EXAM

Midterm Exam

Math 158: Spring 2013

Tuesday, March 17

ANSWERS

Problem 1. Carefully state:

(a) The mean value theorem.

Answer:

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a number $c \in [a, b]$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$.

(b) L'Hôpital's Rule.

Answer:

Suppose $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists. Then $\lim_{x\to c} \frac{f(x)}{g(x)}$ exists and

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

(c) The fundamental theorem of calculus.

Answer:

Let f be function that is integrable on the interval $[a, b]$ and let $c \in [a, b]$. Define $A : [a, b] \to \mathbb{R}$ by $A(x) = \int_c^x f(t)dt$. If f is continuous at x then A is differentiable at x and $A'(x) = f(x)$.

Problem 2. Matching

- $\displaystyle\int_1^\infty \frac{1}{t^4}dt = \frac{1}{3}$

- $\displaystyle\int_1^\infty \frac{1}{1+t^4}dt = \frac{\operatorname{arccoth}(\sqrt{2})}{4\sqrt{2}}$

- $\displaystyle\int_1^\infty \frac{t}{1+t^4} = \frac{\pi}{8}dt$

- $\displaystyle\int_0^\infty \frac{t}{e^t}dt = 1$

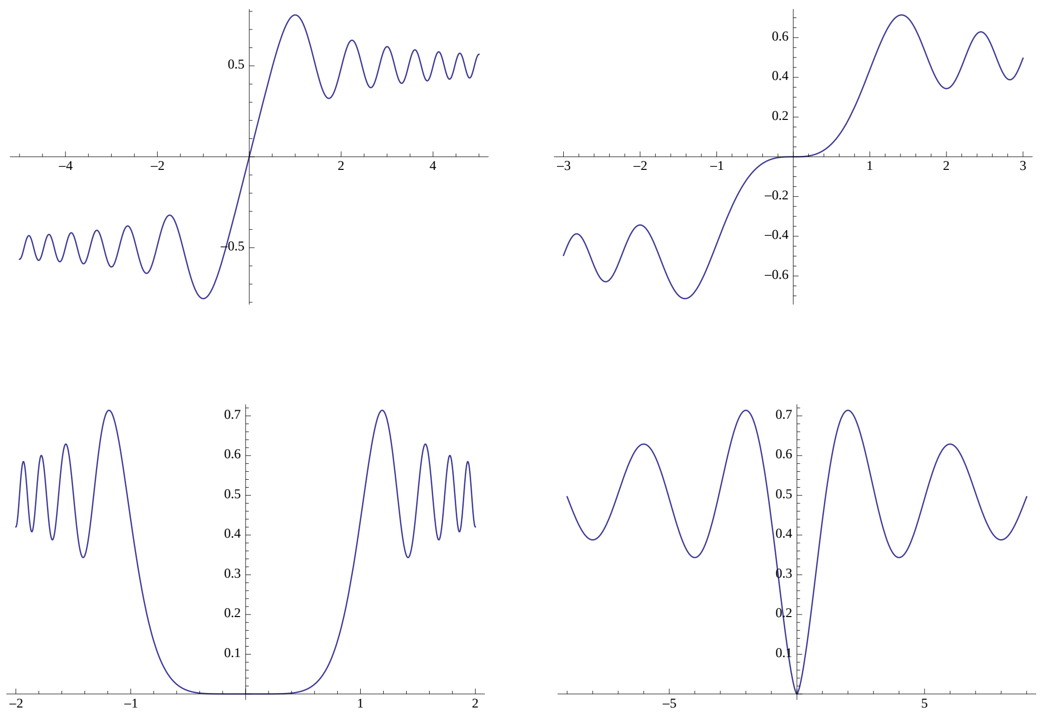
- $\displaystyle\int_0^\infty \frac{t}{e^{t^2}}dt = \frac{1}{2}$

- $\displaystyle\int_0^\infty \frac{1}{e^{t^2}}dt = \frac{\sqrt{\pi}}{2}$

Problem 3. The so called *Fresnel integral sine function* is defined to be

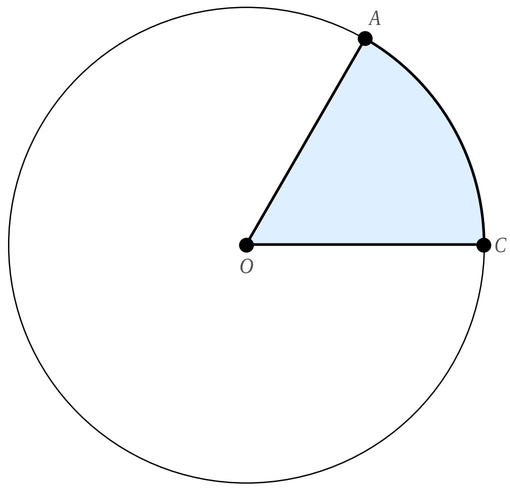
$$S(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right)\, dt.$$

Which is the graph of S?



Answer:
By process of elimination, the answer is the one in the top right. Note that $S'(0) = \sin(0) = 0$, which eliminates the graph on the top left. Moreover, $S'(x) = \sin\left(\frac{\pi x^2}{2}\right) \geq 0$ for $-\sqrt{2} \leq x \leq \sqrt{2}$, so S is increasing on $[-\sqrt{2}, \sqrt{2}]$. That elimates the two pictures on the bottom.

Problem 4. Here is the unit circle. The shaded sector AOC has area $\frac{t}{2}$. Use the integral formula for arclength to compute the length of the circular arc from A to C.



