EXAM

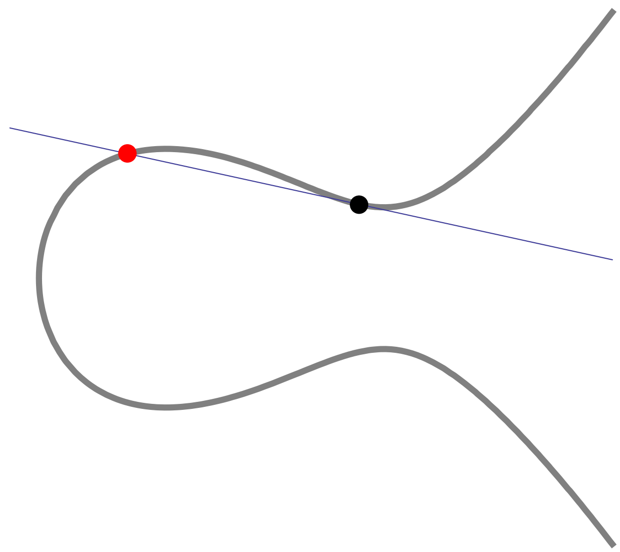
Take Home Exam 1

Math 158: Spring 2013

Due: Tuesday, March 11

ANSWERS

Problem 1. [2 points] Here's a picture of the elliptic curve $y^2 = x^3 - 5x + 8$. The coordinates of the black point are $(1, 2)$. What are the coordinates of the red point?



Answer:
First, differentiate to find the slope of the tangent line to the curve:

$$2y\frac{dy}{dx} = 3x^2 - 5 \Rightarrow \frac{dy}{dx} = \frac{3x^2 - 5}{2y}.$$

So, the tangent line at $(1, 2)$ has slope $-\frac{1}{2}$ and equation

$$y - 2 = -\frac{1}{2}(x - 1).$$

Since the red point lies both on the line and on the curve, its coordinates satisfy both $y^2 = x^3 - 5x + 8$ and $y - 2 = -\frac{1}{2}(x - 1)$. I solved these equations by writing $y = -\frac{1}{2}(x - 1) + 2$ and substituting it into $y^2 = x^3 - 5x + 8$ to get

$$\left(-\frac{1}{2}(x - 1) + 2\right)^2 = x^3 - 5x + 8 \Rightarrow 4x^3 - x^2 - 10x + 7 = 0.$$

Realizing that $x = 1$ is a solution to this equation helps to factor this cubic

$$4x^3 - x^2 - 10x + 7 = (x - 1)^2(4x + 7) = 0$$

and we find the x coordinate of the red point is $-\frac{7}{4}$. It follows that the y coordinate is $\frac{27}{8}$. So, the answer is $\left(\frac{7}{4}, \frac{27}{8}\right)$

Problem 2. **[2 points]** Factoring the identity function.

(a) Find two functions f and g satisfying $f(0) = 0$, $g(0) = 0$ and $x = f(x)g(x)$.

Answer:

There are many ways to do this. Here's one: $f(x) = x^{\frac{1}{3}}$ and $g(x) = x^{\frac{2}{3}}$.

(b) Prove that there do not exist two differentiable functions f and g with $f(0) = 0$, $g(0) = 0$ and $x = f(x)g(x)$.

Answer:

If $x = f(x)g(x)$ and f and g were differentiable, we'd have $1 = f'(x)g(x) + f(x)g'(x)$. Setting $x = 0$ yields $1 = f'(0)g(0) + f(0)g'(0)$. If $f(0) = g(0) = 0$, we get a contradiction $1 = 0$.

Note: Neither $x^{\frac{1}{3}}$ nor $x^{\frac{2}{3}}$ are differentiable at $x = 0$.

Problem 3. **[2 points]** Suppose that $f : [0, 1] \to [0, 1]$ is differentiable and that $f'(x) \neq 1$ for any x. Prove that there is one and only one number $c \in [0, 1]$ with $f(c) = c$.

Answer:

First, we prove there is a number $c \in [0, 1]$ with $f(c) = c$. We only need continuity for this. If $f(0) = 0$ or $f(1) = 1$, we're done. So, assume $f(0) > 0$ and $f(1) < 1$. Since f is differenitable on $[0, 1]$, it is continuous, hence $g(x) = f(x) - x$ is continuous. Since $g(0) = f(0) - 0 > 0$ and $g(1) = f(1) - 1 < 0$, the intermediate value theorem implies that there exists a number $c \in [0, 1]$ with $g(c) = 0$. That is, $f(c) = c$.

