EXAM

Take Home Exam 2

Math 158: Spring 2013

Due: Tuesday, May 13

ANSWERS

Problem 1. Sequences. Find the limit.

(a) $\left\{ \dfrac{(n!)(n^n)}{(2n)!} \right\}$

Answer:

From the solution to problem 4b, the series $\sum \frac{(n!)(n^n)}{(2n)!}$ converges. Therefore, by the n-th term test, the terms $\left\{ \frac{(n!)(n^n)}{(2n)!} \right\} \to 0$.

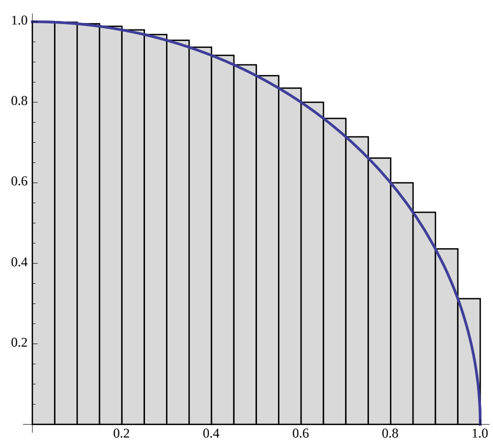
Problem 1.

(b)

$$\begin{aligned}
a_1 &= 1\\
a_2 &= \frac{1}{2}\left(\sqrt{1}+\sqrt{1-\frac{1}{4}}\right)\\
a_3 &= \frac{1}{3}\left(\sqrt{1}+\sqrt{1-\frac{1}{9}}+\sqrt{1-\frac{4}{9}}\right)\\
a_4 &= \frac{1}{4}\left(\sqrt{1}+\sqrt{1-\frac{1}{16}}+\sqrt{1-\frac{4}{16}}+\sqrt{1-\frac{9}{16}}\right)\\
a_5 &= \frac{1}{5}\left(\sqrt{1}+\sqrt{1-\frac{1}{25}}+\sqrt{1-\frac{4}{25}}+\sqrt{1-\frac{9}{25}}+\sqrt{1-\frac{16}{25}}\right)\\
&\vdots
\end{aligned}$$

Answer:

Fix an integer n and let $x_i = \frac{i}{n}$. Define a step function $t_n : [0,1] \to \mathbb{R}$ by $t_n(x) = \sqrt{1-x_{i-1}^2}$ if $x_{i-1} \le x < x_i$. Then $\int_0^1 t_n = a_n$. Here's a picture of t_{20}



Now, f defined by $f(x) = \sqrt{1-x^2}$ is integrable on $[0,1]$ and we have

$$\int_0^1 \sqrt{1-x^2}dx = \inf_n \int_0^1 t_n = \lim_{n\to\infty} \int_0^1 t_n = \lim_{n\to\infty} a_n.$$

Since $\int_0^1 \sqrt{1-x^2}dx = \frac{\pi}{4}$, we have $\{a_n\} \to \frac{\pi}{4}$.

Problem 2. Series. Determine whether they converge absolutely, converge conditionally, or diverge. Give brief, but conclusive, evidence supporting your answers.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log(n)^2)}$

Answer:

This series converges by the integral test. Let $f(x) = \frac{1}{x(\log(x))^2}$. Note that f is decreasing ($f'(x) = -\frac{2+\log(x)}{x^2(\log(x))^2} < 0$ for $x > 1$) and

$$\int_1^n f = \frac{1}{\log(2)} - \frac{1}{\log(n)} \underset{n\to\infty}{\longrightarrow} \frac{1}{\log(2)},$$

which is finite.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!n^n}{(2n)!}$

Answer:

We use the ratio test. Note that $a_n = \frac{n!n^n}{(2n)!} > 0$ and

$$\begin{aligned}\left\{\frac{a_{n+1}}{a_n}\right\} &= \left\{\left(\frac{(n+1)!(n+1)^{n+1}}{(2n+2)!}\right)\left(\frac{(2n)!}{n!n^n}\right)\right\} \\ &= \left\{\frac{(n+1)(n+1)}{(2n+2)(2n+1)}\left(\frac{n+1}{n}\right)^n\right\} \\ &\to \frac{e}{4}.\end{aligned}$$

Since $0 < \frac{e}{4} < 1$, the series converges.

