Mathematical induction

Problem 1. Do either part (a) or part (b)

(a) Let c be any fixed natural number. Prove that $1^c + 2^c + \cdots + n^c > \dfrac{n^{c+1}}{c+1}$ for every $n \in \mathbb{N}$.

Problem 1. Continued.

(b) Let F_k denote the k-th Fibonacci number, and let c be any fixed natural number. Prove that $F_{n+c} = F_c F_{n+1} + F_{c-1} F_n$ for every $n \in \mathbb{N}$.

Integration

Problem 2.

(a) Let $s : [a, b] \to \mathbb{R}$ be a step function. Define $\int_a^b s$.

(b) Let $f : [a, b] \to \mathbb{R}$ be any bounded function. Define the statement "f is integrable", and the expression $\int_a^b f$.

Problem 3. True or false:

(a) If x is any rational number and y is any irrational number, then $x + y$ is irrational.

(b) If x is any rational number and y is any irrational number, then xy is irrational.

(c) For any function $f : X \to Y$ and any $C \subseteq Y$, $f^{-1}(\overline{C}) = \overline{f^{-1}(C)}$.

(d) For any function $f : X \to Y$ and any $A \subseteq X$, $f(\overline{A}) = \overline{f^{(}A)}$.

(e) If A is any nonempty subset of real numbers bounded below, then there is an element $z \in A$ with $z \leq a$ for all $a \in A$.

(f) If A is any nonempty subset of natural numbers, then there is an element $z \in A$ with $z \leq a$ for all $a \in A$.

(g) Let $g : [0,1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} x & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 1 & \text{otherwise.} \end{cases}$$

Then g is integrable and $\int_0^1 g = 1$.

(h) Let $g : [0,1] \to \mathbb{R}$ defined by as above. There are step function s and t with $\int_0^1 s = \underline{I}(g)$ and $\int_0^1 t = \overline{I}(g)$.

(i) Let $h(x) = [\sqrt{x}]$. Then h is a step function and $\int_1^{16} h = 34$.

(j) If f is decreasing on $[a,b]$ then f is integrable and $(b-a)f(b) \leq \int_a^b f \leq (b-a)f(a)$.

Problem 4. Bonus. Suppose that instead of the usual definition, we had defined the integral of a step function s by the formula

$$\int_a^b s = \sum_{i=1}^{n} \left(s_i^2\right)(x_i - x_{i-1})$$

where $\{x_i\}$ is a partition of $[a, b]$ for which s takes the constant value s_i on the open subinterval (x_{i-1}, x_i). Then, a different theory of integration would result, with possibly different properties. Of the following two familiar properties of the integral, only one is true under the modified definition. For two bonus points, prove the claim that is true and disprove the claim that is false:

Claim. *For all step functions s and all constants* $c \in \mathbb{R}$, $\int_a^b cs = c \int_a^b s$.

Claim. *For all step functions s and all* $a, b, c \in \mathbb{R}$ *with* $a < b < c$, $\int_a^c s = \int_a^b s + \int_b^c s$.

EXAM

Midterm

Math 157

October 26, 2004

- Show your work, but not your scratchwork. Neatness counts.
- Problems 1 and 2 are worth five points each, and each part of problem 3 is worth one point for a total of 20 possible points.

Success!