Problem 1. [2 points each] Compute.

(a) Write $\frac{1}{3+4i}$ in the form $a + bi$.

(b) Write $-2 + 2\sqrt{3}i$ in polar form $z = re^{i\theta}$.

Problem 2. **[4 points]** Suppose an aircraft was on a runway in Norfolk Virgina in the morning and six hours later it was on a runway on Midway Island. Prove that the aircraft created at least two sonic booms that day.

Hint: A sonic boom is the sound associated when the pressure waves created by an object traveling through air converge into a single shock wave. This happens when the object travels at speed of sound, about 761 mph. Norfolk and Midway are over 5600 miles apart.

Problem 3. [2 points each]

(a) Define the number e.

(b) Prove that $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$.

Problem 3. Continued.

(c) Prove that: if $f : \mathbb{R} \to \mathbb{R}$ satisfies $f' = f$ then $f(x) = Ae^x$ for some constant A.

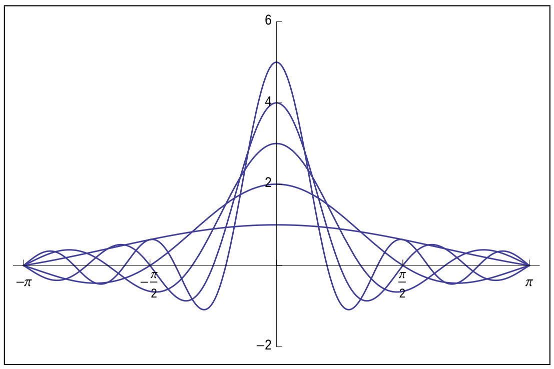
Hint: One of the simplest ways to do it is to prove that $\frac{f(x)}{e^x}$ is constant.

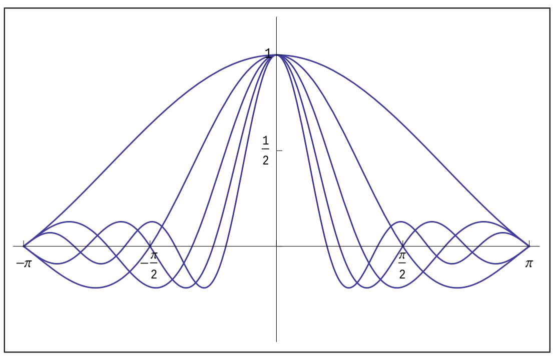
Problem 4. [4 points] Consider the three sequence of functions defined by

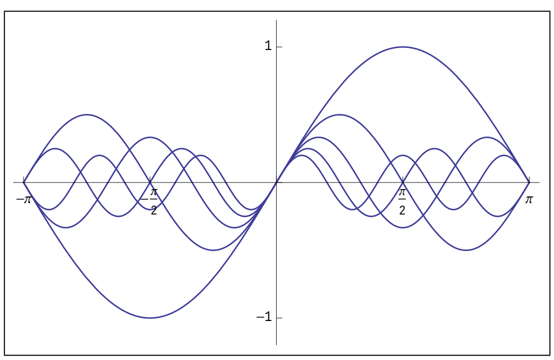
$$a_n(x) = \frac{\sin(nx)}{n}, \qquad b_n(x) = \frac{\sin(nx)}{nx}, \text{ and } c_n(x) = \frac{\sin(nx)}{x}.$$

Each of the pictures below shows a sketch of the graphs of the first five functions of the one of the sequences. Label each picture with $\{a_n\}, \{b_n\}$, or $\{c_n\}$ and say whether the sequence of functions pictured

- converges uniformly on $[-\pi, \pi]$,
- converges pointwise on $[-\pi, \pi]$,
- does not converge on $[-\pi, \pi]$.