Problem 2. Continued.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$.

Answer:

Note that this is a series of positive terms. We do a limit comparison test with the divergent harmonic series:

$$\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n^{1+\frac{1}{n}}}} = \lim_{n\to\infty} n^{\frac{1}{n}} = \lim_{n\to\infty} \exp\left(\frac{1}{n}\log(n)\right) = 1.$$

Since $1 \neq 0, \infty$, we conclude that the series in question diverges.

(d) $\displaystyle\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right)$

Answer:

This series diverges: we use a limit comparision with the harmonic series. Note that $\log\left(1+\frac{1}{n}\right) > 0$ and

$$\lim_{n\to\infty} \frac{\log\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = 1.$$

Since $1 \neq 0, \infty$, the limit comparison test says that $\sum_{n=1}^{\infty} \log\left(1+\frac{1}{n}\right)$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ do the same thing, which is diverge.

Problem 2. Continued.

(e) $\displaystyle\sum_{n=2}^{\infty} \log\left(\frac{(2n-1)(n-1)}{(n)(2n-3)}\right)$

Answer:

Here, we examine the n-th partial sum:

$$\begin{aligned} s_n &= \log\left(\frac{(3)(1)}{(2)(1)}\right) + \log\left(\frac{(5)(2)}{(3)(3)}\right) + \log\left(\frac{(7)(3)}{(4)(5)}\right) + \log\left(\frac{(9)(4)}{(5)(7)}\right) + \cdots + \log\left(\frac{(2n-1)(n-1)}{(n)(2n-3)}\right) \\ &= \log\left(\frac{(3)(1)(5)(2)(7)(3)(9)(4)\cdots(2n-1)(n-1)}{(2)(1)(3)(3)(4)(5)(5)(7)\cdots(n)(2n-3)}\right) \\ &= \log\left(\frac{2n-1}{n}\right). \end{aligned}$$

As $n \to \infty$, $s_n \to \log(2)$. That is, the series converges and has the sum $\log(2)$.

(f) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{3n}$

Answer:

As $n \to \infty$, the n-th term $\left(\frac{n}{n+1}\right)^{3n} \to \frac{1}{e^3} \neq 0$. Therefore, by the n-th term test, the series $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{3n}$ diverges.

Problem 2. Continued.

(g) $\displaystyle\sum_{k=1}^{\infty} \frac{e^k k!}{k^k}$

Answer:

Starting with the right inequality of (3) in part (b) of the last problem

$$\frac{k^k}{e^{k-1}} < k! \Rightarrow \frac{e^{k-1}k!}{k^k} > 1 \Rightarrow \frac{e^k k!}{k^k} > e.$$

Therefore,

$$\left\{ \frac{e^k k!}{k^k} \right\} \nrightarrow 0$$

and the series diverges by the n-th term test.

(h) $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{e^k k!}$

Answer:

Note that

$$\sum_k \frac{1}{k} \text{ diverges } \Rightarrow \sum_k \frac{1}{ek} \text{ diverges } \Rightarrow \sum_{k=1}^{\infty} \frac{k^k}{e^k k!} \text{ diverges.}$$

The last implication follows from the ordinary comparison test and the inequality:

$$\frac{1}{ek} < \frac{k^k}{e^k k!}$$

which is obtained by dividing $(k-1)! < \dfrac{k^k}{e^{k-1}}$ (which is the left hand part of (3) in part (b) of the last problem) by $\frac{1}{(k!)(e)}$.

Problem 3. Prove or disprove:

(a) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.

Answer:

This statement is false. For example, $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ converges but $\sum_{n=1}^{\infty}\left((-1)^{n+1}\frac{1}{\sqrt{n}}\right)^2 = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

(b) Suppose $a_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n$ converges. Then $\sum_{n=1}^{\infty} \sin(a_n)$ converges also.

Answer:

This is true. If $\sum_{n=1}^{\infty} a_n$ then $a_n \to 0$, so there exists a natural number N so that if $n \geq N$, $a_n < \pi$. Then, for $n > N$, both a_n and $\sin(a_n)$ are positive and the limit comparison test applies. Since $\{a_n\} \to 0$, we have

$$\left\{\frac{\sin a_n}{a_n}\right\} \to 1$$

and we can conclude that $\sum_{n=1}^{\infty} \sin(a_n)$ converges also.

Problem 3. Continued.