Answer:
The arc in question is part of the graph of $f(x) = \sqrt{1-x^2}$. We compute $\sqrt{1+f'(x)^2}$:

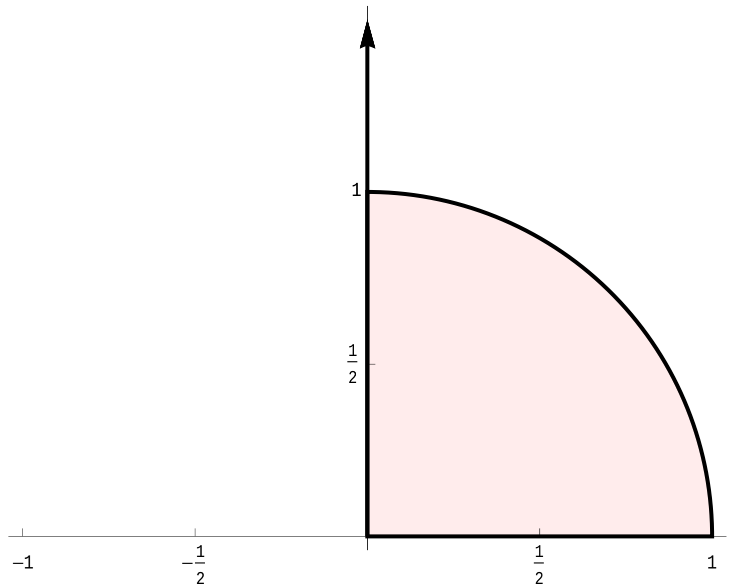
$$\begin{aligned}
\sqrt{1+f'(x)^2} &= \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2} \\
&= \sqrt{1+\frac{x^2}{1-x^2}} \\
&= \sqrt{\frac{1}{1-x^2}} \\
&= \frac{1}{\sqrt{1-x^2}}
\end{aligned}$$

The arc is the part of the graph over the interval $[\cos(t), 1]$ so the length of the arc is given by

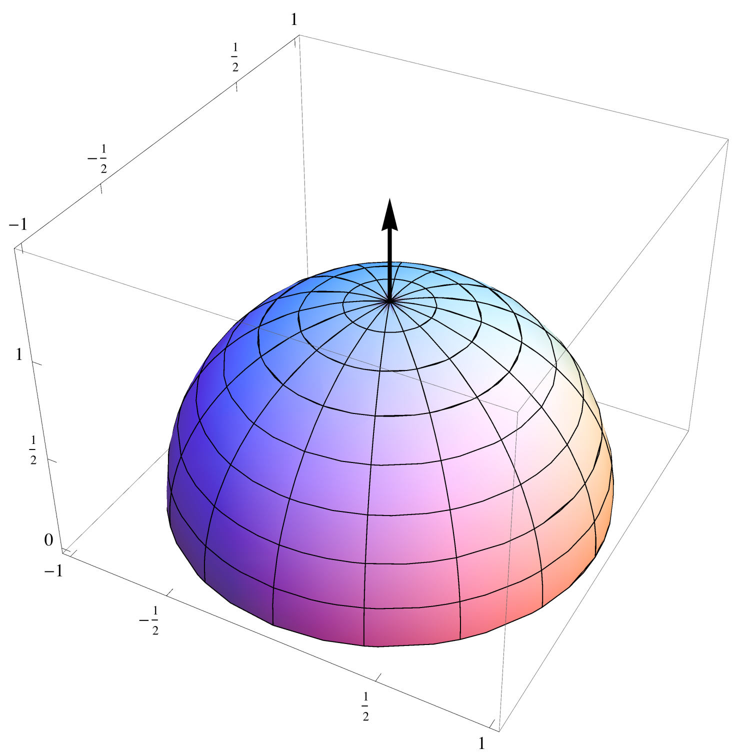
$$\begin{aligned}
\int_{\cos(t)}^{1} \frac{1}{\sqrt{1-x^2}} &= \arcsin(t)\Big]_{\cos(t)}^{1} \\
&= \arcsin(1) - \arcsin(\cos(t)) \\
&= \frac{\pi}{2} - \arcsin\left((\sin\left(\frac{\pi}{2} - t\right)\right) \\
&= \frac{\pi}{2} - \frac{\pi}{2} + t \\
&= t
\end{aligned}$$

as expected.

Problem 5.
Consider the region in the first quadrant bound by the unit circle.



If we rotate this region around the y-axis we obtain the upper half of the unit hemisphere.



Your problem: Find the volume of the northern hemisphere of the unit sphere in two different ways:

Problem 5. Continued.

(a) By discs.

Answer:

Rotating a horizontal slice of the quarter circle at y yields a disc of radius $r = x$, which has area $\pi r^2 = \pi(x^2) = \pi(1 - y^2)$. So we have

$$\begin{aligned} Volume &= \int_0^1 \pi(1 - y^2)dy \\ &= \pi y - \frac{y^3}{3}\Big]_0^1 \\ &= \frac{2}{3}\pi. \end{aligned}$$

(b) By shells.

Answer:

Rotating a vertical slice of the quarter circle at x yields a cylindrical shell of height $y = \sqrt{1 - x^2}$ and radius x, which has area $2\pi rh = 2\pi x\sqrt{1 - x^2}$. So we have

$$\begin{aligned} Volume &= \int_0^1 2\pi x\sqrt{1 - x^2}dx \\ &= -\frac{2}{3}\pi \left(1 - x^2\right)^{3/2}\Big]_0^1 \\ &= \frac{2}{3}\pi. \end{aligned}$$