Problem 5. **[2 points each]** It's easy to check that $\dfrac{1}{n^2+n} = \dfrac{1}{n} - \dfrac{1}{n+1}$. Use this fact to compute

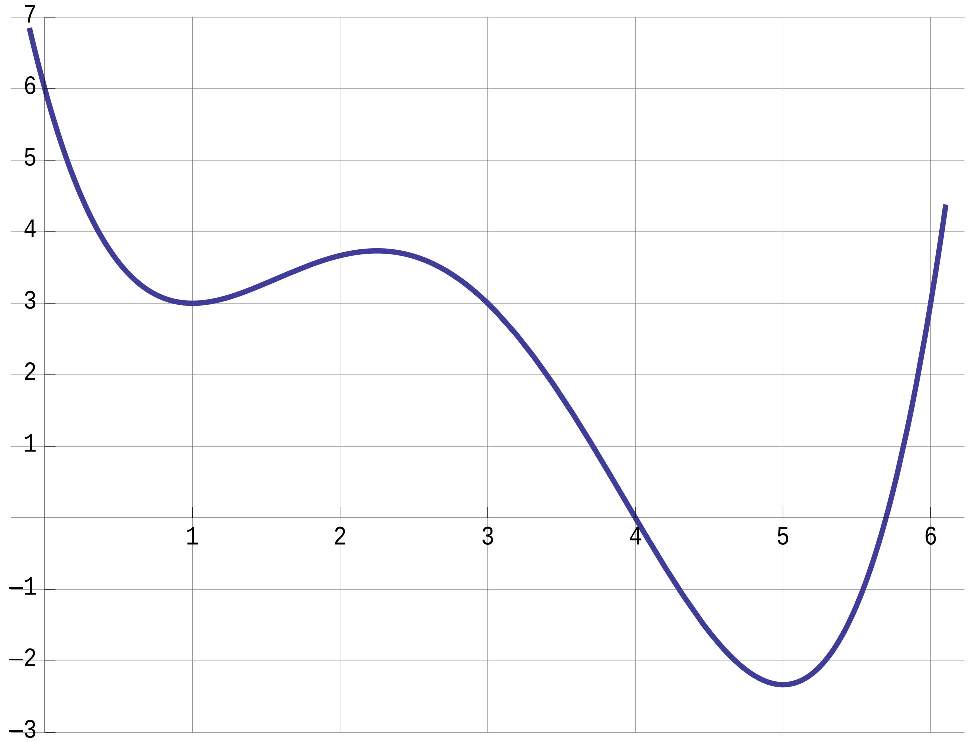
(a) $\displaystyle\int_1^\infty \frac{dx}{x^2+x}$

(b) $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+n}$

Matching [1 point each]

Problem 6. **[1 point each]** Below the graph of function f is sketched. Define g by

$$g(x) = \int_0^x f(t)dt \text{ for } 0 \leq x \leq 6.$$



(a) $\int_0^6 f$

(b) $\int_0^6 f'$

(c) g has an absolute maximum at

(d) $g(0)$

(e) $g'(0)$

(f) $g''(0)$

Here are the answers (out of order): -7.5 $\quad -3$ $\quad 0$ $\quad 3$ $\quad 4$ $\quad 6$ $\quad 10.2$

Problem 7. **[2 points each]** Let $f(x) = e^{\sin(x)}$ and let $p(x)$ be the degree two Taylor polynomial for f centered at zero.

(a) Find (by any method) the polynomial p.

(b) Carefully state Taylor's remainder formula for the difference $f(x) - p(x)$.

Problem 7.

(c) Compute $\int_{-\frac{1}{2}}^{\frac{1}{2}} p(x)\,dx$ and use the fact that $|f'''(\theta)| < \frac{5}{2}$ for $-\frac{1}{2} \leq \theta \leq \frac{1}{2}$ to find a bound on the error

$$\left| \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\,dx - \int_{-\frac{1}{2}}^{\frac{1}{2}} p(x)\,dx \right|.$$

EXAM

Final Exam

Math 158: Spring 2014

May 20, 2014

- Make sure you answers are clearly and carefully written. Proofread!
- Neatness counts.
- There are a total of 34 points on this exam. For your information: $21/32 \approx 62\%$, $25/35 \approx 73\%$, $31/34 \approx 91\%$

Success!