(c) If $\left|\frac{a_{n+1}}{a_n}\right| \to \frac{1}{3}$ then for every $k \in \mathbb{N}$, $\sum_{n=1}^{\infty} n^k a_n$ converges.

Answer:

True. Apply the ratio test to $\sum_{n=1}^{\infty} |n^k a_n|$:

$$\frac{(n+1)^k |a_{n+1}|}{n^k |a_n|} = \left(\frac{n+1}{n}\right)^k \left|\frac{a_{n+1}}{a_n}\right| \to (1)\left(\frac{1}{3}\right) = \frac{1}{3}.$$

(d) For any sequence $\{a_n\}$ there exists $k \in \mathbb{N}$ so that $\sum_{n=1}^{\infty} \frac{a_n}{k^n}$ converges.

Answer:

This is False. $\sum_{n=1}^{\infty} \frac{n!}{k^n}$ diverges by the ratio test since

$$\frac{\frac{(n+1)!}{k^{n+1}}}{\frac{n!}{k^n}} = \frac{(n+1)k}{n} \to \infty$$

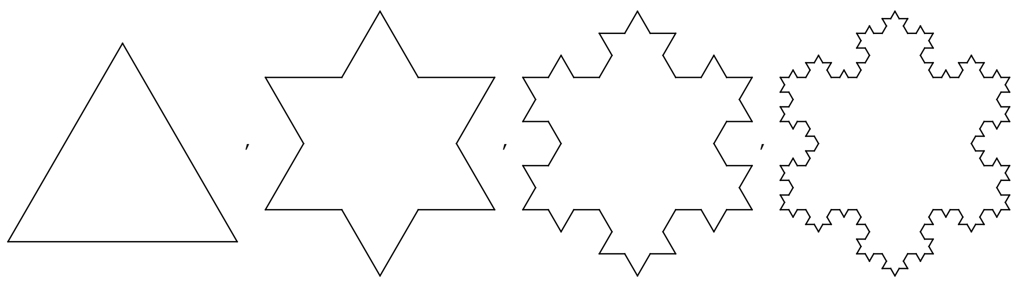
for every number k.

Problem 4. In 1904, H. Koch introduced the now famous "Koch curve." It is defined as the limit of a sequence of simple curves C_n which are defined recursively:

- C_1 is an equilateral triangle.
- From the curve C_{n-1} we obtain the curve C_n by placing a smaller equilateral triangle onto the middle third of each straight side, pointing outward and erasing the base of the new triangle (the old middle third).

Now, let L_n be the length of C_n (say you start with an equilateral triangle with sides of length one, so $L_1 = 3$). Let A_n be the area enclosed by C_n. Decide whether $\{L_n\}$ and $\{A_n\}$ converge.

Answer:
Here's a picture of C_1, C_2, C_3, C_4:



First, we show that the perimeter tends to infinity. Consider C_1 which has perimeter $L_1 = 3$. Since we get C_2 from C_1 by replacing each edge by four smaller edges, $\frac{1}{3}$ as long, we have $L_2 = \frac{4}{3}L_1 = \left(\frac{4}{3}\right) 3$ Then, we get C_3 from C_2 by replacing each edge of C_2 by four edges $\frac{1}{3}$ as long giving $L_3 = \frac{4}{3}L_2 = \left(\frac{4}{3}\right)^2 3$. Iterating, we find that

$$L_n = \left(\frac{4}{3}\right)^{n-1} 3,$$

and as $n \to \infty$, we see that $\{L_n\} \to \infty$.
Now, let us consider area. We can compute A_n by adding the areas of little triangles to the area of A_{n-1}. The number of triangles we add to get C_n from C_{n-1} equals the number of sides of C_{n-1}. To determine the number of sides, note that C_1 has 3, C_2 has $(3)(4)$, C_3 has $(3)(4^2)$, and in general, the number of sides of C_k is $3(4)^{k-1}$. So, we have

$$A_n = A_{n-1} + (3)(4^{n-2})(\text{the area of each little triangle added}).$$

To figure out the area of each little triangle added, note that those added at step k are $\frac{1}{9}$ the size of those added at step $k-1$. So, we can determine A_n recursively

$$A_n = A_1 + \left(\frac{1}{3}A_1 + \frac{1}{3}\left(\frac{4}{9}\right)A_1 + \frac{1}{3}\left(\frac{4}{9}\right)^2 A_1 + \cdots + \frac{1}{3}\left(\frac{4}{9}\right)^{n-2} A_1\right)$$