Now suppose f is differentiable and $f'(x) \neq 1$ for any x. Assume that $f(c) = c$ and $f(d) = d$. If $c \neq d$, the mean value theorem says there exists a number x between c and d with $f'(x) = \frac{f(d)-f(c)}{d-c} = 1$, which isn't possible if there are no numbers x with $f'(x) = 1$. Therefore $c = d$.

Problem 4. **[2 points]** The standard normal curve is given by

$$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

Give a good sketch of this curve. Be sure to indicate the points at which the curve changes concavity.

Answer:
Let $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$. Note that $f(x) > 0$ for all x. We compute

$$f'(x) = -\frac{1}{\sqrt{2\pi}} x e^{-\frac{x^2}{2}}.$$

Note that $f(0) = 0$, $f(x) > 0$ if $x < 0$ and $f(x) < 0$ if $x > 0$. Therefore,

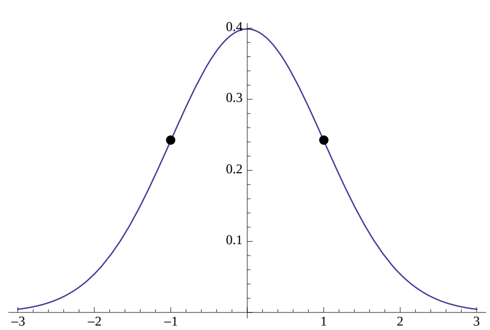
- f is increasing on the interval $(-\infty, 0)$
- f attains a maximum value of $f(0) = \frac{1}{\sqrt{2\pi}}$ at $x = 0$, and
- f is decreasing on the interval $(0, \infty)$

We compute

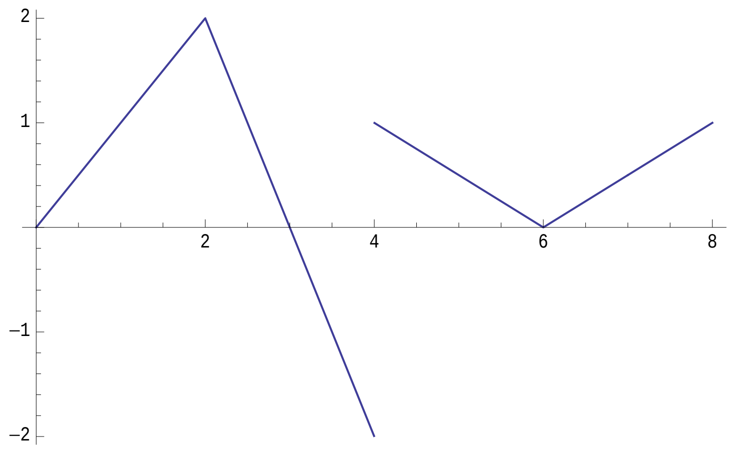
$$f''(x) = -\frac{1}{\sqrt{2\pi}} \left(e^{-\frac{x^2}{2}} - x^2 e^{-\frac{x^2}{2}} \right) = \frac{1}{\sqrt{2\pi}} \left(x^2 - 1 \right) e^{-\frac{x^2}{2}}.$$

Since the exponential function is always positive, the sign of $f''(x)$ equals the sign of $x^2 - 1$, which is negative on $(-1, 1)$ and positive elsewhere. Therefore

- f is concave up on $(\infty, -1) \cup (1, \infty)$, and
- f is concave down on the interval $(-1, 1)$.
- f changes concavity at the points $\left(-1, \frac{1}{\sqrt{2e\pi}}\right)$ and $\left(1, \frac{1}{\sqrt{2e\pi}}\right)$.



