EXAM

Challenge Final Exam

Math 158: Spring 2014

May 20, 2014

ANSWERS

Problem 1. Suppose F^+ is a subset of a field F. There are three "order axioms" that F and F^+ might satisfy:

A1. If $x, y \in F^+$ then $x + y \in F^+$ and $xy \in F^+$.

A2. For every nonzero $x \in F$, either $x \in F^+$ or $-x \in F^+$, but not both.

A3. $0 \notin F^+$.

The real numbers $\mathbb{R}$ and the positive reals $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ satisfy axioms **A1**, **A2**, **A3**.

If we define the set $\mathbb{C}^+ = \{z \in \mathbb{C} \,|\, \text{Real}(z) > 0\}$, then which of the order axioms are violated?

Answer:

- Axiom 1 is violated since you have $1 + 2i \in \mathbb{C}^+$, but $(1 + 2i)(1 + 2i) = -3 + 4i \notin \mathbb{C}^+$.
- Axiom 2 is violated since you have the number $3i \in \mathbb{C}$, but neither $3i$ nor $-3i$ are in $\mathbb{C}^+$.
- Axiom 3 is satisfied since $0 \notin \mathbb{C}^+$.

Problem 2. Prove that if $\{f_n\}$ is a sequence of integrable functions that converges uniformly to f on an interval $[a, b]$ then the sequence of numbers $\left\{\int_a^b f_n\right\} \to \int_a^b f$.

Answer:
Suppose $\{f_n\} \to f$ uniformly on $[a, b]$. To prove that $\left\{\int_a^b f_n\right\} \to \int_a^b f$, let $\epsilon > 0$ be given. Since $\{f_n\} \to f$ uniformly on $[a, b]$, there exists $N \in \mathbb{N}$ so that if $n \geq N$, $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$ for all $x \in [a, b]$. Now for $n \geq N$ we have

$$\begin{aligned} \left| \int_a^b f_n(x)\, dx - \int_a^b f(x)\, dx \right| &\leq \int_a^b |f_n(x) - f(x)|\, dx \\ &\leq \int_a^b \frac{\epsilon}{b-a}\, dx \\ &= \epsilon. \end{aligned}$$

This proves that $\left\{\int_a^b f_n\right\} \to \int_a^b f$.

Problem 3. Prove an "improved" n-th term test for divergence:

Suppose $\{a_n\}$ *is a sequence of nonnegative numbers and* $\{na_n\} \to L$. *If* $L \neq 0$ *then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Answer:
By the limit comparison test, if $\lim_{n\to\infty} na_n = \lim_{n\to\infty} \frac{a_n}{\frac{1}{n}} = L \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ and the series $\sum_{n=1}^{\infty} \frac{1}{n}$ do the same thing, which is diverge.

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$. It's not too hard to see that $|f(s) - f(t)| < |s - t|$ for all $s, t \in \mathbb{R}$ then f is continuous. Prove that if $|f(s) - f(t)| < |s - t|^2$ for all $s, t \in \mathbb{R}$ then f is constant.

Answer:
Since $|f(s) - f(t)| \leq |s - t|$, we have $|f(x + h) - f(x)| \leq |h|^2$. Therefore,

$$\left| \frac{f(x+h) - f(x)}{h} \right| \leq |h| \Rightarrow -|h| \leq \frac{f(x+h) - f(x)}{h} \leq |h|.$$

So, by the squeeze theorem,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = 0.$$

Therefore, f is constant.

Problem 5. You already know that $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{1}{k!} = e$ and $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$. Prove that for every $n \in \mathbb{N}$,

$$\left(1+\frac{1}{n}\right)^n \leq \sum_{k=0}^{n} \frac{1}{k!}$$

Answer:
We use the binomial theorem to expand

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= 1 + \binom{n}{1}\left(\frac{1}{n}\right) + \binom{n}{2}\left(\frac{1}{n}\right)^2 + \cdots \left(\frac{1}{n}\right)^n \\ &= 1 + 1 + \frac{(n)(n-1)}{(2!)(n)(n)} + \frac{(n)(n-1)(n-2)}{(3!)(n)(n)(n)} + \cdots + \frac{n!}{n!(n^n)} \\ &\leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}.\end{aligned}$$

The last inequality follows from using the fact that for each term, we have $\frac{(n)(n-1)(n-2)\cdots(n-k)}{k!n^k} < \frac{1}{k!}$ since $\frac{(n)(n-1)(n-2)\cdots(n-k)}{n^k} < \frac{(n)(n)\cdots(n)}{n^k} = 1$.

Problem 6. Prove that for all x, $1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 \leq e^x$.

Note: it is *not* true that $1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 \leq e^x$ for all x.

Answer:
Let $f(x) = e^x$. Note that $1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$ is the degree three Taylor approximation for $f(x)$ centered at 0. By Taylor's theorem

$$f(x) - \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3\right) = \frac{f^{(4)x^4}(\theta)}{4!}$$

for some θ between 0 and x. The result follows from observing that $f^{(4)}(\theta)x^4 = e^\theta x^4 \geq 0$.
