Math 157 Fall 2013 Homework 4 - selected answers

Problem 1 (Section I 3.5 Exercise 5)**.** First, let me record what I will use. By applying Theorem 1.22 with $a = 0$, we know that if $b > 0$ and $c < 0$, then $bc < 0$. This proves that if $a > 0$ and $b > 0$, then the quotient $\frac{a}{b} > 0$ (otherwise, if $c = \frac{a}{b} < 0$, the product $cb = a < 0$). Also, by exercise 4, we know that if $a > 0$, then $\frac{1}{a} > 0$.

Now suppose that $0 < a < b$. This means that $a > 0$, $b > 0$ and $0 < b - a$. Since $0 < a$ and $0 < b$ we know $0 < ab$. Since $0 < b - a$ and $0 < ab$, we have

$$0 < \frac{b-a}{ab} = \frac{1}{a} - \frac{1}{b}.$$

This proves that $\frac{1}{b} < \frac{1}{a}$. Also, since $0 < b$, we have $0 < \frac{1}{b}$, so we get $0 < \frac{1}{b} < \frac{1}{a}$.

Problem 2 (Section I 3.12 Exercise 1)**.** Suppose $x < y$. Let $z = \frac{x+y}{2}$. Then $x < z < y$. To see this, first note that

$$x < y \Rightarrow \frac{x}{2} < \frac{y}{2}.$$

Then,

$$x = \frac{x}{2} + \frac{x}{2} < \frac{x}{2} + \frac{y}{2} = z$$

and

$$z = \frac{x}{2} + \frac{y}{2} < \frac{y}{2} + \frac{y}{2} = y.$$

Problem 3 (Section I 3.12 Exercise 3)**.** Let $x > 0$. Since $\mathbb{N}$ is unbounded, there exists $n \in \mathbb{N}$ with $\frac{1}{x} < n$. Then, $\frac{1}{n} < x$.

Problem 4 (Section I 3.12, part of Exercise 7)**.** I will prove that if x is rational and y is irrational, then $x + y$ is irrational.

First, I'll prove that $x, y \in \mathbb{Q} \Rightarrow x - y \in \mathbb{Q}$. To prove this, suppose $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$ with $b \neq 0$ and $y = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$ with $d \neq 0$. Then $x - y = \frac{ad-bc}{bd}$. Since $a, b, c, d \in \mathbb{Z}$, so is $ad - bc$ and bd. Since $b, d \neq 0$, the product $bd \neq 0$. So, $x - y = \frac{ad-bc}{bd} \in \mathbb{Q}$.

Now if $x + y$ is rational and x is rational, then the difference $x + y - y = x$ is rational also. Therefore, if x is rational and y is irrational, it's impossible for $x + y$ to be rational.

Problem 5. $\sqrt{3}$ is irrational. So is $-\sqrt{3}$. But the sum $\sqrt{3} - \sqrt{3} = 0$ is rational.