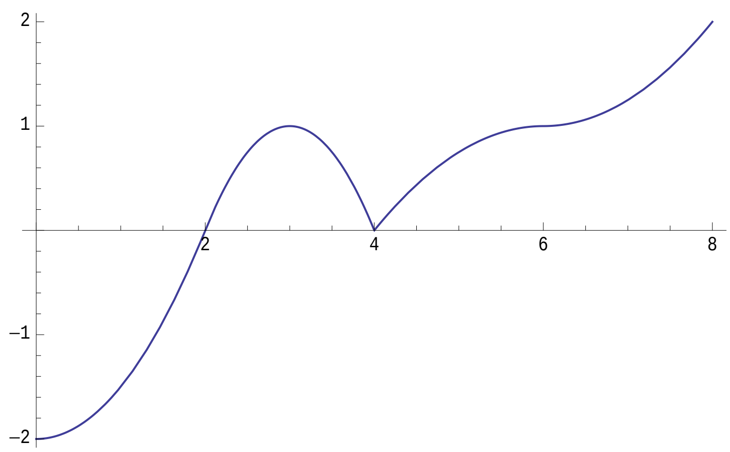
Problem 5. **[2 points]** For any $x > 0$, let $A(x) = \int_2^x f(t)dt$ where f is the function whose graph is sketched below:



Give a good sketch of the graph of A. Indicate the value of A and A' at $x = 0, 1, 2, 3, 4, 5, 6, 7, 8$ and indicate on which intervals A is concave up and concave down.

Answer:
From the fundamental theorem of calculus, $A'(x) = f(x)$, which can be read from the graph. Then, we can determine concavity by looking at $A''(x) = f'(x)$ and see that A is concave up when f is increasing (on the intervals $(0, 2)$ and $(6, 8)$) and see that A is concave down when f is decreasing (on the intervals $(2, 6)$.)
To find the values of A, I computed areas of triangles, keeping care with the signs. Here's a table, and a sketch:

x	0	1	2	3	4	5	6	7	8
$A'(x)$	0	1	2	0	DNE	$\frac{1}{2}$	0	$\frac{1}{2}$	1
$A(x)$	-2	$-\frac{3}{2}$	0	1	0	$\frac{3}{4}$	1	$\frac{5}{4}$	2



Problem 6. [2 points] Suppose $a < b < c < d < e$.

(a) Find the minimum of $f(x) = (x-a)^2 + (x-b)^2 + (x-c)^2 + (x-d)^2 + (x-e)^2$

Answer:

f is a degree two polynomial, it has one minimum where $f'(x) = 0$. A computation shows the minimum is at the average of the numbers a, b, c, d, e:

$$f'(x) = 2(x-a) + 2(x-b) + 2(x-c) + 2(x-d) + 2(x-e) = 0$$
$$\Rightarrow 5x = (a+b+c+d+e) \Rightarrow x = \frac{1}{5}(a+b+c+d+e).$$

So, the minimum value is

$$f\left(\frac{1}{5}(a+b+c+d+e)\right).$$

(c) **[Bonus point]** Prove that if $a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} + \frac{e}{5} = 0$ then $a + bx + cx^2 + dx^3 + ex^4 = 0$ for some $x \in [0, 1]$.

Answer:

Note if f is continuous on $[a, b]$ and $\int_a^b f = 0$, then there must exist a number $x \in [a, b]$ with $f(x) = 0$. If $f(x) > 0$ and continuous, then $\int_a^b f(x) > 0$. Similarly, if $f(x) < 0$ and continuous, $\int_a^b f(x) < 0$. Thus, if $\int_a^b f = 0$ and f is continuous, we know f can't be strictly positive, nor strictly negative, and so by the intermediate value theorem, f must be zero for at least one $x \in [a, b]$.

Now, applying this to

$$\int_0^1 a + bx + cx^2 + dx^3 + ex^4 dx = a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4} + \frac{e}{5}$$

proves the statement.

Problem 6. Continued.

(b) Find the minimum of $g(x) = |x-a| + |x-b| + |x-c| + |x-d| + |x-e|$

Answer:

The graph of g is continuous and piecewise linear—it's a straight line with nonzero slope on each subinterval $(-\infty, a)$, (a, b), (b, c), (c, d), (d, e), (e, ∞). The minimum of g must occur where the derivative doesn't exist; i.e. one of the points $\{a, b, c, d, e\}$. To figure out which, we'll evaluate g at each point.

$$
\begin{aligned}
g(a) &= (b-a) + (c-a) + (d-a) + (e-a) \\
g(b) &= (b-a) + (c-b) + (d-b) + (e-b) \\
g(c) &= (c-a) + (c-b) + (d-c) + (e-c) \\
g(d) &= (d-a) + (d-b) + (d-c) + (e-d) \\
g(e) &= (e-a) + (e-b) + (e-c) + (e-d)
\end{aligned}
$$