As $n \to \infty$, what is in parentheses becomes a convergent geometric series with ratio $r = \frac{4}{9}$ and we have

$$\lim_{n\to\infty} A_n = A_1 + \frac{\frac{1}{3}A_1}{1-\frac{4}{9}} = \frac{8}{5}A_1 = \frac{2\sqrt{3}}{10}.$$

Problem 5. Use the power series for $\arctan(x)$ to approximate π to 5 decimal places. (Hint: The series you get converges too slowly at $x = 1$. First check that $4\arctan\left(\frac{1}{2}\right) + 4\arctan\left(\frac{1}{3}\right) = \pi$ and ...)

Answer:
Recall that $\tan(x+y) = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$. So,

$$\tan\left(\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right)\right) = \frac{\frac{1}{2}+\frac{1}{3}}{1-\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1.$$

Thus, $\arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \frac{\pi}{4}$.
If we want to approximate π accurate to five decimal places, we need to approximate the sum $4\arctan\left(\frac{1}{2}\right) + 4\arctan\left(\frac{1}{3}\right)$ with an error less than .000005. We have

$$4\arctan\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right) - \frac{4}{3}\left(\frac{1}{2}\right)^3 + \frac{4}{5}\left(\frac{1}{2}\right)^5 - \frac{4}{7}\left(\frac{1}{2}\right)^7 + \cdots$$

which is a convergent alternating series of terms of decreasing magnitude, hence can be approximated by $4\left(\frac{1}{2}\right) - \frac{4}{3}\left(\frac{1}{2}\right)^3 + \cdots - \frac{4}{15}\left(\frac{1}{2}\right)^{15}$ with an error less than $\frac{4}{17}\left(\frac{1}{2}\right)^{17} = \frac{1}{557056} < .0000002$.
Likewise,

$$4\arctan\left(\frac{1}{3}\right) = 4\left(\frac{1}{3}\right) - \frac{4}{3}\left(\frac{1}{3}\right)^3 + \frac{4}{5}\left(\frac{1}{3}\right)^5 - \frac{4}{7}\left(\frac{1}{3}\right)^7 + \cdots$$

is a convergent alternating series of terms of decreasing magnitude, hence can be approximated by $4\left(\frac{1}{3}\right) - \frac{4}{3}\left(\frac{1}{3}\right)^3 + \cdots + -\frac{4}{11}\left(\frac{1}{3}\right)^{11}$ with an error less than $\frac{4}{13}\left(\frac{1}{3}\right)^{13} = \frac{4}{20726199} < .0000002$.
Therefore, the sum

$$4\left(\frac{1}{2}\right) - \frac{4}{3}\left(\frac{1}{2}\right)^3 + \cdots - \frac{4}{15}\left(\frac{1}{2}\right)^{15} + 4\left(\frac{1}{3}\right) - \frac{4}{3}\left(\frac{1}{3}\right)^3 + \cdots - \frac{4}{11}\left(\frac{1}{3}\right)^{11}$$
$$= \frac{13964621526980227}{4445076601405440} = 3.141592998\ldots$$

is an approximation of π accurate to five decimal places.

Remark: Here's a computation about the accuracy of this approximation, in case more details are desired:

$$\left|\left(4\left(\frac{1}{2}\right) + \cdots - \frac{4}{15}\left(\frac{1}{2}\right)^{15} + 4\left(\frac{1}{3}\right) + \cdots - \frac{4}{11}\left(\frac{1}{3}\right)^{11}\right) - \pi\right|$$
$$= \left|\left(4\left(\frac{1}{2}\right) + \cdots - \frac{4}{15}\left(\frac{1}{2}\right)^{15} + 4\left(\frac{1}{3}\right) + \cdots - \frac{4}{11}\left(\frac{1}{3}\right)^{11}\right) - \left(4\arctan\left(\frac{1}{3}\right) + 4\arctan\left(\frac{1}{2}\right)\right)\right|$$
$$\leq \left|4\left(\frac{1}{2}\right) + \cdots - \frac{4}{15}\left(\frac{1}{2}\right)^{15} - 4\arctan\left(\frac{1}{3}\right)\right| + \left|4\left(\frac{1}{3}\right) + \cdots - \frac{4}{11}\left(\frac{1}{3}\right)^{11} - 4\arctan\left(\frac{1}{2}\right)\right|$$
$$< .000002 + .0000002$$
$$< .0000005$$

So 3.141592998 is an approximation of π accurate to at least five decimal places (it is, in fact, accurate to six decimal places).

