EXAM

Final Exam

Math 158

May 26, 2005

ANSWERS

Problem 1. Let $f(x) = x^3 \arctan\left(x^2\right) \sinh\left(6x^7\right)$. Find $f^{(50)}(0)$.

Answer:
The number $f^{(50)}(0)$ is easily determined from the coefficient of x^{50} in the power series for f. The coefficient of x^{50} can be determined by looking at the power series

$$x^3 \arctan(x^2) = x^5 - \frac{x^9}{3} + \frac{x^{13}}{5} - \frac{x^{17}}{7} + \frac{x^{21}}{9} - \frac{x^{25}}{11} + \frac{x^{29}}{13} - + \cdots$$

and

$$\sinh(6x^7) = 6x^7 + 36x^{21} + \frac{324x^{35}}{5} + \frac{1944x^{49}}{35} + \cdots$$

and realizing that the only term of order x^{50} in the product $f(x) = x^3 \arctan\left(x^2\right) \sinh\left(6x^7\right)$ is

$$\left(\frac{x^{29}}{13}\right)\left(36x^{21}\right) = \frac{36}{13}x^{50}.$$

Therefore, we know

$$\frac{f^{(50)}(0)}{50!} = \frac{36}{13} \Rightarrow f^{(50)}(0) = (50!)\frac{36}{13}.$$

Problem 2. Matching

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^{n+\frac{1}{2}}} = \frac{\pi}{6}.$

Answer:
$\arctan\left(\frac{1}{\sqrt{3}}\right)$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{i^n}{2^n} = \frac{-1+2i}{5}$

Answer:

Convergent geometric series with ratio $\frac{i}{2}$.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n2^n} = \log(2).$

Answer:

$-\log(1-x)$ when $x = \frac{1}{2}$.

Problem 2. Continued.

(d) $\displaystyle\int_1^\infty \frac{dx}{1+x^2} = \frac{\pi}{4}.$

Answer:

$\arctan'(x) = \frac{1}{1+x^2}.$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{i^n}{n2^n} = \ln\left(\frac{2}{\sqrt{5}}\right) + i\arctan\left(\frac{1}{2}\right).$

Answer:

$-\log(1-x)$ when $x = \frac{i}{2}$.

(f) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n(\pi i)^{2n}}{(2n)!} = \frac{e^\pi + e^{-\pi}}{2}.$

Answer:

$\cos(i\pi) = \cosh(\pi).$

(g) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{i}{n!}\right) = \infty$

Answer:

The harmonic series diverges.

(h) $\displaystyle\int_1^\infty \frac{dx}{x^2} = 1$

Answer:

compute

(i) $\displaystyle\sum_{n=0}^{\infty} \frac{\pi^n i^n}{n!2^n} = i$

Answer:

$\exp\left(\frac{\pi i}{2}\right).$

(j) $\displaystyle\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$

Answer:

Process of elimination.

Problem 3. Define precisely:

(a) The sequence of functions $\{u_k\}$ converges pointwise to f on the set S.

Answer:

The sequence of functions $\{u_k\}$ converges pointwise to f on the set S if and only if for each $x \in S$ the sequence of numbers $\{u_k(x)\}$ converges to the number $f(x)$.

(b) The sequence of functions $\{u_k\}$ converges uniformly to f on the set S.

Answer:

The sequence of functions $\{u_k\}$ converges uniformly to f on the set S if and only if for each $\epsilon > 0$ there exists a natural number N so that for all $n \geq N$ and for all $x \in S$

$$|u_k(x) - f(x)| < \epsilon.$$

Problem 4. Define three sequences of functions by

$$f_n(x) = \frac{2nx}{1+n^2x^2}, \quad g_n(x) = \frac{2x}{1+n^2x^2}, \quad h_n(x) = \frac{2n}{1+n^2x^2}.$$

All three sequences of functions $\{f_n\}$, $\{g_n\}$, and $\{h_n\}$ converge pointwise to 0 on $(0,1]$.

(a) Which one converges uniformly? Prove it.

Answer:

The sequence of functions $\{g_n(x)\}$ converges uniformly to zero on $[0,1]$. To prove it, we use the fact that $0 \le g_n(x) \le \frac{1}{n}$ for all $0 \le x \le 1$. Therefore, if $\epsilon > 0$ has been given, choose N to be a natural number greater than $\frac{1}{\epsilon}$. Then, if $n > N$, we have $n > \epsilon$ and $|g_n(x)| \le \frac{1}{n} < \epsilon$ for all $x \in [0,1]$.