Since $a < b$, the last three terms of $g(a)$ are greater than the last three terms of $g(b)$, so we find that

$$g(a) > g(b).$$

Since $d < e$, the first three terms of $g(e)$ are greater than the first three of $g(d)$, so we find that

$$g(e) > g(d).$$

Now, by combining the first two terms of $g(b)$, the first and third terms of $g(c)$, and last two terms of $g(d)$ we find that

$$
\begin{aligned}
g(b) &= (c-a) + (d-b) + (e-b) \\
g(c) &= (c-a) + (d-a) + (e-c) \\
g(d) &= (d-a) + (d-b) + (e-c)
\end{aligned}
$$

Comparing $g(b)$ with $g(c)$, we find that

$$g(b) > g(c)$$

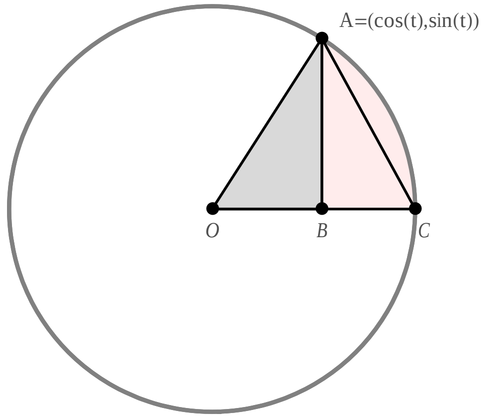
becasue the last two terms of $g(b)$ are greater than the last two terms of $g(c)$. Comparing $g(c)$ with $g(d)$, we find that

$$g(c) < g(d)$$

because the first two terms of $g(d)$ are greater than the first two terms of $g(c)$.

Conclusion: The minimum of g occurs at c and the minimum value is $d + e - a - b$.

Problem 7. Here is the unit circle $x^2 + y^2 = 1$.



(a) **[1 point]** Express the area of the sector OAC in terms of t.

Answer:

First, we compute the area of the region ABC (with the circular part):

$$\begin{aligned} Area(ABC) &= \int_{x=\cos(t)}^{x=\cos(0)} y dx = \int_t^0 \sin(u)(-\sin(u)du) \\ &= \int_0^t \sin^2(u)du \\ &= \frac{u}{2} - \frac{\cos(2u)}{4}\Big]_0^t \\ &= \frac{t}{2} - \frac{\cos(2t)}{4}. \end{aligned}$$

Now, we add the area of the triangle OAB which is $\frac{1}{2}\sin(t)\cos(t)$ to get the area of sector OAC

$$Area(OBC) = Area(ABC) + Area(OAB) = \frac{t}{2} - \frac{\cos(2t)}{4} + \frac{1}{2}\sin(t)\cos(t) = \frac{t}{2}.$$

To make the last simplification, recall that $\cos(2t) = 2\sin(t)\cos(t)$.

Problem 7. Continued.

(b) **[1 point]** Suppose the point A is moving counterclockwise around the circle so that the area of the sector OAC is increasing at a constant rate. Determine the rate is the area of triangle OAC changing at the moment that it is equilateral.

Answer:

It will be helpful to have some notation. Let a be the area of triangle OAC and let b be the area of sector OAC. Let s represent time, expressed in some units. The quantities a and b depend on time and so they are functions of s.

The area a (as a function of t) is given by $a(t) = \frac{1}{2}\left(\overline{OC}\right)\left(\overline{AB}\right) = \frac{1}{2}\sin(t)$. The rate at which a is changing is given by

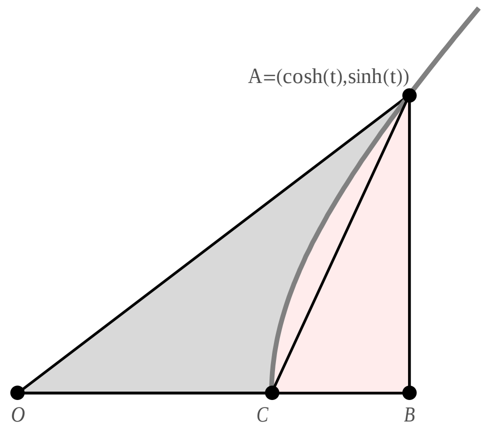
$$\frac{da}{ds} = \frac{da}{dt}\frac{dt}{ds} = \frac{1}{2}\cos(t)\frac{dt}{ds}.$$

