EXAM

Final Exam

Math 158: Spring 2014

May 20, 2014

ANSWERS

Problem 1. **[2 points each]** Compute.

(a) Write $\dfrac{1}{3+4i}$ in the form $a+bi$.

Answer:
$z^{-1} = \frac{\bar{z}}{|z|^2}$ so $\dfrac{1}{3+4i} = \frac{3-4i}{25} = \frac{3}{25} - \frac{4}{25}i$.

(b) Write $-2+2\sqrt{3}i$ in polar form $z = re^{i\theta}$.

Answer:
$r = |z| = \sqrt{4+12} = 4$ and θ satisfies $\cos(\theta) = -2$ and $\sin(\theta) = 2\sqrt{3}$ so $\theta = \frac{2\pi}{3}$. Thus $-2+2\sqrt{3}i = 4e^{\frac{2\pi}{3}i}$.

Problem 2. **[4 points]** Suppose an aircraft was on a runway in Norfolk Virgina in the morning and six hours later it was on a runway on Midway Island. Prove that the aircraft created at least two sonic booms that day.

Hint: A sonic boom is the sound associated when the pressure waves created by an object traveling through air converge into a single shock wave. This happens when the object travels at speed of sound, about 761 mph. Norfolk and Midway are over 5600 miles apart.

Answer:
By the mean value theorem, there must be some time during the day, say T hours after takeoff, when the aircraft was travelling at

$$\frac{\text{total distand}}{\text{total time}} > \frac{5600mi}{6hrs} > 900mi/hr.$$

Since the aircraft begins at time 0 with zero velocity and at time T it is travelling at over 900 mi/hr, and the speed of sound is between 0 and 900 the intermediate value theorem says that the there was some time before T when the aircraft was travelling at the speed of sound. Again, since the aircraft was travelling at over $900mi/hr$ at time T and then later the aircraft is at rest, the intermediate value theorem again there was some point after T when the aircraft was travelling at the speed of sound.

Problem 3. [2 points each]

(a) Define the number e.

Answer:

The number e is the unique number satisfying $\ln(e) = 1$.

(b) Prove that $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$.

Answer:

First, note that

$$\lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} \stackrel{L}{=} \lim_{x\to\infty}\frac{\left(\frac{1}{1+\frac{1}{x}}\right)\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty}\left(\frac{1}{1+\frac{1}{x}}\right) = 1.$$

So,

$$\ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right) = \lim_{x\to\infty}\ln\left(\left(1+\frac{1}{x}\right)^x\right) = \ln(1).$$

Since $\ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right) = 1$ and e is the unique number for which $\ln(e) = 1$, we conclude that

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e.$$

(c) Prove that: if $f : \mathbb{R} \to \mathbb{R}$ satisfies $f' = f$ then $f(x) = Ae^x$ for some constant A.

Answer:

Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f' = f$. Since

$$\left(\frac{f(x)}{e^x}\right)' = \frac{e^x f'(x) - f(x)e^x}{(e^x)^2} = \frac{e^x f(x) - f(x)e^x}{(e^x)^2} = 0$$

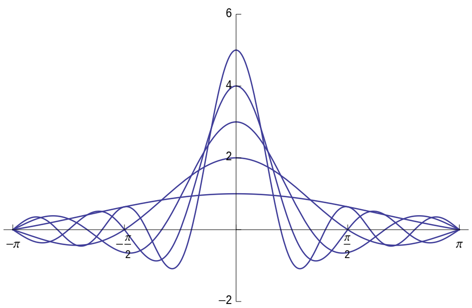
we know that $\frac{f(x)}{e^x}$ is constant, say $\frac{f(x)}{e^x} = A$. Therefore, $f(x) = Ae^x$.