To see that $0 \le g_n(x) \le \frac{1}{n}$ for all $x \in [0,1]$, notice that for each n, the function g_n is continuous on $[0,1]$ and therefore has a maximum. Since g_n is differentiable, the maximum must occur at $x = 0$, $x = 1$, or at a critical point x where $g_n'(x) = 0$. A quick computation reveals that

$$g_n'(x) = \frac{2 - 2n^2x^2}{(n^2x^2+1)^2}$$

and we see that $g_n'(x) = 0$ when $x = \frac{1}{n}$. Checking $g(0) = 0$, $g\left(\frac{1}{n}\right) = \frac{1}{n}$ and $g(1) = \frac{2}{1+n^2}$ reveals the maximum of g_n is $\frac{1}{n}$.

(b) For which does $\lim_{n\to\infty} \int_0^1 u_n \neq \int_0^1 \lim_{n\to\infty} u_n$?

Answer:

The answer is $\{h_n(x)\}$ since

$$\lim_{n\to\infty} \int_0^1 h_n = \lim_{n\to\infty} \int_0^1 \frac{2n}{n^2x^2+1}\,dx = \lim_{n\to\infty} 2\arctan(n) = \pi \neq 0.$$

Remark: We confirm that there are no other correct answers to (a) or (b).

Since $\{g_n(x)\}$ converges uniformly to zero on $[0,1]$, we know $\lim_{n\to\infty} \int_0^1 g_n = \int_0^1 \lim_{n\to\infty} g_n$. So $\{g_n\}$ isn't an answer to (b).

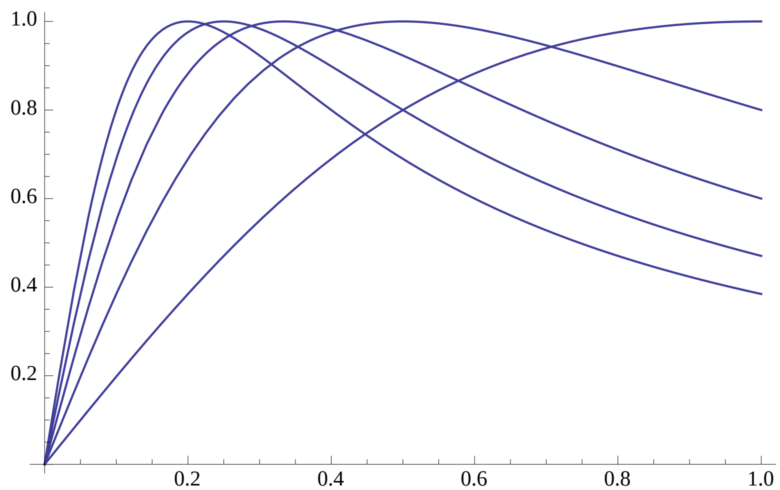
To see that $\{f_n\}$ isn't an answer to (b), we compute

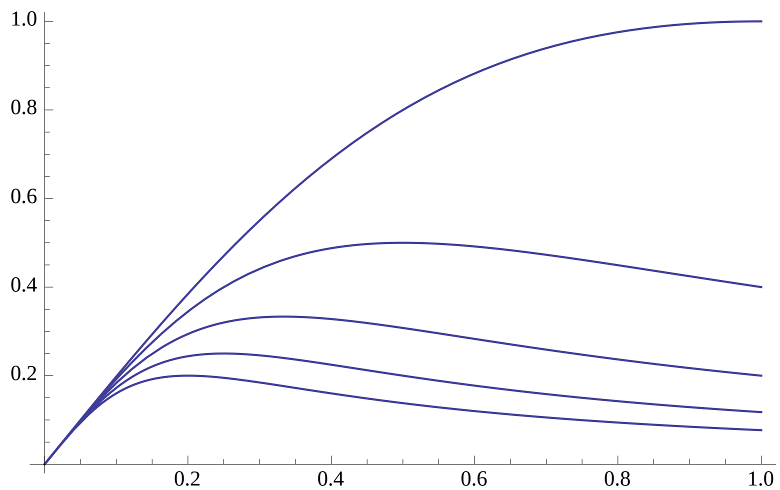
$$\lim_{n\to\infty} \int_0^1 f_n = \lim_{n\to\infty} \int_0^1 \frac{2nx\,dx}{n^2x^2+1} = \lim_{n\to\infty} \frac{\ln(1+n^2)}{n} = 0.$$