The area b as a function of t was computed above to be $b(t) = \frac{t}{2}$. The rate at which b is changing (which we are told is constant) is given by

$$\frac{db}{ds} = \frac{db}{dt}\frac{dt}{ds} = \frac{1}{2}\frac{dt}{ds}.$$

Putting these together, we see that $\frac{da}{ds} = \cos(t)\frac{db}{ds}$. At the moment that the triangle is equilateral, $\cos(t) = \frac{1}{2}$ and we find that the rate at which the area of the triangle is increasing is $\frac{ds}{dt} = \frac{1}{4}\frac{dt}{ds}$, which is half the rate at which the area of the sector is increasing.

Problem 7. Continued. Here is the unit hyperbola $x^2 - y^2 = 1$.



(c) **[1 point]** Express the area of the hyperbolic sector OAC in terms of t.

Answer:

First, we compute the area of the region ABC (including the curved piece):

$$\begin{aligned} Area(ABC) &= \int_{x=\cosh(0)}^{x=\cosh(t)} y dx = \int_t^0 \sinh(u)(\sinh(u)du) \\ &= \int_0^t \sinh^2(u)du \\ &= -\frac{u}{2} + \frac{\sinh(2u)}{4}\Big]_0^t \\ &= -\frac{t}{2} + \frac{\sinh(2t)}{4}. \end{aligned}$$

The area of the triangle OAB is $\frac{1}{2}\cosh(t)\sinh(t)$. To get the answer, we subtract

$$Area(OBC) = Area(OAB) - Area(ABC) = \frac{1}{2}\sinh(t)\cosh(t) - \frac{t}{2} + \frac{\sinh(2t)}{4} = \frac{t}{2}.$$

To make the last simplification, recall that $\sinh(2t) = 2\sinh(t)\cosh(t)$.

Problem 7. Continued.

(d) **[Bonus point]** Suppose the point A is moving away from the origin so that the area of the sector OAC is increasing at a constant rate. Determine the rate is the area of triangle OAB changing at the moment that it is isoceles.

Answer:

The triangle OAB will never be isoceles since $\overline{OB} = \cosh(t)$ and $\overline{AB} = \sinh(t)$ are never equal (and the hypotenuse $\overline{OA}$ is always longer than both $\overline{OB}$ and $\overline{AB}$.)

So, instead, we'll compute the rate at which the area of triangle OAC is changing at the moment that it is isoceles.

It will be helpful to have some notation. Let a be the area of triangle OAC and let b be the area of sector OAC. Let s represent time, expressed in some units. The quantities a and b depend on time and so they are functions of s.

The area a (as a function of t) is given by $a(t) = \frac{1}{2}\left(\overline{OC}\right)\left(\overline{AB}\right) = \frac{1}{2}\sinh(t)$. The rate at which a is changing is given by

$$\frac{da}{ds} = \frac{da}{dt}\frac{dt}{ds} = \frac{1}{2}\cosh(t)\frac{dt}{ds}.$$

The area b as a function of t was computed above to be $b(t) = \frac{t}{2}$. The rate at which b is changing (which we are told is constant) is given by

$$\frac{db}{ds} = \frac{db}{dt}\frac{dt}{ds} = \frac{1}{2}\frac{dt}{ds}.$$

Putting these together, we see that $\frac{da}{ds} = \cosh(t)\frac{db}{ds}$. At the moment that the triangle is isoceles,

$$\overline{AC} = \overline{OC} \Leftrightarrow \sqrt{(\cosh(t)-1)^2 + \sinh(t)^2)} = 1.$$

Solving for $\cosh(t)$:

$$\begin{aligned}\sqrt{(\cosh(t)-1)^2 + \sinh(t)^2)} = 1 &\Rightarrow (\cosh(t)-1)^2 + \sinh(t)^2 = 1\\ &\Rightarrow (\cosh(t)-1)^2 + \cosh(t)^2 - 1 = 1\\ &\Rightarrow 2\cosh(t)^2 - 2\cosh(t) - 1 = 0\\ &\Rightarrow \cosh(t) = \frac{1}{2}\left(1 \pm \sqrt{3}\right)\end{aligned}$$

Since $\cosh(t) > 0$, we find that $\cosh(t) = \frac{1}{2}\left(1+\sqrt{3}\right)$ at the moment that the triangle OAC is isoceles.