Problem 6. Recall, the Fibonacci numbers f_n are defined inductively by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n > 1$. Define the "Fibonacci series" to be the power series

$$\sum_{n=0}^{\infty} f_n x^n = x + x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots$$

Prove that the Fibonacci series converges for $-\frac{2}{1+\sqrt{5}} < x < \frac{2}{1+\sqrt{5}}$ and for those x for which it converges

$$\sum_{n=0}^{\infty} f_n x^n = \frac{x}{1-x-x^2}.$$

Answer:
First we check that $x^2 f(x) + x f(x) - f(x) = -x$.

$$\begin{aligned}
x^2 f(x) + x f(x) - f(x) &= x^2 \sum_{n=0}^{\infty} f_n x^n + x \sum_{n=0}^{\infty} f_n x^n - \sum_{n=0}^{\infty} f_n x^n \\
&= \sum_{n=2}^{\infty} f_{n-2} x^n + \sum_{n=1}^{\infty} f_{n-1} x^n - \sum_{n=0}^{\infty} f_n x^n \\
&= \sum_{n=2}^{\infty} f_{n-2} x^n + \left(x + \sum_{n=2}^{\infty} f_{n-1} x^n \right) - \left(x + x^2 + \sum_{n=2}^{\infty} f_n x^n \right) \\
&= x^2 - (x + x^2) + \sum_{n=2}^{\infty} (f_{n-1} + f_n - f_{n+1}) x^n \\
&= -x.
\end{aligned}$$

The computation above shows that if the series converges to $f(x)$, then f must satisfy

$$x^2 f(x) + x f(x) - f(x) = -x \Rightarrow f(x) = \frac{x}{1-(x^2+x)}.$$

Now, consider the geometric series $\sum_{n=0}^{\infty} xr$ with ratio $r = x^2 + x$. This series converges and has the sum $\frac{x}{1-(x^2+x)}$ for

$$|r| < 1 \Leftrightarrow |x^2 + x| < 1 \Leftrightarrow -\phi < x < \frac{1}{\phi}$$

where ϕ is the famous golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ and $\frac{1}{\phi} = \frac{2}{1+\sqrt{5}} = \frac{1-\sqrt{5}}{2}$.
Remark: The series $\sum_{n=0}^{\infty} x(x+x^2)^n$ differs from the Fibonacci series by a re-arrangement. However, since (both) converge absolutely, this does not affect the sum. To be rigorous, pick any number s with $0 < s < \frac{1}{\phi}$. Then, the series $\sum_{n=0}^{\infty} s(s^2+s)$ converges absolutely at s. In particular, the re-arrangement $\sum_{n=0}^{\infty} f_n s^n$ converges absolutely. Since it's a power series centered at 0, it must converge for all x with $|x| < s$. This holds for all $s < \frac{1}{\phi}$, therefore the series $\sum_{n=0}^{\infty} f_n s^n$ converges for all x with $|x| < \frac{1}{\phi}$.

Problem 7. When asked to approximate $\int_0^{\frac{1}{2}} \sqrt{1+2x^3}dx$, two students responded by correctly approximating $\sqrt{1+2x^3}$ by a Taylor polynomial $\approx 1+x^3$ and then computing

$$\int_0^{\frac{1}{2}} (1+x^3)dx = \frac{33}{64} = 0.515625.$$

When asked to bound the error in their approximations, one student wrote:

Let $f(x) = \sqrt{1+2x^3}$*. By Taylor's theorem, the remainder is given by*

$$R_3(x) := \sqrt{1+2x^3} - (1+x^3) = \frac{f^{(4)}(c)}{4!}\left(\frac{1}{2}\right)^4 \text{ for some } c \in \left[0, \frac{1}{2}\right].$$

