Limits

L'Hôpital's rule

Theorem 1 (L'Hôpital's Rule, $\frac{0}{0}$ form, a simple version)**.** *Suppose that*

$$\lim_{x\to c} f(x) = 0 \text{ and } \lim_{x\to c} g(x) = 0,$$

that f and g are differentiable at c, and that $g'(c) \neq 0$. Then,

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}.$$

Proof. First note that since f and g are differentiable at c, they are continuous at c. This, together with the hypothesis that $\lim\limits_{x\to c} f(x) = 0$ and $\lim\limits_{x\to c} g(x) = 0$, imply that $f(c) = 0$ and $g(c) = 0$. Now,

$$\frac{f'(c)}{g'(c)} = \frac{\lim_{x\to c} \frac{f(x)-f(c)}{x-c}}{\lim_{x\to c} \frac{g(x)-g(c)}{x-c}} = \lim_{x\to c} \frac{f(x)-f(c)}{g(x)-g(c)} = \lim_{x\to c} \frac{f(x)}{g(x)}.$$

□

Now, suppose in addition to the hypotheses above, that f' and g' are defined and continuous in a neighborhood of c. Then, we have $\frac{f'(c)}{g'(c)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$ and we can rewrite the conclusion of the theorem above as

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

However, one can say more. Even if $\frac{f'(c)}{g'(c)}$ does not exist[1] one can still conclude that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the right hand side exists. We summarize:

[1]There are several reasons that $\frac{f'(c)}{g'(c)}$ might not exist: $f'(c)$ might not exist, $g'(c)$ might not exist, or $g'(c)$ might equal zero.

Theorem 2 (L'Hôpital's Rule, $\frac{0}{0}$ form, strong version)**.** *Suppose that*

$$\lim_{x\to c} f(x) = 0 \text{ and } \lim_{x\to c} g(x) = 0$$

and suppose that $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ *exists (or equals* $\pm\infty$*), then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

Proof. We give a proof in the case that $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists and is finite. Note that we may set $f(c) = 0$ and $g(c) = 0$. Then, for $x \neq c$, f and g are continuous on the interval $[c, x]$ (or the interval $[x, c]$, depending on whether $x > c$ or $x < c$), and differentiable on the interval (c, x) (or the interval (x, c)). Therefore, we may apply Caucy's mean value theorem to obtain a number d, between x and c with

$$\frac{f'(d)}{g'(d)} = \frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f(x)}{g(x)}.$$

Since as $x \to c$, we have $d \to c$, we conclude that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{d\to c} \frac{f'(d)}{g'(d)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

In the last equality, we have just changed the variable d to the variable x. □

Now, we can use this $\frac{0}{0}$ form of L'Hôpital's rule to get another version:

Theorem 3 (L'Hôpital's Rule, $\frac{\infty}{\infty}$ form)**.** *Suppose that*

$$\lim_{x\to c} f(x) = \pm\infty \text{ and } \lim_{x\to c} g(x) = \pm\infty$$

and suppose that $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ *exists (or is* $\pm\infty$*), then*

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}.$$

Proof. We only provide a proof under the special assumption that $\lim_{x\to c} \frac{f(x)}{g(x)}$ exists and is nonzero, call it L, and that $\lim_{x\to c} \frac{f'(x)}{g'(x)} = M$ is finite.

Consider $L = \lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$. Since $\lim_{x\to c} \frac{1}{g(x)} = 0$ and $\lim_{x\to c} \frac{1}{f(x)} = 0$; and

$$\lim_{x\to c} \frac{\frac{d}{dx}\frac{1}{g(x)}}{\frac{d}{dx}\frac{1}{f(x)}} = \lim_{x\to c} \frac{\frac{g'(x)}{(g(x))^2}}{\frac{f'(x)}{(f(x))^2}} == \lim_{x\to c} \frac{(g(x))^2}{(f(x))^2}\frac{f'(x)}{g'(x)} = \frac{M}{L^2}.$$

Therefore, by the $\frac{0}{0}$ form of L'Hôpital's rule, we can conclude that

$$L = \frac{M}{L^2} \Rightarrow L = M.$$

□

There are many more versions of L'Hôpital's rule. There are versions where $x \to c$ is replaced with $x \to c^-$, $x \to c^+$, $x \to \infty$, or $x \to -\infty$. The proofs above all work for the one sided versions, but you need to modify the argument for the $x \to \pm\infty$ versions.

Problem 1. L'Hôpital's rule *does not apply* without the assumption that $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists in either the finite or infinite sense. Give an example where $\lim_{x\to c} f(x) = 0$, $\lim_{x\to c} g(x) = 0$, $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and is finite and where $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ does not exist. Give another example where $\lim_{x\to c} f(x) = \pm\infty$, $\lim_{x\to c} g(x) = \pm\infty$, $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and is finite and where $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ does not exist.