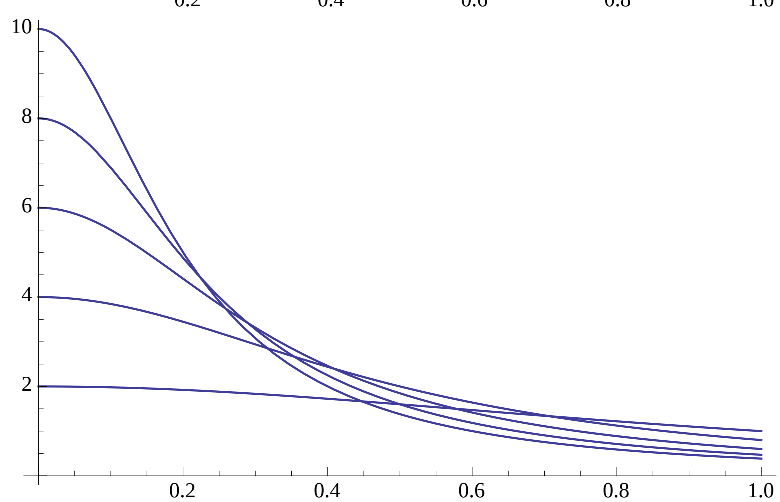
To see that $\{f_n\}$ isn't an answer for (a), note that $f_n\left(\frac{1}{n}\right) = 1$ so it is impossible for $|f_n(x) - 0| < \epsilon$ for all $x \in (0,1]$ if $\epsilon < 1$.

Finally $\{h_n\}$ isn't an answer to (a) since $\lim_{n\to\infty} \int_0^1 h_n \neq 0$.

Problem 4. Continued. Here are plots of $\{f_n\}$, $\{g_n\}$, and $\{h_n\}$:







Problem 5. For any complex number $z \neq 0$, we define $\log(z) = \ln|z| + i\arg(z)$. Then, recalling that $e^{x+iy} := e^x(\cos(y) + i\sin(y))$, we can define z^w for any $z, w \in \mathbb{C}$, $z \neq 0$ by

$$z^w := e^{w\log(z)}.$$

(a) Compute $\log(-1)$ and $(-1)^i$.

Answer:

$\log(-1) = \ln(|-1|) + i\arg(-1) = \ln(1) + i\pi = i\pi$ and $(-1)^i = \exp(i\log(-1) = \exp(i(i\pi)) = \frac{1}{e^\pi}$.

(b) Prove that $z^{w_1}z^{w_2} = z^{w_1+w_2}$.

Answer:

First note that $e^{(x_1+iy_1)}e^{(x_2+iy_2)} = e^{(x_1+x_2+i(y_1+y_2))}$:

$$\begin{aligned} e^{(x_1+iy_1)}e^{(x_2+iy_2)} &= e^{x_1}(\cos(y_1) + i\sin(y_1))\, e^{x_2}(\cos(y_2) + i\sin(y_2)) \\ &= e^{x_1+x_2}(\cos(y_1)\cos(y_2) - \sin(y_1)\sin(y_2) + i\sin(y_1)\cos(y_2) + i\cos(y_1)\sin(y_2)) \\ &= e^{x_1+x_2}(\cos(y_1+y_2) + i\sin(y_1+y_2)) \\ &= e^{(x_1+x_2+i(y_1+y_2))}. \end{aligned}$$

$$z^{w_1}z^{w_2} = e^{w_1\log(z)}e^{w_2\log(z)} = e^{(w_1+w_2)\log(z)} = z^{w_1+w_2}.$$

(c) Prove or disprove: $(z_1^w)(z_2^w) = (z_1z_2)^w$.

Answer:

False. For example $(-1)^i(-1)^i = \left(\frac{1}{e^\pi}\right)\left(\frac{1}{e^\pi}\right) = \frac{1}{e^{2\pi}}$ which does not equal $((-1)(-1))^i = 1^i = e^{i\log(1)} = e^0 = 1$.

Problem 6.

(a) Find a power series for $\sqrt{1+x}$ centered at $x=0$.

Answer:

We use the binomial expansion $\sqrt{1+x} = 1 + \binom{\frac{1}{2}}{1}x + \binom{\frac{1}{2}}{2}x^2 + \binom{\frac{1}{2}}{3}x^3 + \cdots$ to get

$$1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} - \frac{21x^6}{1024} + - \cdots$$

(b) Compute $\lim\limits_{x\to 0} \dfrac{\sqrt{1+2x^3} - 1 - x^3}{x^6}$.

Answer:

It will be convenient to have the power series:

$$\sqrt{1+2x^3} = 1 + x^3 - \frac{x^6}{2} + \frac{x^9}{2} - \frac{5x^{12}}{8} + \cdots$$

So,

$$\lim_{x\to 0} \frac{\sqrt{1+2x^3} - 1 - x^3}{x^6} = \lim_{x\to 0} -\frac{1}{2} + \frac{x^3}{2} - \frac{5x^6}{8} + \cdots = -\frac{1}{2}.$$

(c) Approximate $\int_0^{\frac{1}{2}} \sqrt{1+2x^3}$ with an error less than $\dfrac{1}{20480}$.

