Problem 1. Suppose F^+ is a subset of a field F. There are three "order axioms" that F and F^+ might satisfy:

A1. If $x, y \in F^+$ then $x + y \in F^+$ and $xy \in F^+$.

A2. For every nonzero $x \in F$, either $x \in F^+$ or $-x \in F^+$, but not both.

A3. $0 \notin F^+$.

The real numbers $\mathbb{R}$ and the positive reals $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ satisfy axioms **A1**, **A2**, **A3**.

If we define the set $\mathbb{C}^+ = \{z \in \mathbb{C} \,|\, \mathrm{Real}(z) > 0\}$, then which of the order axioms are violated?

Answer:

Problem 2. Prove that if $\{f_n\}$ is a sequence of integrable functions that converges uniformly to f on an interval $[a, b]$ then the sequence of numbers $\left\{\int_a^b f_n\right\} \to \int_a^b f$.

Answer:

Problem 3. Prove an "improved" n-th term test for divergence:

> *Suppose* $\{a_n\}$ *is a sequence of nonnegative numbers and* $\{na_n\} \to L$. *If* $L \neq 0$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Answer:

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$. It's not too hard to see that $|f(s) - f(t)| < |s - t|$ for all $s, t \in \mathbb{R}$ then f is continuous. Prove that if $|f(s) - f(t)| < |s - t|^2$ for all $s, t \in \mathbb{R}$ then f is constant.

Answer:

Problem 5. You already know that $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} = e$ and $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$. Prove that for every $n \in \mathbb{N}$,

$$\left(1 + \frac{1}{n}\right)^n \leq \sum_{k=0}^{n} \frac{1}{k!}$$

Answer:

Problem 6. Prove that for all x, $1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \leq e^x$.

Note: it is *not* true that $1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 \leq e^x$ for all x.

Answer:

EXAM

Challenge Final Exam

Math 158: Spring 2014

May 20, 2014

- Make sure you answers are clearly and carefully written. Proofread!
- Neatness counts.
- There are a total of 34 points on this exam. For your information: $21/32 \approx 62\%$, $25/35 \approx 73\%$, $31/34 \approx 91\%$

Success!