Problem 2. The squeeze theorem will tell you that

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

So, there must exist a number $\delta > 0$ so that if $0 < |x| < \delta$ then

$$\left| x \sin\left(\frac{1}{x}\right)\right| < \frac{1}{1000}.$$

One student, doing a few calculations on a calculator, noticed that $\sin\left(\frac{1}{0.106}\right) = -0.00918417$ and answered that $\delta = .106$ works. Is he correct?

Problem 3. State and prove Cauchy's mean value theorem

Problem 4. True or false. If true, give a rigorous proof using the definition of limits. If false, supply an example showing the statement is false:

(a) If $\lim_{x\to c} f(x) = 0$ then $\lim_{x\to c} \frac{1}{f(x)} = \infty$.

(b) If $\lim_{x\to c} f(x) = \infty$ then $\lim_{x\to c} \frac{1}{f(x)} = 0$.

Problem 5. Give an example of two functions f and g satisfying

(a) $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = \infty$,

(b) $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and is finite, and

(c) $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ does not exist.

Problem 6. Give an example of two functions f and g (or prove that no such example exists)

(a) with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ exists and is finite, and $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ does not exist.

(b) where neither $\lim_{x\to c} f(x)$ nor $\lim_{x\to c} f(x)$ exist and $\lim_{x\to c} f(x) + g(x)$ exists and is finite.

(c) where neither $\lim_{x\to c} f(x)$ nor $\lim_{x\to c} f(x)$ exist and $\lim_{x\to c} f(x)g(x)$ exists and is finite.

(d) for which $\lim_{x\to c} f(x)$ does not exist, $\lim_{x\to c} f(x)$ exists and is finite and $\lim_{x\to c} f(x) + g(x)$ exists and is finite.

(e) or which $\lim_{x\to c} f(x)$ does not exist, $\lim_{x\to c} f(x)$ exists and is finite and $\lim_{x\to c} f(x)g(x)$ exists and is finite.

Problem 7. Compute

(a) $\lim_{x\to 0} \frac{\exp(x)-1}{x}$

(b) $\lim_{x\to 0} \frac{\exp(x)-1-x}{x^2}$

(c) $\lim_{x\to 0} \frac{\log(1+x)}{x}$

(d) $\lim_{x\to 0} \frac{\log(1+x)-x}{x^2}$

(e) $\lim_{x\to 0} \frac{\cos(x)-1}{x}$

(f) $\lim_{x\to 0} \frac{\cos(x)-1}{x^2}$

(g) $\lim_{x\to 0} \frac{\sin(x)}{x}$

(h) $\lim_{x\to 0} \frac{\sin(x)-x}{x^2}$

(i) $\lim_{x\to 0} \frac{\sin(x)-x}{x^3}$

(j) $\lim_{x\to 0} \frac{\cosh(x)-1}{x}$

(k) $\lim_{x\to 0} \frac{\cosh(x)-1}{x^2}$

(l) $\lim_{x\to 0} \frac{\sinh(x)}{x}$

(m) $\lim_{x\to 0} \frac{\sinh(x)-x}{x^2}$

(n) $\lim_{x\to 0} \frac{\sinh(x)-x}{x^3}$

(o) $\lim_{x\to 0} \frac{\arcsin(x)}{x}$

(p) $\lim_{x\to 0} \frac{\arcsin(x)-x}{x^2}$

(q) $\lim_{x\to 0} \frac{\arcsin(x)-x}{x^3}$

(r) $\lim_{x\to 0} \frac{\arctan(x)}{x}$

(s) $\lim_{x\to 0} \frac{\arctan(x)-x}{x^2}$

(t) $\lim_{x\to 0} \frac{\arctan(x)-x}{x^3}$

Problem 8. If S is a set and $R \subseteq S \times S$, one may call R a relation on the set S. One says that a is related to b, and writes aRb if the pair $(a,b) \in R$. If R is a relation on A we say that

- R is reflexive if and only if for all $a \in S$, aRa.
- R is symmetric if and only if for all $a,b \in S$, $aRb \Rightarrow bRa$.
- R is transitive if and only if for all $a,b,c \in S$, aRb and $bRc \Rightarrow aRc$.
- R is said to satisfy the trichotomy law if and only if for all $a,b \in S$ exactly one of the following hold: aRb, bRa, or $a = b$.

For example, $<$ defines a relation on the real numbers that is transitive, and satisfies the trichotomy law, but $<$ is neither reflexive nor symmetric. The relation $<$ is not reflexive since, for example, it is not true that $2 < 2$. The relation $<$ is not symmetric since, for example, $1 < 8$ but it is not true that $8 < 1$.