Problem 4. [4 points] Consider the three sequence of functions defined by

$$a_n(x) = \frac{\sin(nx)}{n}, \qquad b_n(x) = \frac{\sin(nx)}{nx}, \text{ and } c_n(x) = \frac{\sin(nx)}{x}.$$

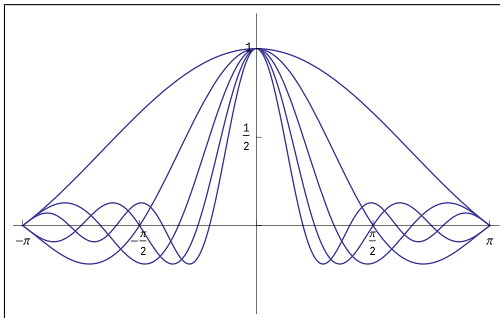
Each of the pictures below shows a sketch of the graphs of the first five functions of the one of the sequences. Label each picture with $\{a_n\}, \{b_n\}$, or $\{c_n\}$ and say whether the sequence of functions pictured

- converges uniformly on $[-\pi, \pi]$,
- converges pointwise on $[-\pi, \pi]$,
- does not converge on $[-\pi, \pi]$.



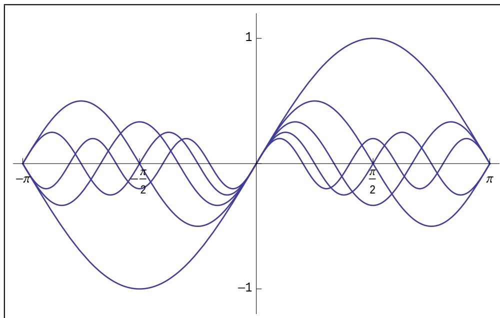
$\{c_n\}$ does not converge on $[-\pi, \pi]$.

Note that $\{c_n(0)\} = \{n\}$ does not converge.



$\{b_n\}$ converges pointwise on $[-\pi, \pi]$.

Note that $\{b_n\} \to f$ where $f(x) = 0$ for $x \neq 0$ and $f(0) = 1$. Since $\{b_n\}$ is a sequence of continuous functions, if it converged uniformly, the limit would have to be continuous, which it is not.



$\{a_n\}$ converges uniformly on $[-\pi, \pi]$.

Note that $|a_n| < \frac{1}{n}$ for all $x \in [\pi, \pi]$.

Problem 5. **[2 points each]** It's easy to check that $\frac{1}{n^2+n} = \frac{1}{n} - \frac{1}{n+1}$. Use this fact to compute

(a) $\int_1^\infty \frac{dx}{x^2+x}$

Answer:

$$\begin{aligned}\int_1^\infty \frac{dx}{x^2+x} &= \int_1^\infty \left(\frac{1}{x} - \frac{1}{x+1}\right) \\ &= \lim_{B\to\infty} \int_1^B \left(\frac{1}{x} - \frac{1}{x+1}\right) \\ &= \lim_{B\to\infty} \ln(B) - \ln(B+1) + \ln(2) \\ &= \lim_{B\to\infty} \ln(\frac{B}{B+1} + \ln(2) \\ &= \ln(2).\end{aligned}$$

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$

Answer:

Look at the n-th partial sum of $\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$