Therefore, $\frac{da}{ds} = \frac{1}{2}\left(1+\sqrt{3}\right)\frac{db}{ds}$.

Problem 8. [2 points] Compute $\int_0^\pi \sin^n(x)dx$ for $n = 0, 1, 2, \ldots$.

Answer:
First, let us record a formula for integrating powers of the sine function:

$$\int \sin^k(t)dt = -\frac{1}{k}\sin^{k-1}(t)\cos(t) + \frac{k-1}{k}\int \sin^{k-2}(t)dt.$$

A way to deduce this formula is outlined as exercise 8 in section 5.10 in Apostol's book, and I do it as a footnote to this problem. Integrating over $[0, \pi]$ yields nice reduction formula:

$$\int_0^\pi \sin^k(t)dt = \frac{k-1}{k}\int_0^\pi \sin^{k-2}(t)dt. \tag{1}$$

Let $c_n = \int_0^\pi \sin^n(t)dt$. Notice that

$$c_1 = \int_0^\pi \sin(t) = 2$$

and

$$c_2 = \int_0^\pi \sin^2(t)dt = \int_0^\pi \frac{1-\cos(2t)}{2}dt = \frac{\pi}{2}.$$

Applying the reduction (1), one finds that c_n satisfies the recursion

$$c_n = \frac{n-1}{n}c_{n-1}$$

which allows us to determine c_n for every $n \in \mathbb{N}$:

$$\begin{aligned}
c_1 &= 2 \\
c_2 &= \frac{\pi}{2} \\
c_3 &= \frac{2}{3}c_1 = \frac{4}{3} \\
c_4 &= \frac{3}{4}c_2 = \frac{3}{8}\pi \\
c_5 &= \frac{4}{5}c_3 = \frac{16}{15} \\
c_6 &= \frac{5}{6}c_4 = \frac{5}{16}\pi \\
&\vdots \\
c_k &= \begin{cases} 2\left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\cdots\left(\frac{k-1}{k}\right) & \text{if } k \text{ is odd,} \\ \left(\frac{\pi}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{7}{8}\right)\cdots\left(\frac{k-1}{k}\right) & \text{if } k \text{ is even.} \end{cases}
\end{aligned}$$

Problem 9. **[All correct: 3 points. Eight correct: 1 points]** Compute. Interpret any integrals above as limits if necessary. For example,

$$\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \lim_{x\to 1^-} \int_0^x \frac{dt}{\sqrt{1-t^2}}.$$

(a) $\displaystyle\int_0^1 \frac{tdt}{\sqrt{1+t^2}} = \sqrt{2}-1$

Answer:

Let $u = 1+t^2$. Then $du = 2tdt \Rightarrow tdt = \frac{1}{2}du$ and $\int_0^1$ becomes $\int_1^2$. We have

$$\int_0^1 \frac{tdt}{\sqrt{1+t^2}} = \int_1^2 \frac{\frac{1}{2}du}{\sqrt{u}} = \sqrt{u}\Big]_1^2 = \sqrt{2}-1.$$

(b) $\displaystyle\int_0^1 \frac{tdt}{\sqrt{1-t^2}} = 1$

Answer:

Here, the integral is only defined over intervals contained in $(-1,1)$, so we interpret $\displaystyle\int_0^1 \frac{tdt}{\sqrt{1-t^2}}$ as $\displaystyle\lim_{x\to 1^-}\int_0^x \frac{tdt}{\sqrt{1-t^2}}$. Now, we use the substitution $u = 1-t^2$ which leads to $du = -2tdt \Rightarrow tdt = -\frac{1}{2}du$ and $\lim_{x\to 1^-}\int_0^x$ becomes $\lim_{y\to 0^+}\int_1^y$.