Since

$$f^{(4)}(x) = -\frac{180x^2}{(1+2x^3)^{\frac{3}{2}}} + \frac{972x^5}{(1+2x^3)^{\frac{5}{2}}} - \frac{1215x^8}{(1+2x^3)^{\frac{7}{2}}}$$

has a maximum absolute value of approximately 17.69612 *we have*

$$\left|\int_0^{\frac{1}{2}} \sqrt{1+2x^3} - \int_0^{\frac{1}{2}} (1+x^3)dx\right| \leq \int_0^{\frac{1}{2}} \left|\sqrt{1+2x^3} - (1+x^3)\right| dx$$

$$= \int_0^{\frac{1}{2}} |R_3(x)|dx = \frac{1}{2}R_3(x) \leq \frac{1}{2}\left(\frac{18}{4!}\left(\frac{1}{2}\right)^4\right) = \frac{3}{128}$$

The other student used another method to obtain a better error bound of $\frac{1}{1792}$. Your problem: justify the statement $\int_0^{\frac{1}{2}} \sqrt{1+2x^3} \approx \frac{33}{64} = 0.515625$ with an error less than $\frac{1}{1792}$.

Answer:
Note that

$$\begin{aligned}\int_0^{\frac{1}{2}} \sqrt{1+2x^3} &= \int_0^{\frac{1}{2}} \left(1 + x^3 - \frac{x^6}{2} + - \cdots\right) dx \\ &= \left(x + \frac{x^4}{4} - \frac{x^7}{14} + - \cdots\right)\Big]_0^{\frac{1}{2}} \\ &= \frac{1}{2} + \frac{1}{64} - \frac{1}{1792} + - \cdots\end{aligned}$$

Since this is a convergent alternating series with decreasing terms, it can be approximated by the sum of the first two terms, with an error less than the third. That is

$$\int_0^{\frac{1}{2}} \sqrt{1+2x^3} \approx \frac{1}{2} + \frac{1}{64} = \frac{33}{64}$$

with an error less than $\frac{1}{1792}$.

Problem 8. Here's a remarkable fact:

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$$

and an outline of how to prove it.

(a) First show that

$$\log(1) + \log(2) + \cdots + \log(n-1) < \int_1^n \log(x)dx < \log(2) + \log(3) + \cdots + \log(n).$$

Answer:

The $\log$ function is increasing, which gives for each fixed $k \in \mathbb{N}$ the inequality

$$\log(k) < \log(x) < \log(k+1) \text{ for all } k < x < k+1.$$

Integrating over $[k, k+1]$ gives

$$\log(k) < \int_k^{k+1} \log(x)dx < \log(k+1).$$

Summing from $k = 1$ to $k = n - 1$ gives the result.

Problem 8. Continued.

(b) Then show that

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}.$$

Answer:

Let's write the conclusion from part (a) as

$$\log((n-1)!) < \int_1^n \log(x)dx < \log(n!). \tag{1}$$

In the center we have

$$\int_1^n \log(x)dx = x\log(x) - x\Big]_1^n = n\log(n) - n + 1 = \log(n^n) - (n-1)$$

Write $\log(n^n) - (n-1) = \log(n^n) - \log(\exp(n-1)) = \log\left(\frac{n^n}{e^{n-1}}\right)$ and substitute in the middle of (1) to get

$$\log((n-1)!) < \log\left(\frac{n^n}{e^{n-1}}\right) < \log(n!). \tag{2}$$

Since the $\log$ function is strictly increasing, $\log(a) < \log(b) \Leftrightarrow a < b$ so (2) implies

$$(n-1)! < \frac{n^n}{e^{n-1}} < n! \tag{3}$$

Replacing n by $n+1$ produces

$$n! < \frac{(n+1)^{(n+1)}}{e^n} < (n+1)!. \tag{4}$$

The right part of (3) and the left part of (4) gives the result.

(c) You should be able to find your way to conclude that $\lim\limits_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$.

Answer:

Part (b) and some algebra gives

$$\frac{n}{e} < \sqrt[n]{n!} < \frac{(n+1)^{1+\frac{1}{n}}}{e^{1+\frac{1}{n}}} \Rightarrow \frac{ne^{1+\frac{1}{n}}}{(n+1)^{1+\frac{1}{n}}} < \frac{n}{\sqrt[n]{n!}} < e.$$

Since,

$$\left\{\frac{ne^{1+\frac{1}{n}}}{(n+1)^{1+\frac{1}{n}}}\right\} \to e,$$

the squeeze theorem gives, $\lim\limits_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$.