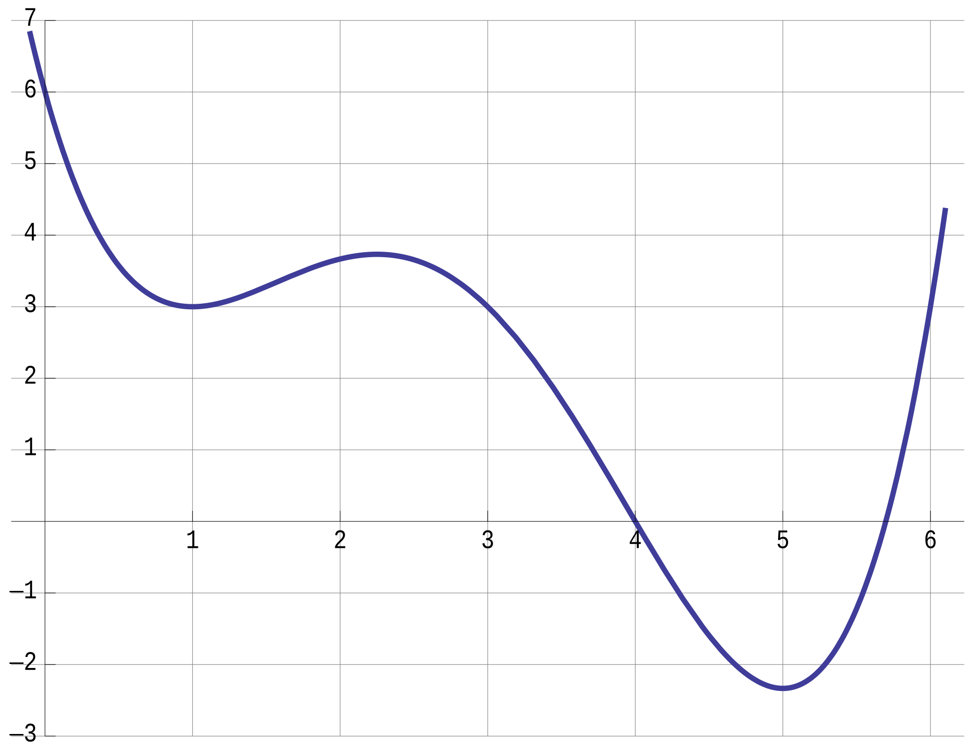
$$\begin{aligned}s_n &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \frac{1}{1} - \frac{1}{n+1}\end{aligned}$$

and we see that $\{s_n\} \to 1$.

Matching [1 point each]

Problem 6. [1 point each] Below the graph of function f is sketched. Define g by

$$g(x) = \int_0^x f(t)dt \text{ for } 0 \leq x \leq 6.$$



(a) $\int_0^6 f = 10.2$ This is the exact answer, I approximated it by inspection, then took the answer from the list.

(b) $\int_0^6 f' = -3$ by the fundamental theorem of calculus $\int_0^6 f' = f(6) - f(0) = 3 - 6$.

(c) g has an absolute maximum at 4. By looking at where f is positive and where f is negative, we see that g increases to 4, then decreases (then increases again near 6, but just a bit).

(d) $g(0) = \int_0^0 f = 0$.

(e) $g'(0) = 6$ by FTC $g'(0) = f(0) = 6$

(f) $g''(0) = f'(0) = -7.5$. Again, this answer is exact, but I approximated by inspection and then took the answer from the list.

Here are the answers (out of order): -7.5 -3 0 3 4 6 10.2

Problem 7. **[2 points each]** Let $f(x) = e^{\sin(x)}$ and let $p(x)$ be the degree two Taylor polynomial for f centered at zero.

(a) Find (by any method) the polynomial p.

Answer:

The power series for f is

$$\begin{aligned}\exp(\sin(x)) &= \exp\left(x - \frac{x^3}{3!} + \cdots\right)\\ &= 1 + \left(x - \frac{x^3}{3!} + \cdots\right) + \frac{1}{2!}\left(x - \frac{x^3}{3!} + \cdots\right)^2 + + \frac{1}{3!}\left(x - \frac{x^3}{3!} + \cdots\right)^3 + \cdots\\ &= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots\end{aligned}$$

Therefore, the degree two Taylor polynomial for f is

$$p(x) = 1 + x + \frac{x^2}{2}.$$

(b) Carefully state Taylor's remainder formula for the difference $f(x) - p(x)$.