$$\begin{aligned}\int_0^1 \frac{tdt}{\sqrt{1-t^2}} &= \lim_{x\to 1^-}\int_0^x \frac{tdt}{\sqrt{1-t^2}} \\ &= \lim_{y\to 0^+}\int_1^y -\frac{\frac{1}{2}du}{\sqrt{u}} \\ &= -\sqrt{u}\Big]_1^y \\ &= \lim_{y\to 0^+} -\sqrt{y}-(-1) \\ &= 1.\end{aligned}$$

(c) $\displaystyle\int_0^1 \frac{tdt}{1+t^2} = \log(\sqrt{2})$

Answer:

Here, we use the same substitution as in part (a): $u = 1+t^2$. Then

$$\int_0^1 \frac{tdt}{1+t^2} = \int_1^2 \frac{\frac{1}{2}du}{u} = \frac{1}{2}\log(u)\Big]_1^2 = \frac{1}{2}\log 2.$$

Problem 9. Continued.

(d) $\displaystyle\int_0^1 \frac{tdt}{1-t^2} = \infty$

Answer:

First, we must interpret this integral as $\lim_{x\to 1^-} \int_0^x \frac{tdt}{1-t^2}$. Then, we make a substitution as in part (b):

$$\begin{aligned}
\int_0^1 \frac{tdt}{1-t^2} &= \lim_{x\to 1^-} \int_0^x \frac{tdt}{1-t^2} \\
&= \lim_{y\to 0^+} \int_1^y -\frac{\frac{1}{2}du}{u} \\
&= -\frac{1}{2}\log(u)\Big]_1^y \\
&= \lim_{y\to 0^+} -\frac{1}{2}\log(y) - \left(-\frac{1}{2}\log(1)\right) \\
&= \infty.
\end{aligned}$$

(e) $\displaystyle\int_0^1 \frac{dt}{\sqrt{1+t^2}} = \ln(1+\sqrt{2})$.

Answer:

If you remember that $\operatorname{arcsinh}'(t) = \frac{1}{\sqrt{1+t^2}}$, then this problem is very simple. If you don't, you can make the substitution $t = \sinh(u)$. Here, we change variables in the limit of integration: $t = 0 \Rightarrow u = 0$, and $t = 1 \Rightarrow u = \operatorname{arcsinh}(1)$. So, we get

$$\int_0^1 \frac{dt}{\sqrt{1+t^2}} = \int_0^{\operatorname{arcsinh}(1)} \frac{\cosh(u)du}{\cosh(u)} = \int_0^{\operatorname{arcsinh}(1)} du = \operatorname{arcsinh}(1).$$

It's not necessary, but one can rewrite $\operatorname{arcsinh}(1)$ as $\ln(1+\sqrt{2})$.

Remark: If you want to express the $\arcsin(t)$ in terms of logs but can't remember the formula, here's how to invert the hyperbolic sine. Write $u = \operatorname{arcsinh}(t) \Leftrightarrow t = \sinh(u) = t = \frac{1}{2}(e^u - e^{-u})$. Multiply by e^u and simplify to get $(e^u)^2 - 2te^u - 1 = 0$ and use the quadratic formula to get $e^u = t + \sqrt{1+t^2}$.

Problem 9. Continued.

(f) $\displaystyle\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \frac{\pi}{2}$

Answer:

First note that this integral is only defined over intervals within $(-1,1)$, so we interpret the integral as a limit: $\displaystyle\int_0^1 \frac{dt}{\sqrt{1-t^2}}$ as $\displaystyle\lim_{x\to 1^-}\int_0^x \frac{dt}{\sqrt{1-t^2}}$. If you remember that $\arcsin'(t) = \frac{1}{\sqrt{1-t^2}}$, then this problem can be solved as follows:

$$\int_0^1 \frac{dt}{\sqrt{1-t^2}} = \lim_{x\to 1^-}\int_0^x \frac{dt}{\sqrt{1-t^2}} = \lim_{x\ to 1^-} \arcsin(x) - \arcsin(0) = \frac{\pi}{2}.$$

If you don't remember that $\arcsin'(t) = \frac{1}{\sqrt{1-t^2}}$, use the substitution $t = \sin(u)$. Then $dt = \cos(u)du$ and $\lim_{x\to 1^-}\int_0^x$ becomes $\lim_{y\to\frac{\pi}{2}^-}\int_0^y$.

