Problem 1. Sequences. Find the limit.

(a) $\left\{ \dfrac{(n!)(n^n)}{(2n)!} \right\}$

(b)

$$
\begin{aligned}
a_1 &= 1 \\
a_2 &= \frac{1}{2}\left(\sqrt{1} + \sqrt{1 - \frac{1}{4}}\right) \\
a_3 &= \frac{1}{3}\left(\sqrt{1} + \sqrt{1 - \frac{1}{9}} + \sqrt{1 - \frac{4}{9}}\right) \\
a_4 &= \frac{1}{4}\left(\sqrt{1} + \sqrt{1 - \frac{1}{16}} + \sqrt{1 - \frac{4}{16}} + \sqrt{1 - \frac{9}{16}}\right) \\
a_5 &= \frac{1}{5}\left(\sqrt{1} + \sqrt{1 - \frac{1}{25}} + \sqrt{1 - \frac{4}{25}} + \sqrt{1 - \frac{9}{25}} + \sqrt{1 - \frac{16}{25}}\right) \\
&\vdots
\end{aligned}
$$

Problem 2. Series. Determine whether they converge absolutely, converge conditionally, or diverge. Give brief, but conclusive, evidence supporting your answers.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log(n)^2)}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!n^n}{(2n)!}$

Problem 2. Continued.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$.

(d) $\displaystyle\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right)$

Problem 3. Continued.

(e) $\sum_{n=2}^{\infty} \log\left(\frac{(2n-1)(n-1)}{(n)(2n-3)}\right)$

(f) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{3n}$

Problem 4. Continued.

(g) $\displaystyle\sum_{k=1}^{\infty} \frac{e^k k!}{k^k}$

(h) $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{e^k k!}$

Problem 5. Prove or disprove:

(a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.

(b) Suppose $a_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ converges. Then $\sum_{n=1}^{\infty} \sin(a_n)$ converges also.

Problem 5. Continued.

(c) If $\left|\frac{a_{n+1}}{a_n}\right| \to \frac{1}{3}$ then for every $k \in \mathbb{N}$, $\sum_{n=1}^{\infty} n^k a_n$ converges.

(d) For any sequence $\{a_n\}$ there exists $k \in \mathbb{N}$ so that $\sum_{n=1}^{\infty} \frac{a_n}{k^n}$ converges.

Problem 6. In 1904, H. Koch introduced the now famous "Koch curve." It is defined as the limit of a sequence of simple curves C_n which are defined recursively:

- C_1 is an equilateral triangle.
- From the curve C_{n-1} we obtain the curve C_n by placing a smaller equilateral triangle onto the middle third of each straight side, pointing outward and erasing the base of the new triangle (the old middle third).

Now, let L_n be the length of C_n (say you start with an equilateral triangle with sides of length one, so $L_1 = 3$). Let A_n be the area enclosed by C_n. Decide whether $\{L_n\}$ and $\{A_n\}$ converge.

Problem 7. Use the power series for $\arctan(x)$ to approximate π to 5 decimal places. (Hint: The series you get converges too slowly at $x = 1$. First check that $4\arctan\left(\frac{1}{2}\right) + 4\arctan\left(\frac{1}{3}\right) = \pi$ and ...)

Problem 8. Recall, the Fibonacci numbers f_n are defined inductively by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. Define the "Fibonacci series" to be the power series

$$\sum_{n=0}^{\infty} f_n x^n = x + x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots$$

Prove that the Fibonacci series converges for $-\frac{2}{1+\sqrt{5}} < x < \frac{2}{1+\sqrt{5}}$ and for those x for which it converges

$$\sum_{n=0}^{\infty} f_n x^n = \frac{x}{1 - x - x^2}.$$

Problem 9. When asked to approximate $\int_0^{\frac{1}{2}} \sqrt{1+2x^3}dx$, two students responded by correctly approximating $\sqrt{1+2x^3}$ by a Taylor polynomial $\approx 1+x^3$ and then computing

$$\int_0^{\frac{1}{2}} (1+x^3)dx = \frac{33}{64} = 0.515625.$$

When asked to bound the error in their approximations, one student wrote:

Let $f(x) = \sqrt{1+2x^3}$. *By Taylor's theorem, the remainder is given by*

$$R_3(x) := \sqrt{1+2x^3} - (1+x^3) = \frac{f^{(4)}(c)}{4!}\left(\frac{1}{2}\right)^4 \textit{ for some } c \in \left[0, \frac{1}{2}\right].$$

Since

$$f^{(4)}(x) = -\frac{180x^2}{(1+2x^3)^{\frac{3}{2}}} + \frac{972x^5}{(1+2x^3)^{\frac{5}{2}}} - \frac{1215x^8}{(1+2x^3)^{\frac{7}{2}}}$$

has a maximum absolute value of approximately 17.69612 *we have*

$$\left|\int_0^{\frac{1}{2}} \sqrt{1+2x^3} - \int_0^{\frac{1}{2}} (1+x^3)dx\right| \leq \int_0^{\frac{1}{2}} \left|\sqrt{1+2x^3} - (1+x^3)\right| dx$$

$$= \int_0^{\frac{1}{2}} |R_3(x)|dx = \frac{1}{2}R_3(x) \leq \frac{1}{2}\left(\frac{18}{4!}\left(\frac{1}{2}\right)^4\right) = \frac{3}{128}$$

The other student used another method to obtain a better error bound of $\frac{1}{1792}$. Your problem: justify the statement $\int_0^{\frac{1}{2}} \sqrt{1+2x^3} \approx \frac{33}{64} = 0.515625$ with an error less than $\frac{1}{1792}$.

Problem 10. Here's a remarkable fact:

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$$

and an outline of how to prove it.

(a) First show that

$$\log(1) + \log(2) + \cdots + \log(n-1) < \int_1^n \log(x)dx < \log(2) + \log(3) + \cdots + \log(n).$$

(b) Then show that

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}.$$

(c) You should be able to find your way to conclude that $\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$.

EXAM

Take Home Exam 2

Math 158: Spring 2013

Due: Tuesday, May 13

- This exam is due at the beginning of class on Thursday, May 15. But I urge you to complete it by Tuesday, May 13.
- If you use any resources besides your book or your notes, please cite them.
- Make sure you answers are clearly and carefully written. Neatness counts!

Success!