For another example, consider the following relation on the set $\mathbb{Z}$:

$$a|b \text{ if and only if } b \text{ is divisible by } a.$$

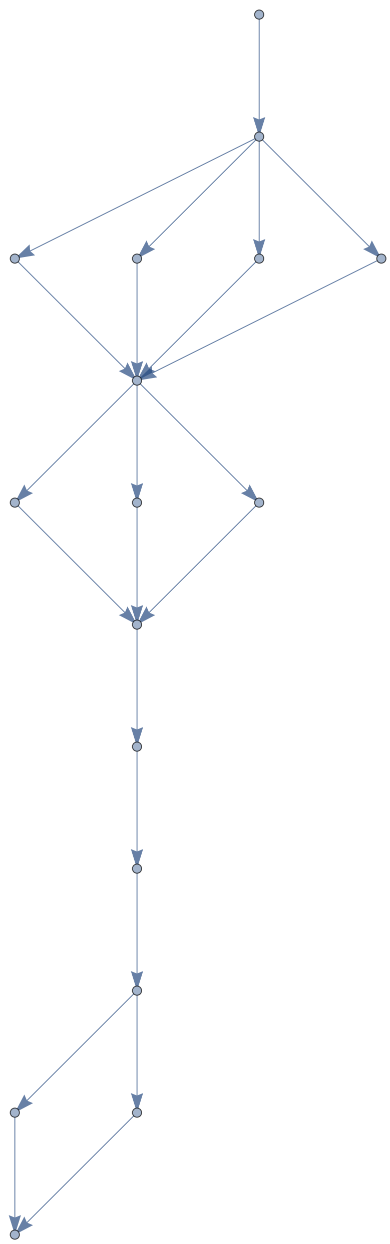
One can check that this relation is reflexive, symmetric, and transitive, but does not satisfy the trichotomy law.

Now, for your problem: consider a new relation $\ll$ on the set of functions. Let us define

$$f \ll g \text{ if and only if } \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

(a) Which apply to $\ll$: reflexive, symmetric, transitive, trichotomy law ?

(b) Label the vertices in the picture below according to the rule: there should be a directed path from f to g whenever $f \ll g$.



Here are the labels:

- $\exp(x)$
- $\log(x)$
- $\log(\log(x)$
- x
- $(\exp(x))^{\frac{1}{50}}$
- $(\exp(x))^{50}$
- $\exp(x^{\frac{1}{50}})$
- $\exp(x^{50})$
- x^{50}
- $x^{\frac{1}{50}}$
- $50x$
- $\exp(50x)$
- $\log(e^{50x})$
- $\log(x^{50})$
- $\log(50x)$
- $\log\left(x^{\frac{1}{50}}\right)$
- $(\log(x))^{\frac{1}{50}}$

Volume and arclength

Problem 9. Review the computations of the n dimensional volume of an n dimensional sphere of radius R for $n = 2, 3, 4$.

(a) Go on and find the volumes of 5, 6, and 7 dimensional spheres of radius R.

(b) Make a conjecture about the formula for the n dimensional volume of an n dimensional sphere.

(c) Try to prove your conjecture.

Problem 10 (Challenge)**.** Review the derivation for the arclength of a curve as an integral and compute the circumference of the unit circle.

(a) If we redefine the notion of distance between two points in the plane, we get a different formula for the arclength of a curve. For example, the "d_n distance" between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is defined to be

$$d_n(P, Q) = (|x_2 - x_1|^n + |y_2 - y_1|^n)^{\frac{1}{n}} \text{ for } n \geq 1.$$

In this terminology, the "ordinary" distance is the d_2 distance. Give a formula for the arclength of a curve as an integral using the d_n distance.

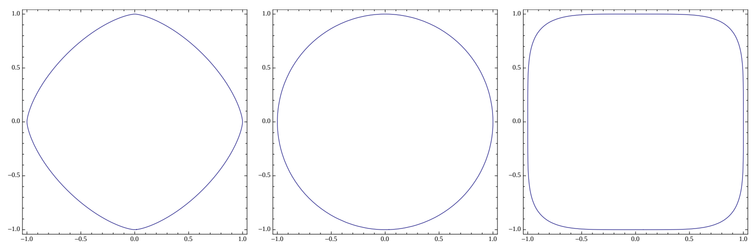
(b) The unit circle C is defined to be the set of all points in $\mathbb{R}^2$ whose distance from the origin O is one:

$$C = \{P \in \mathbb{R}^2 : d(P, O) = 1\} = \left\{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = 1\right\}$$
$$= \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

Define C_n to be the set of all points in $\mathbb{R}^2$ whose d_n distance from the origin O is one:

$$C_n = \{P \in \mathbb{R}^2 : d_n(P, O) = 1\} = \left\{(x, y) \in \mathbb{R}^2 : (|x|^n + |y|^n)^{\frac{1}{n}} = 1\right\}$$
$$= \{(x, y) \in \mathbb{R}^2 : x^n + y^n = 1\}.$$

Sketch pictures of C_n for several n. Here is a picture of $C_{\frac{3}{2}}$, C_2 (the ordinary circle), and C_5:



(c) Now define $\pi(n)$ to be arclength of the upper half of the unit circle C_n. This gives a different universal constant for each n. Do some computations, say compute $\pi_1, \pi_{\frac{3}{2}}, \pi_3, \pi_{100}$, and $\lim_{n\to\infty} \pi_n$, to get some idea of the range of values of π.