$$\begin{aligned}
\int_0^1 \frac{dt}{\sqrt{1-t^2}} &= \lim_{x\to 1^-}\int_0^x \frac{dt}{\sqrt{1-t^2}} \\
&= \lim_{y\to\frac{\pi}{2}^-}\int_0^y \frac{\cos(u)du}{\cos(u)} \\
&= \lim_{y\to\frac{\pi}{2}^-}\int_0^y du \\
&= \lim_{y\to\frac{\pi}{2}^-} y \\
&= \frac{\pi}{2}.
\end{aligned}$$

(g) $\displaystyle\int_0^1 \sqrt{1-t^2}dt = \frac{\pi}{2}$

Answer:

This is the area of the upper half of the unit circle, which is $\frac{\pi}{2}$. If you didn't realize this, you could make the substitution $t = \cos(u)$ and proceed (this will lead to a computation just like the one in 7(a)).

Problem 9. Continued.

(h) $\displaystyle\int_0^1 \sqrt{1+t^2}dt = \frac{\sqrt{2}}{2} + \frac{\log(1+\sqrt{2})}{2}.$

Answer:

I recommend using the substitution $t = \sinh(u)$. Then, $1 + t^2 = \sin^2(u) + 1 = \cosh^2(u)$ and $dt = \cosh(u)du$. Let's find the general antiderivative first, then compute the definite integral:

$$\begin{aligned}
\int \sqrt{1+t^2}\,dt &= \int \sqrt{\cosh^2(u)}\cosh(u)\,du \\
&= \int \cosh^2(u)\,du \\
&= \int \left(\frac{e^{2u}+2+e^{-2u}}{4}\right)\,dt \\
&= \frac{e^{2u}+4u-e^{-2u}}{8} + C
\end{aligned}$$

To finish, and put things in terms of t again, use the fact that $t = \sinh(u) \Rightarrow u = \ln(t+\sqrt{1+t^2})$ and use a little algebra

$$\begin{aligned}
&= \frac{(t+\sqrt{1+t^2})^2 + 4\ln(t+\sqrt{1+t^2}) - (t+\sqrt{1+t^2})^{-2}}{8} + C \\
&= \frac{t\sqrt{1+t^2}}{2} + \frac{\ln(t+\sqrt{1+t^2})}{2} + C.
\end{aligned}$$

So, we have

$$\int_0^1 \sqrt{1+t^2}dt = \frac{t\sqrt{1+t^2}}{2} + \frac{\ln(t+\sqrt{1+t^2})}{2}\Bigg]_0^1 = \frac{\sqrt{2}}{2} + \frac{\log(1+\sqrt{2})}{2}.$$

(i) $\displaystyle\int_0^1 \frac{dt}{1+t^2} = \frac{\pi}{4}.$

Answer:

Let $t = \tan(u)$. Here $t = \tan(u) \Rightarrow 1 + t^2 = \sec^2(u)$ and $dt = \sec^2(u)du$. Changing the limits of integration, $t = 0 \Rightarrow u = 0$ and $t = 1 \Rightarrow u = \frac{\pi}{4}$. We compute:

$$\int_0^1 \frac{dt}{1+t^2} = \int_0^{\frac{\pi}{4}} \frac{\sec^2(u)}{\sec^2(u)}du = u\Big]_0^{\frac{\pi}{4}} = \frac{\pi}{4}.$$

Problem 9. Continued.

(j) $\displaystyle\int_0^1 \frac{dt}{1-t^2} = \infty$

Answer:

First note that this integral is only defined over intervals within $(-1, 1)$, so we interpret the integral as a limit: $\displaystyle\int_0^1 \frac{dt}{1-t^2}$ as $\displaystyle\lim_{x\to 1^-}\int_0^x \frac{dt}{1-t^2}$. Then, use the fact that

$$\frac{1}{1-t^2} = \frac{\frac{1}{2}}{1+t} + \frac{\frac{1}{2}}{1-t}.$$

$$\begin{aligned}
\int_0^1 \frac{dt}{1-t^2} &= \lim_{x\to 1^-}\int_0^x \frac{dt}{1-t^2} \\
&= \lim_{x\to 1^-}\int_0^x \frac{\frac{1}{2}dt}{1+t} + \frac{\frac{1}{2}dt}{1-t} \\
&= \lim_{x\to 1^-} \frac{1}{2}\log(1+t) - \frac{1}{2}\log(1-t)\Big]_0^x \\
&= \infty
\end{aligned}$$