# Math 157 Fall 2013 Homework 6 - Selected answers

**Problem 1.** Read Chapter 1, through section 1.8. (pages 48-61).

(a) In Section 1.5, do exercises 2,3,4,8,9,10, and 11.

**Answer.** Exercise 2. Let $f(x) = 1 + x$ and $g(x) = 1 - x$. Then

$$\begin{aligned}
f(2) + g(2) &= 3 + -1 = 2 \\
f(2) - g(2) &= 3 - -1 = 4 \\
f(2)g(2) &= (3)(-1) = -3 \\
\frac{f(2)}{g(2)} &= \frac{3}{-1} = -3 \\
g(f(2)) &= g(3) = 1 - 3 = -2 \\
f(g(2)) &= f(-1) = -1 + 1 = 0 \\
f(a) + g(-a) &= 1 + a + (1 - -a) = 2 + 2a \\
f(t)g(-t) &= (1 + t)(1 + t).
\end{aligned}$$

**Answer.** Exercise 9. Let

$$f(x) = \sum_{k=0}^{n} c_k x^k = c_0 + c_1 x + c_2 x^2 + \cdots + c_k x^k.$$

(a) If $n \geq 1$ and $f(0) = 0$, the $f(x) = xg(x)$ where $g$ is a polynomial of degree $n - 1$. To see this, note that for any polynomial, $f(0) = c_0$. So, if $f(0) = 0$, then $c_0 = 0$. So,

$$\begin{aligned}
f(x) &= \sum_{k=1}^{n} c_k x^k = c_1 x + c_2 x^2 + \cdots + c_k x^k \\
&= x\left(c_1 + c_2 x + \cdots c_k x^{k-1}\right) \\
&xg(x)
\end{aligned}$$

where

$$g(x) = \left(c_1 + c_2 x + \cdots c_k x^{k-1}\right) = \sum_{k=0}^{n} c_{k+1} x^k.$$

(b) For each real number $a$, the function $p$ defined by $p(x) = f(x + a)$ is a polynomial of degree $n$. To see this, simply substitute $x + a$ for $x$ in $f(x)$, expand the expressions $(x + a)^k$, and collect the powers of $x$. More explicitly,

$$\begin{aligned}
p(x) &= f(x + a) \\
&= c_0 + c_1(x + a) + c_2(x + a)^2 + \cdots + c_n(x + a)^n \\
&= c_0 + (c_1 x + c_1 a) + (c_2 x^2 + 2c_2 a x + c_2 a^2) + (c_3 x^3 + 3c_3 a x^2 + 3c_3 a^2 x + c_3 a^3) \\
&\qquad + \cdots + (c_n x^n + nc_n a x^{n-1} + \cdots + nc_n a^{n-1} x + c_n a^n) \\
&= (c_0 + c_1 a + c_2 a^2 + c_3 a^3 + \cdots + c_n a^n) + (c_1 + 2c_2 a + 3c_3 a^2 + \cdots + nc_n a^{n-1})x + \\
&\qquad + (c_2 + 3c_3 a + \cdots)x^2 + \cdots + c_n x^n
\end{aligned}$$

which is a polynomial of degree $n$ also.

(c) If $n \geq 1$ and $f(a) = 0$, then there exists a polynomial $h$ of degree $n-1$ with $f(x) = (x-a)h(x)$. To see this, let $p(x) = f(x+a)$. Note that $f(x) = p(x-a)$. Since $f(a) = 0$, we have $p(0) = 0$ and since $p$ is a polynomial of degree $n$, part (a) implies that $p$ factors as

$$p(x) = xq(x)$$

for some polynomial $q(x)$ of degree $n-1$. Therefore $f(x) = p(x-a) = (x-a)q(x-a)$. Since $q(x)$ is a polynomial of degree $n-1$, so is $h(x) := q(x-a)$. Thus, we have found a polynomial $h$ of degree $n-1$ so that

$$f(x) = (x-a)h(x)$$

(d) It follows that if $f$ is a polynomial of degree $n$, then $f(x) = 0$ for at most $n$ distinct real numbers. I'll walk through a slow argument of this fact.

