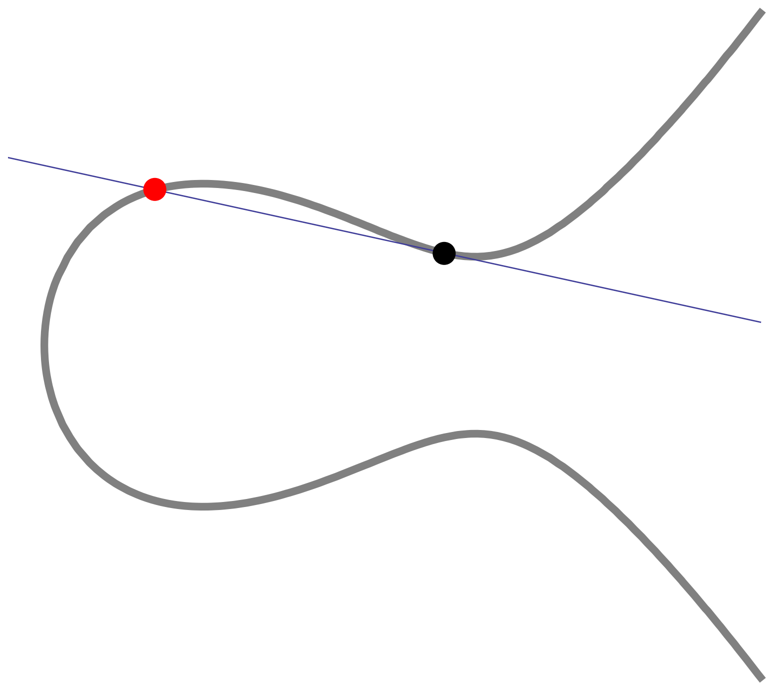
Problem 1. **[2 points]** Here's a picture of the elliptic curve $y^2 = x^3 - 5x + 8$. The coordinates of the black point are $(1, 2)$. What are the coordinates of the red point?



Problem 2. **[2 points]** Factoring the identity function.

(a) Find two functions f and g satisfying $f(0) = 0$, $g(0) = 0$ and $x = f(x)g(x)$.

(b) Prove that there do not exist two differentiable functions f and g with $f(0) = 0$, $g(0) = 0$ and $x = f(x)g(x)$.

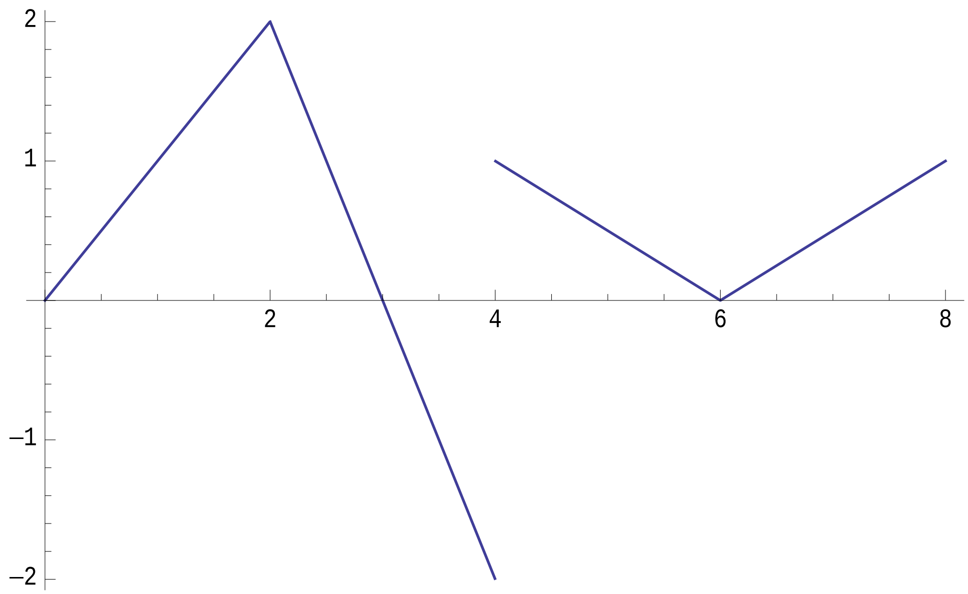
Problem 3. **[2 points]** Suppose that $f : [0,1] \to [0,1]$ is differentiable and that $f'(x) \neq 1$ for any x. Prove that there is one and only one number $c \in [0,1]$ with $f(c) = c$.

Problem 4. **[2 points]** The standard normal curve is given by

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Give a good sketch of this curve. Be sure to indicate the points at which the curve changes concavity.