Answer:

Taylor's theorem says that for any x, there exists a number θ between 0 and x with

$$e^{\sin(x)} - \left(1 + x + \frac{x^2}{2}\right) = \frac{f'''(\theta)x^3}{3!}.$$

(c) Compute $\int_{-\frac{1}{2}}^{\frac{1}{2}} p(x)\,dx$ and use the fact that $|f'''(\theta)| < \frac{5}{2}$ for $-\frac{1}{2} \leq \theta \leq \frac{1}{2}$ to find a bound on the error

$$\left|\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\,dx - \int_{-\frac{1}{2}}^{\frac{1}{2}} p(x)\,dx\right|.$$

Answer:

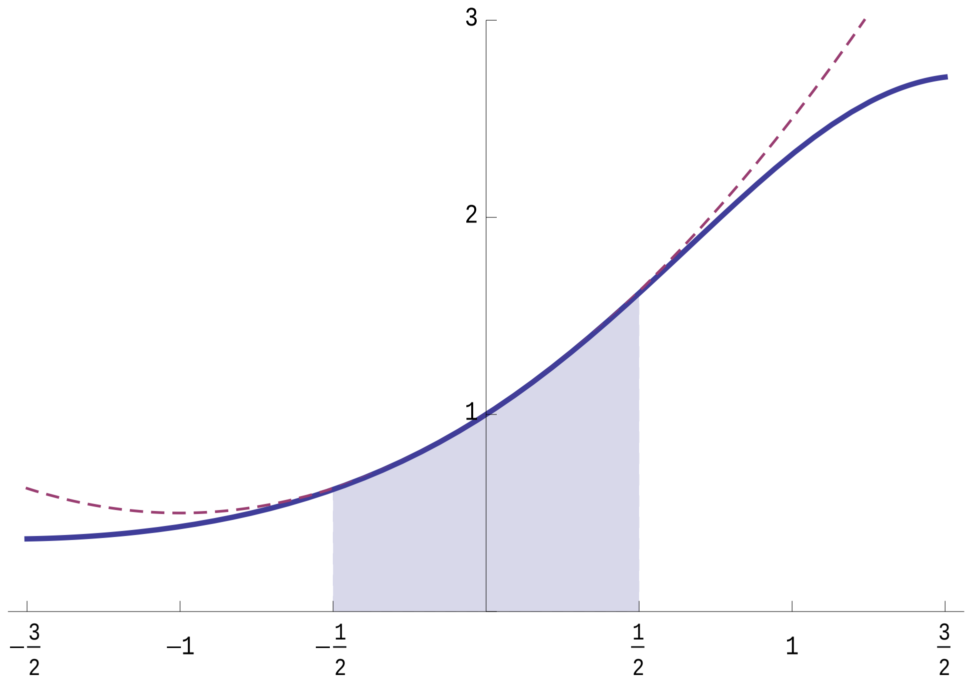
We compute

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} p(x)\,dx = x + \frac{x^2}{2} + \frac{x^3}{6}\Bigg]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{25}{24}.$$

And we bound the error:

$$\begin{aligned}\left|\int_{-\frac{1}{2}}^{\frac{1}{2}} p(x)\,dx - \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\,dx\right| &\leq \int_{-\frac{1}{2}}^{\frac{1}{2}} |f(x) - p(x)|\,dx\\ &\leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \left|\frac{f'''(\theta)x^3}{3!}\right|\,dx \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\frac{5}{2}\right)\left(\frac{1}{2}\right)^3\left(\frac{1}{3!}\right)\,dx = \frac{5}{96}.\end{aligned}$$

Problem 7. *Here's a footnote to problem 7.*
It's not possible to find an antiderivative for $e^{\sin(x)}$ in terms of elementary functions, but the method outlined in this problem is a very good way to approximate $\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{\sin(x)}\,dx$.
In the picture below, the graph of f (the solid cuve) is sketched with the graph of p (the dashed curve) and it looks like $\int_{-\frac{1}{2}}^{\frac{1}{2}} p(x)$ might be a very good approximation for $\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{\sin(x)}\,dx$.



In fact, although the bound on the error found above is $\frac{5}{96} = .0520833\ldots$, the actual error $\left|\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\,dx - \int_{-\frac{1}{2}}^{\frac{1}{2}} p(x)\,dx\right|$ is less than .002. Using the fourth order Taylor polynomial $1 + x + \frac{x^2}{2} - \frac{x^4}{8}$ yields $\frac{1997}{1920}$ for an approximation of $\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\,dx$, which is accurate to over 6 decimal places.