If $n \geq 2$, and $f(x) = 0$ for two distinct values of $x$, say $x = a$ and $x = b$, then there exists a polynomial $g$ of degree $n-2$ so that $f(x) = (x-a)(x-b)g(x)$. To see this, use the previous part to write $f(x) = (x-a)h(x)$ for a polynomial $h$ of degree $n-1$. Then, since $f(b) = 0$, we have $(b-a)h(b) = 0$. Since $b - a \neq 0$ if $a$ and $b$ are distinct, we must have $h(b) = 0$. Then, the previous part says that there exists a polynomial $g$ of degree $n-2$ so that $h(x) = (x-b)g(x)$. Then,

$$f(x) = (x-a)(x-b)g(x).$$

Continuing, one sees that if $f(x) = 0$ for $k$ distinct values of $x$, say $x = a_1, \ldots, a_k$, then there exists a polynomial $g$ of degree $n-k$ so that

$$f(x) = (x-a_1)\cdots(x-a_k)g(x).$$

Note that if the degree of $f$ is $n$, then $f(x) = 0$ for at most $n$ distinct values of $x$. To see this, note that if $f(x) = 0$ for $n > 0$ distinct values of $x$, call them $a_1, \ldots, a_n$, then

$$f(x) = (x-a_1)(x-a_2)\cdots(x-a_n)g(x)$$

for some polynomial $g$ of degree zero. Since $g$ is a degree zero polynomial, $g(x) = c$ for some constant $c$ and $c \neq 0$ since degree of $f$ is $n > 0$. So the polynomial $f$ has the form

$$f(x) = c(x-a_1)(x-a_2)\cdots(x-a_n)$$

for some constants $a_1, \dots, a_n$ and some nonzero constant $c$. Then, for any number $a_{n+1}$ distinct from $a_1, \dots, a_n$, we have $(a_{n+1} - a_i) \neq 0$, so the righthand side $c(x - a_1)(x - a_2) \cdots (x - a_n)$ is not zero at $x = a_{n+1}$. That is, $f(x)$ cannot be zero for any number $a_{n+1}$ distinct from $a_1, \dots, a_n$.

Another way to summarize this is to say that if $f$ is a polynomial of degree less than or equal to $n$ and $f(x) = 0$ for $n+1$ distinct values of $x$, then $f(x) = 0$ for all $x$.

(e) Now suppose that $g$ is a polynomial of $m \geq n$, where $n$ is the degree of the polynomial $f$. If $f(x) = g(x)$ for $m+1$ distinct values of $x$, then $f = g$. To see this, note that the function $h$ defined by $h(x) = f(x) - g(x)$ is a polynomial of degree $\leq m$. If $f(x) = g(x)$ for $m+1$ distinct values of $x$, then $h(x) = 0$ for more than $m+1$ distinct numbers, hence $h(x) = 0$.

(b) In Section 1.7, do exercises 1,2,3, and 6.

# Functions

**Definition 1.** We say that a set of ordered pairs $f \subseteq X \times Y$ is a *function from $X$ to $Y$* if and only if for all $x \in X$ there exists one and only one $y \in Y$ so that $(x, y) \in f$. We usually write $f : X \to Y$ if $f \subseteq X \times Y$ is a function and we write $y = f(x)$ if $(x, y) \in f$. The set $X$ is called *the domain of $f$* and the set $Y$ is called *the codomain of $f$*.

It is common to think of a function $f : X \to Y$ a "rule" that associates $x \in X$ to $y \in Y$ whenever $y = f(x)$ and to refer to the set of ordered pairs $f \subseteq X \times Y$ as the *graph* of the function. By this convention the "rule" is referred to as the function $f$ and the set $graph(f) = \{(x, y) \in X \times Y : f(x) = y\}$ is the *graph of $f$*.

