ does not exist and $\lim_{x\to a} g(x)$ exists then it is impossible that for $\lim_{x\to a} f(x) + g(x)$ to exist.

(b) If $\lim_{x\to a} f(x)$ does not exist and $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a} f(x)g(x)$ does not exist.

Answer. False. For example, let $f(x) = 0$ for $x \in \mathbb{Q}$ and $f(x) = 1$ for $x \in \mathbb{R} \setminus \mathbb{Q}$. Let $g(x) = 1$ for $x \in \mathbb{Q}$ and $g(x) = 0$ for $x \in \mathbb{R} \setminus \mathbb{Q}$. Note that neither $\lim_{x\to 3} f(x)$ nor $\lim_{x\to 3} g(x)$ exist. However, the function $f(x)g(x) = 0$, so $\lim_{x\to 3} f(x)g(x) = \lim_{x\to 3} 0 = 0$ exists.

(c) If $\lim_{x\to a} f(x)$ does not exist and $\lim_{x\to a} g(x) = L$, then $\lim_{x\to a} f(x)g(x)$ does not exist.

Answer. False. For example, let $f(x) = 0$ for $x \in \mathbb{Q}$ and $f(x) = 1$ for $x \in \mathbb{R} \setminus \mathbb{Q}$. Let $g(x) = 0$ for all $x \in \mathbb{R}$. Note that $\lim_{x\to 3} f(x)$ does not exist and $\lim_{x\to 3} g(x) = 0$ exists. Also, the function $f(x)g(x) = 0$, so $\lim_{x\to 3} f(x)g(x) = \lim_{x\to 3} 0 = 0$ exists.

(d) If $f(x) \geq 0$ for all x in an interval $[a, b]$ and $\int_a^b f = 0$, then $f = 0$.

Answer. False. Let $f(x) = 1$ for $x = \frac{1}{2}$ and let $f(x) = 0$ for all other x. Then $\int_0^1 f(x) = 0$, but $f \neq 0$.

Problem 3. [1 point each] Definitions and theorems

(a) Let f be a function defined on an open neighborhood of c. Define the statement "f is continuous at c."

(b) State Bolzano's theorem.

(c) State the intermediate value theorem.

(d) State the mean value theorem for integrals.

Answer. See the textbook.

Problem 4. [Bonus 2 points] Prove:

Theorem. *Suppose f is continuous on $[a, b]$ for some numbers $a < b$ and that $f(x) \geq 0$ for all $x \in [a, b]$. If $\int_a^b f = 0$ then $f(x) = 0$ for all $x \in [a, b]$.*

Answer. Let $a < b$ and let f be a continuous function on $[a, b]$ satisfying $f(x) \geq 0$ for all $x \in [a, b]$.

Suppose that there is a number c with $f(c) \neq 0$. Say $f(c) = y > 0$. Since f is continuous, there exists a number $\delta > 0$ so that $f(x) > \frac{y}{2}$ for all $x \in (c-\delta, c+\delta) \subset [a, b]$. Therefore, the function s defined by $s(x) = \frac{y}{2}$ for $c - \delta < x < c + \delta$ and $s(x) = 0$ for all other x satisfies

$$s(x) \leq f(x) \text{ for all } x \in [a, b].$$

Thus,

$$\int_a^b f \geq \int_a^b s = \frac{y(b-a)}{2} > 0.$$

This proves that if it is not true that $f(x) = 0$ for all $x \in [a, b]$, then $\int_a^b f \neq 0$.

Therefore, if $\int_a^b f = 0$ then $f(x) = 0$ for all $x \in [a, b]$.