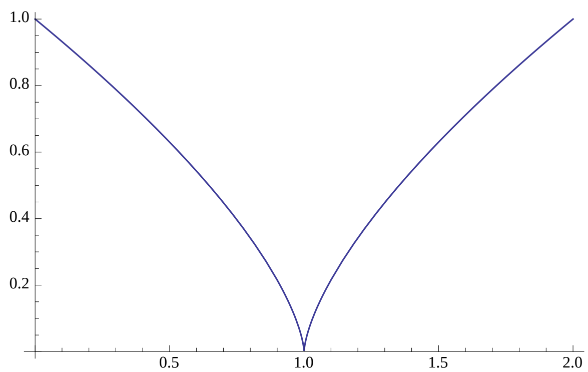
Some new problems

Problem 1. Read through section 4.20 in the text and do exercises 1, 8, 9 in section 4.19 (page 191) and exercises 2, 3, 19, 21, 28 in section 4.21 (page 194-196)

Problem 2. True or False: Let $f(x) = (x-1)^{\frac{2}{3}}$. Since $f(0) = 1$ and $f(2) = 1$, there is some point $c \in (0,2)$ with $f'(c) = 0$.

Answer. False. One might be tempted to apply the mean value theorem, but it doesn't apply since f is not differentiable on $(0,2)$ ($f'(1)$ does not exist).

Here's a plot of $y = (x-1)^{\frac{2}{3}}$.



Problem 3. Prove:

(a) If f satisfies $|f(s) - f(t)| \le |s - t|$ for all s, t, then f is continuous.

Answer. The statement "f is continuous at the number t" means that for all $\epsilon > 0$ there exists a $\delta > 0$ so that if $|s-t| < \delta$ then $|f(s) - f(t)| < \epsilon$. So, let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. Then if $|s-t| < \delta$ we have $|s-t| < \epsilon$. Since $|f(s) - f(t)| < |s-t|$, we have $|f(s) - f(t)| < \epsilon$ as needed.

(b) If f satisfies $|f(s) - f(t)| \le |s-t|^2$ for all s, t, then f is constant.

Answer. Since $|f(s) - f(t)| \le |s-t|^2$, we have $|f(x+h) - f(x)| \le |h|^2$. Therefore,

$$\left|\frac{f(x+h) - f(x)}{h}\right| \le |h| \Rightarrow -|h| \le \frac{f(x+h) - f(x)}{h} \le |h|.$$

So, by the squeeze theorem,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 0.$$

Therefore, f is constant.

Problem 4. It is hard, or maybe even impossible, to determine

$$\int_{-\frac{\sqrt{\pi}}{2}}^{-\frac{\sqrt{\pi}}{2}} \cos(x^2)dx$$

exactly. But $g(x) = \cos(x^2)$ can be approximated by a polynomial. Your problem: find a polynomial $p(x) = ax^4 + bx^3 + cx^2 + dx + e$ that satisfies $p(0) = g(0)$, $p'(0) = g'(0)$, $p''(0) = g''(0)$, $p'''(0) = g'''(0)$, and $p''''(0) = g''''(0)$. Use a computer to graph g and p in the same picture. Approximate $\int_{-\frac{\sqrt{\pi}}{2}}^{-\frac{\sqrt{\pi}}{2}} \cos(x^2)dx$ by computing

$$\int_{-\frac{\sqrt{\pi}}{2}}^{-\frac{\sqrt{\pi}}{2}} p(x)dx$$

instead.

Answer. For $g(x) = \cos(x^2)$, we have

$$\begin{aligned}
g(x) &= \cos\left(x^2\right) & g(0) &= 1\\
g'(x) &= -2x\sin\left(x^2\right) & g'(0) &= 0\\
g''(x) &= -4x^2\cos\left(x^2\right) - 2\sin\left(x^2\right) & g''(0) &= 0\\
g^{(3)}(x) &= 8x^3\sin\left(x^2\right) - 12x\cos\left(x^2\right) & g^{(3)}(0) &= 0\\
g^{(4)}(x) &= 16x^4\cos\left(x^2\right) + 48x^2\sin\left(x^2\right) - 12\cos\left(x^2\right) & g^{(4)}(0) &= -12\\
g^{(5)}(x) &= -32x^5\sin\left(x^2\right) + 160x^3\cos\left(x^2\right) + 120x\sin\left(x^2\right) & g^{(5)}(0) &= 0
\end{aligned}$$

If $p(x) = ax^4 + bx^3 + cx^2 + dx + e$, then

$$\begin{aligned}
p(x) &= ax^4 + bx^3 + cx^2 + dx + e & p(0) &= e\\
p'(x) &= 4ax^3 + 3bx^2 + 2cx + d & p'(0) &= d\\
p''(x) &= 12ax^2 + 6bx + 2c & p''(0) &= 2c\\
p^{(3)}(x) &= 24ax + 6b & p^{(3)}(0) &= 6b\\
p^{(4)}(x) &= 24a & p^{(4)}(0) &= 24a\\
p^{(5)}(x) &= 0 & p^{(5)}(0) &= 0
\end{aligned}$$

Matching coefficients, we see that

$$\begin{aligned}
p(0) = g(0) &\Rightarrow e = 1\\
p'(0) = g'(0) &\Rightarrow d = 0\\
p''(0) = g''(0) &\Rightarrow c = 0\\
p^{(3)}(0) = g^{(3)}(0) &\Rightarrow b = 0\\
p^{(4)}(0) = g^{(4)}(0) &\Rightarrow a = -\frac{1}{2}
\end{aligned}$$