Problem 5. **[2 points]** For any $x > 0$, let $A(x) = \int_2^x f(t)dt$ where f is the function whose graph is sketched below:



Give a good sketch of the graph of A. Indicate the value of A and A' at $x = 0, 1, 2, 3, 4, 5, 6, 7, 8$ and indicate on which intervals A is concave up and concave down.

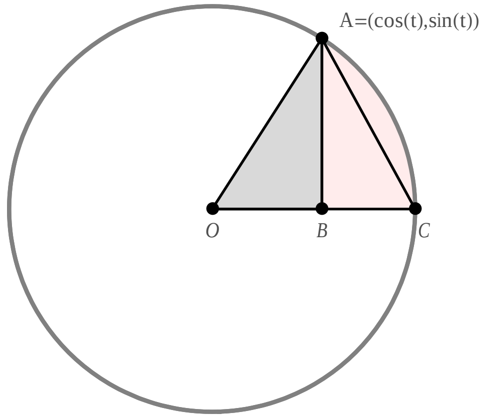
Problem 6. [2 points] Suppose $a < b < c < d < e$.

(a) Find the minimum of $f(x) = (x-a)^2 + (x-b)^2 + (x-c)^2 + (x-d)^2 + (x-e)^2$

(b) Find the minimum of $g(x) = |x-a| + |x-b| + |x-c| + |x-d| + |x-e|$

(c) **[Bonus point]** Prove that if $a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} + \frac{e}{5} = 0$ then $a + bx + cx^2 + dx^3 + ex^4 = 0$ for some $x \in [0, 1]$.

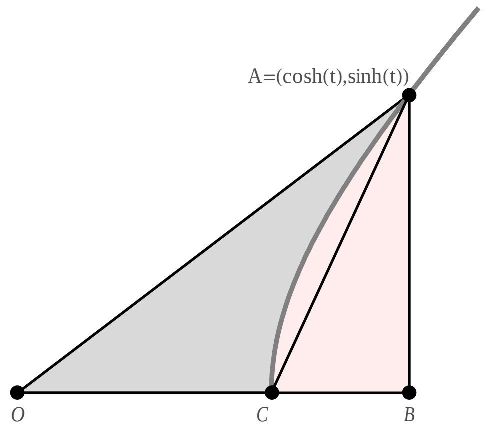
Problem 7. Here is the unit circle $x^2 + y^2 = 1$.



(a) **[1 point]** Express the area of the sector OAC in terms of t.

(b) **[1 point]** Suppose the point A is moving counterclockwise around the circle so that the area of the sector OAC is increasing at a constant rate. Determine the rate is the area of triangle OAC changing at the moment that it is isoceles.

Problem 7. **Continued.** Here is the unit hyperbola $x^2 - y^2 = 1$.



(b) **[1 point]** Express the area of the hyperbolic sector OAC in terms of t.

(c) **[Bonus point]** Suppose the point A is moving away from the origin so that the area of the sector OAC is increasing at a constant rate. Determine the rate is the area of triangle OAB changing at the moment that it is isoceles.

Problem 8. **[2 points]** Compute $\int_0^{\pi} \sin^n(x)dx$ for $n = 0, 1, 2, \ldots$.

Problem 9. **[All correct: 3 points. Eight correct: 1 points]** Compute. Interpret any integrals above as limits if necessary. For example,

$$\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \lim_{x \to 1^-} \int_0^x \frac{dt}{\sqrt{1-t^2}}.$$

(a) $\displaystyle\int_0^1 \frac{tdt}{\sqrt{1+t^2}}$

(b) $\displaystyle\int_0^1 \frac{tdt}{\sqrt{1-t^2}}$

(c) $\displaystyle\int_0^1 \frac{tdt}{1+t^2}$

(d) $\displaystyle\int_0^1 \frac{tdt}{1-t^2}$

(e) $\displaystyle\int_0^1 \frac{dt}{\sqrt{1+t^2}}$

Problem 9. Continued.

(f) $\displaystyle\int_0^1 \frac{dt}{\sqrt{1-t^2}}$

(g) $\displaystyle\int_0^1 \sqrt{1-t^2}dt$

(h) $\displaystyle\int_0^1 \sqrt{1+t^2}dt$

(i) $\displaystyle\int_0^1 \frac{dt}{1+t^2}$

(j) $\displaystyle\int_0^1 \frac{dt}{1-t^2}$

EXAM

Take Home Exam 1

Math 158: Spring 2013

Due: Tuesday, March 11

- This exam is due at the beginning of class on Tuesday, March 11.
- You are allowed to use your book or your notes, but you may not consult with any person about the exam.

Success!