Problem 1. Let $f(x) = x^3 \arctan\left(x^2\right) \sinh\left(6x^7\right)$. Find $f^{(50)}(0)$.

Problem 2. Matching

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^{n+\frac{1}{2}}}.$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{i^n}{2^n}.$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n}.$

(d) $\displaystyle\int_1^{\infty} \frac{dx}{1+x^2}.$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{i^n}{n2^n}.$

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(\pi i)^{2n}}{(2n)!}.$

(g) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{i}{n!}\right).$

(h) $\displaystyle\int_1^{\infty} \frac{dx}{x^2}.$

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{\pi^n i^n}{n!2^n}.$

(j) $\displaystyle\int_0^{\infty} e^{-x^2} dx.$

The answers are

$$\frac{\pi}{6}, \quad \ln(2), \quad 1, \quad \infty, \quad \frac{\sqrt{\pi}}{2}, \quad i, \quad \frac{-1+2i}{5}, \quad \ln\left(\frac{2}{\sqrt{5}}\right) + i \arctan\left(\frac{1}{2}\right), \quad \frac{\pi}{4}, \quad \frac{e^{\pi}+e^{-\pi}}{2}.$$

Problem 3. Define precisely:

(a) The sequence of functions $\{u_k\}$ converges pointwise to f on the set S.

(b) The sequence of functions $\{u_k\}$ converges uniformly to f on the set S.

Problem 4. Define three sequences of functions by

$$f_n(x) = \frac{2nx}{1+n^2x^2}, \quad g_n(x) = \frac{2x}{1+n^2x^2}, \quad h_n(x) = \frac{2n}{1+n^2x^2}.$$

All three sequences of functions $\{f_n\}$, $\{g_n\}$, and $\{h_n\}$ converge pointwise to 0 on $(0,1]$.

(a) Which one converges uniformly? Prove it.

(b) For which does $\lim_{n\to\infty} \int_0^1 u_n \neq \int_0^1 \lim_{n\to\infty} u_n$?

Problem 5. For any complex number $z \neq 0$, we define $\log(z) = \ln|z| + i\arg(z)$. Then, recalling that $e^{x+iy} := e^x\left(\cos(y) + i\sin(y)\right)$, we can define z^w for any $z, w \in \mathbb{C}$, $z \neq 0$ by

$$z^w := e^{w\log(z)}.$$

(a) Compute $\log(-1)$ and $(-1)^i$.

(b) Prove that $z^{w_1}z^{w_2} = z^{w_1+w_2}$.

(c) Prove or disprove: $(z_1^w)(z_2^w) = (z_1z_2)^w$.

Problem 6.

(a) Find a power series for $\sqrt{1+x}$ centered at $x = 0$.

Problem 6. Continued.

(b) Compute $\lim_{x\to 0} \frac{\sqrt{1+2x^3}-1-x^3}{x^6}$.

(c) Approximate $\int_0^{\frac{1}{2}} \sqrt{1+2x^3}$ with an error less than $\frac{1}{20480}$.

Problem 7. True or False. Right answer $+1$, wrong answer -2.

(a) The sequence of functions $\{x^n\}$ converges uniformly to 0 on the set $(0, 1]$.

(b) $\sin\left(x^2\right)\cos\left(x^2\right) = \frac{1}{2}\left(x^2 - \frac{2^3}{3!}x^6 + \frac{2^5}{5!}x^{10} - \frac{2^7}{7!}x^{14} + \cdots\right)$ for all x.

(c) $\sin(i\theta) = i\sinh(\theta)$.

(d) If $\sum_{n=0}^{\infty} a_n(-4)^n$ converges absolutely, then $\sum_{n=0}^{\infty} a_n 4^n$ converge absolutely.

(e) Suppose that a_n is a decreasing sequence of positive numbers. Then the sequence $\{t_n\}$, defined by $t_n = a_1 - a_2 + a_3 - + \cdots + a_{2n-1} - a_{2n}$, converges.

(f) If $a_n > 0$ for all n and $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{1 + a_n}$ diverges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges too.

(g) If $\{u_k\}$ is a sequence of increasing functions converging uniformly to f on a set S, then the sequence of numbers $\{u_k'(x)\}$ converges to $f'(x)$ for each $x \in S$.

EXAM

Final Exam

Math 158

May 26, 2005