**Definition 2.** Suppose that $f : X \to Y$ is a function.

(a) For any subset $A \subseteq X$, we define the set $f(A) \subseteq Y$ by

$$f(A) = \{y \in Y : \exists x \in A \text{ with } f(x) = y\}.$$

(b) We call $f(X) \subseteq Y$ the *range of $f$.*

(c) For any subset $B \subseteq Y$, we define the set $f^{-1}(B) \subseteq X$ by

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

**Problem 2.** Suppose that $f : X \to Y$ is a function.

(a) For any $A \subseteq X$ and $B \subseteq X$, compare $f(A \cup B)$ and $f(A) \cup f(B)$.

**Answer.**

**Claim 1.** *For any $A, B \subseteq X$, we have $f(A \cup B) = f(A) \cup f(B)$.*

*Proof.* Let $A, B \subseteq X$. To prove that $f(A \cup B) \subseteq f(A) \cup f(B)$ let $y \in f(A \cup B)$. This means there exists an element $x \in A \cup B$ with $f(x) = y$. If $x \in A$, we have $y = f(x) \in f(A) \Rightarrow y \in f(A) \cup f(B)$. If $x \in B$, we have $y = f(x) \in f(B) \Rightarrow y \in f(A) \cup f(B)$.

To prove that $f(A) \cup f(B) \subseteq f(A \cup B)$, let $y \in f(A) \cup f(B)$. If $y \in f(A)$, there exists an element $x \in A$ with $f(x) = y$. Since $x \in A$, we have $x \in A \cup B$, so $y = f(x) \in f(A \cup B)$. If $y \in f(B)$, there exists an element $x \in B$ with $f(x) = y$. Since $x \in B$, we have $x \in A \cup B$, so $y = f(x) \in f(A \cup B)$. □

(b) For any $A \subseteq X$ and $B \subseteq X$, compare $f(A \cap B)$ and $f(A) \cap f(B)$.

**Answer.**

**Claim 2.** *For any $A, B \subseteq X$, we have $f(A \cap B) \subseteq f(A) \cap f(B)$.*

*Proof.* Let $A, B \subseteq X$ and let $y \in f(A \cap B)$. This means there exists an element $x \in A \cap B$ with $f(x) = y$. Since $x \in A$, $y = f(x) \in f(A)$. Since $x \in B$, $y = f(x) \in f(B)$. Together, $y \in f(A)$ and $y \in f(B)$ imply $y \in f(A) \cap f(B)$. □

It may be the case that

$$f(A) \cap f(B) \not\subseteq f(A \cap B).$$

Consider an example: Let $X = \{1, 2, 3\}$, $Y = \{a, b, c\}$ and $f : \{1, 2, 3\} \to \{a, b, c\}$ be defined by

$$f(1) = a, \quad f(2) = a, \quad f(3) = b.$$

For $A = \{1\}$ and $B = \{2\}$, we have

$f(A \cap B) = f(\emptyset) = \emptyset$ and $f(A) \cap f(B) = f(\{1\} \cap f(\{2\}) = \{a\} \cap \{a\} = \{a\}$.

**Claim 3.** *If $f$ is injective, then for any $A, B \subseteq X$, $f(A \cap B = f(A) \cap f(B)$.*

*Proof.* Let $A, B \subseteq X$. We already know that $f(A \cap B) \subseteq f(A) \cap f(B)$. To finish, we need to show that if $f$ is injective then $f(A) \cap f(B) \subseteq f(A \cap B)$.

So, assume $f$ is injective and let $y \in f(A) \cap f(B)$. Since $y \in f(A)$ there exists an element $x \in A$ with $f(x) = y$. Since $y \in f(B)$ there exists an element $z \in B$ with $f(z) = y$. Since $f$ is injective and $f(x) = f(z)$, we have $x = z$. Therefore $x \in A$ and $x = z \in B$, so $x \in A \cap B$. Therefore, $y = f(x) \in f(A \cap B)$. □

(c) For any $C \subseteq Y$ and $D \subseteq Y$, compare $f^{-1}(C \cup D)$ and $f^{-1}(C) \cup f^{-1}(D)$.