Answer:

$$\begin{aligned}
\int_0^{\frac{1}{2}} \sqrt{1+2x^3}\,dx &= \int_0^{\frac{1}{2}} \left(1 + x^3 - \frac{x^6}{2} + \frac{x^9}{2} - \frac{5x^{12}}{8} + \cdots\right) dx \\
&= x + \frac{x^4}{4} - \frac{x^7}{(7)(2)} + \frac{x^{10}}{(10)(2)} - \frac{5x^{13}}{(13)(8)} + \cdots\Big]_0^{\frac{1}{2}} dx \\
&= \frac{1}{2} + \frac{1}{(4)(2^4)} - \frac{1}{(2^7)(7)(2)} + \frac{1}{(2^{10})(10)(2)} - \cdots \\
&\approx \frac{1}{2} + \frac{1}{(4)(2^4)} - \frac{1}{(2^7)(7)(2)} \\
&= \frac{923}{1792}
\end{aligned}$$

The error in approximating the exact expression, the infinite sum, by the sum of the first three terms is less than the fourth term

$$\frac{1}{(2^{10})(10)(2)} = \frac{1}{20480} = 0.0000488281\ldots$$

since the exact expression is the sum of a convergent alternating series whose terms decrease.

Problem 7. True or False. Right answer $+1$, wrong answer -2.

(a) The sequence of functions $\{x^n\}$ converges uniformly to 0 on the set $(0, 1]$.

Answer:

False. In fact, this sequence doesn't even converge pointwise on $(0, 1]$ since $\{f_n(1)\} = \{1\} \to 1 \neq 0$, for $f_n(x) = x^n$.

(b) $\sin\left(x^2\right)\cos\left(x^2\right) = \frac{1}{2}\left(x^2 - \frac{2^3}{3!}x^6 + \frac{2^5}{5!}x^{10} - \frac{2^7}{7!}x^{14} + \cdots\right)$ for all x.

Answer:

False, though it's almost correct since $\sin(2x) = 2\sin(x)\cos(x)$, we have

$$\sin\left(x^2\right)\cos\left(x^2\right) = \frac{1}{2}\sin(2x^2) = \frac{1}{2}\left(2x^2 - \frac{2^3}{3!}x^6 + \frac{2^5}{5!}x^{10} - \frac{2^7}{7!}x^{14} + \cdots\right)$$

So, the statement is false as stated, but would be correct if the first term were $2x^2$ instead of x^2.

(c) $\sin(i\theta) = i\sinh(\theta)$.

Answer:

True.

$$\sin(i\theta) = i\theta - (i^3)\frac{1}{3!}\theta^3 + (i^5)\frac{1}{5!}\theta^5 - (i^7)\frac{1}{7!}\theta^7 + - \cdots$$
$$= i\theta + i\frac{1}{3!}\theta^3 + i\frac{1}{5!}\theta^5 - i\frac{1}{7!}\theta^7 + - \cdots = i\sinh(\theta)$$

(d) If $\sum_{n=0}^{\infty} a_n(-4)^n$ converges absolutely, then $\sum_{n=0}^{\infty} a_n 4^n$ converge absolutely.

Answer:

True. $\sum_{n=0}^{\infty} a_n(-4)^n$ converging absolutely means that $\sum_{n=0}^{\infty} |a_n(-4)^n| = \sum_{n=0}^{\infty} |a_n|(4)^n$ converges, which means that $\sum_{n=0}^{\infty} a_n 4^n$ converge absolutely.

(e) Suppose that a_n is a decreasing sequence of positive numbers. Then the sequence $\{t_n\}$, defined by $t_n = a_1 - a_2 + a_3 - + \cdots + a_{2n-1} - a_{2n}$, converges.

Answer:

False. It is necessary that $a_n \to 0$. For example, if $a_n = 1 - \frac{1}{n}$, then t_n is the $2n$-th partial sum of $\sum_{k=1}^{\infty}(-1)^{n+1}\left(1 - \frac{1}{n}\right)$ which diverges.

Problem 7. Continued.

(f) If $a_n > 0$ for all n and $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ diverges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges too.

Answer:

True by the ordinary comparison test since if $a_n > 0$ we have

$$0 < \frac{a_n}{1+a_n} < a_n.$$

(g) If $\{u_k\}$ is a sequence of increasing functions converging uniformly to f on a set S, then the sequence of numbers $\{u_k'(x)\}$ converges to $f'(x)$ for each $x \in S$.

Answer:

This is false. For example the sequence $\{u_k\}$ given by $u_k(x) = \frac{\sin(kx)}{k}$ converges uniformly to 0 on $(0, \pi)$, but the sequence $\{u_k'(x)\} = \{\cos(nx)\}$ does not converge on $(0, \pi)$.