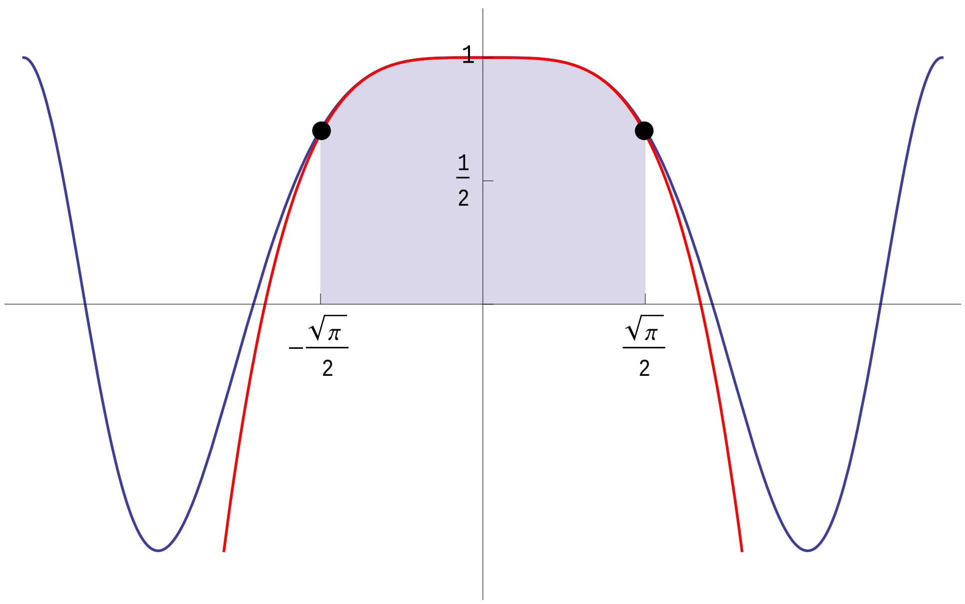
So, the polynomial

$$p(x) = -\frac{1}{2}x^4 + 1$$

has the same value and first-through-fourth derivatives as $g(x) = \cos(x^2)$, so we reason that $p(x)$ is a good approximation for $g(x)$ and

$$\int_{-\frac{\sqrt{\pi}}{2}}^{-\frac{\sqrt{\pi}}{2}} \cos(x^2)dx \approx \int_{-\frac{\sqrt{\pi}}{2}}^{-\frac{\sqrt{\pi}}{2}} \left(-\frac{1}{2}x^4 + 1\right) dx = \sqrt{\pi} - \frac{\pi^{5/2}}{160}.$$

Here's a nice picture:



Problem 5. Suppose that $f(4) = 0$, $g(4) = 0$, $f'(4) = 7$, and $g'(4) = -1$. Prove that $\lim_{x\to 4} \frac{f(x)}{g(x)}$ exists and compute it.

Answer. We begin with $\frac{f'(4)}{g'(4)}$:

$$\begin{aligned}
-7 &= \frac{7}{-1} \\
&= \frac{f'(4)}{g'(4)} \\
&= \frac{\lim_{h\to 0} \frac{f(4+h)-f(4)}{h}}{\lim_{h\to 0} \frac{g(4+h)-g(4)}{h}} \\
&= \lim_{h\to 0} \frac{f(4+h)-f(4)}{g(4+h)-g(4)} \\
&= \lim_{h\to 0} \frac{f(4+h)-0}{g(4+h)-0} \\
&= \lim_{h\to 0} \frac{f(4+h)}{g(4+h)} \\
&= \lim_{x\to 4} \frac{f(x)}{g(x)}.
\end{aligned}$$

Some review problems

Problem 6. Let f be a function defined on an open neighborhood of c. Define the statement "f is differentiable at c" and the number $f'(c)$.

Answer. We say f is differentiable at c provided the limit

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

exists and is finite. If it does, we define the number $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ by $f'(c)$.

Problem 7. Prove that if f is differentiable at $x = c$ then f is continuous at $x = c$.

Answer. We need to prove that $f(c) = \lim_{x\to c} f(x)$, or equivalently, that $\lim_{h\to 0} f(c+h) - f(c) = 0$.

Since f is differentiable at c we know $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists and is finite. Therefore,

$$\lim_{h\to 0} f(c+h) - f(c) = \lim_{h\to 0} \left(\frac{f(c+h)-f(c)}{h}\right)(h) = \lim_{h\to 0} f'(c)h = 0.$$

Problem 8. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable and $f(c) \geq f(x)$ for all $x \in \mathbb{R}$, then $f'(c) = 0$.

Answer. Suppose that $f(c) \geq f(x)$ for all $x \in \mathbb{R}$. If $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = f'(c) > 0$, then, there exists a neighborhood of 0 so that for all h in this neighborhood $\frac{f(c+h)-f(c)}{h} > 0$. But this is impossible since for some h in this neighborhood we will have $h > 0$ and $f(c) > f(c+h)$.

Similarly, if $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = f'(c) < 0$, then, there exists a neighborhood of 0 so that for all h in this neighborhood $\frac{f(c+h)-f(c)}{h} < 0$. But this is impossible since for some h in this neighborhood we will have $h < 0$ and $f(c) > f(c+h)$.

Therefore, if $f'(c)$ exists, it must equal zero.

Problem 9. Prove that if $f'(x) > 0$ for all $x \in (0,1)$, then f is increasing on $(0,1)$.

Answer. Suppose $f'(x) > 0$ for all $x \in (0,1)$. To show that f is increasing on $(0,1)$, choose two numbers $a, b \in (0,1)$ with $a < b$. By the mean value theorem, there exists a number $c \in (a,b)$ with $f'(c) = \frac{f(b)-f(a)}{b-a}$. Since $f'(c) > 0$ and $b - a > 0$ it follows that $f(b) - f(a) > 0 \Rightarrow f(a) < f(b)$, as needed to show that f is increasing.