(d) For any $C \subseteq Y$ and $D \subseteq Y$, compare $f^{-1}(C \cap D)$ and $f^{-1}(C) \cap f^{-1}(D)$.

**Answer.**

**Claim 4.** *For any $C, D \subseteq Y$, we have $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.*

*Proof.* Let $C, D \subseteq Y$. To prove that $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$, let $x \in f^{-1}(C \cap D)$. This means $f(x) \in C \cap D$. So $f(x) \in C$ and $f(x) \in D$. Since $f(x) \in C$, we have $x \in f^{-1}(C)$. Since $f(x) \in D$, we have $x \in f^{-1}(D)$. Together $x \in f^{-1}(C)$ and $x \in f^{-1}(D)$ imply that $x \in f^{-1}(C) \cap f^{-1}(D)$.

To prove that $f^{-1}(C) \cap f^{-1}(D) \subseteq f^{-1}(C \cap D)$, let $x \in f^{-1}(C) \cap f^{-1}(D)$. This means $x \in f^{-1}(C)$ and $x \in f^{-1}(D)$. Since $x \in f^{-1}(C)$, we have $f(x) \in C$. Since $x \in f^{-1}(D)$ we have $f(x) \in D$. Therefore, $f(x) \in C \cap D$. Therefore $x \in f^{-1}(C \cap D)$. □

(e) For any $A \subseteq X$, compare $f(X \setminus A)$ and $Y \setminus f(A)$.

**Answer.** First, we give an example to show that it may be the case that

$$f(X \setminus A) \nsubseteq Y \setminus f(A) \text{ and } Y \setminus f(A) \nsubseteq f(X \setminus A).$$

Let $X = \{1, 2, 3\}$, $Y = \{a, b, c\}$ and $f : \{1, 2, 3\} \to \{a, b, c\}$ be defined by

$$f(1) = a, \quad f(2) = a, \quad f(3) = b.$$

Let $A = \{1\}$. We have

$$f(X \setminus A) = f(\{2, 3\}) = \{a, b\} \text{ and } Y \setminus f(A) = Y \setminus \{a\} = \{b, c\}.$$

**Claim 5.** *If $f : X \to Y$ is injective, then for any $A \subseteq X$ we have $f(X \setminus A) \subseteq Y \setminus f(A)$.*

*Proof.* Assume $f : X \to Y$ is injective and $A \subseteq X$. Let $y \in f(X \setminus A)$. This means there exists $x \in X \setminus A$ with $f(x) = y$. Since $f$ is injective, $x$ is the only element of $X$ with $f(x) = y$. Since $x \notin A$, there is no element $z \in A$ with $f(z) = y$. Thus, $y \notin f(A)$. This says $y \in Y \setminus f(A)$, as needed to prove that $f(X \setminus A) \subseteq Y \setminus f(A)$. □

**Claim 6.** *If $f : X \to Y$ is surjective then for any $A \subseteq X$ we have $Y \setminus f(A) \subseteq f(X \setminus A)$.*

*Proof.* Assume $f : X \to Y$ is surjective and $A \subseteq X$. Let $y \in Y \setminus f(A)$. Because $f$ is onto, there exists $x \in X$ with $f(x) = y$. Note that $x \notin A$ for otherwise $f(x) = y \in f(A)$. Therefore $x \in X \setminus A$. Since $f(x) = y$, we have $y \in f(X \setminus A)$, as needed to prove that $Y \setminus f(A) \subseteq f(X \setminus A)$. □

(f) For any $C \subseteq Y$, compare $f^{-1}(Y \setminus C)$ and $X \setminus f^{-1}(C)$.

(g) For any $A \subseteq X$, compare $f^{-1}(f(A))$ and $A$.

**Answer.**

**Claim 7.** *For any $A \subseteq X$, we have $A \subseteq f^{-1}(f(A))$.*

*Proof.* Let $x \in A$. Then $f(x) \in f(A)$. Since $f(x) \in f(A)$, the element $x \in f^{-1}(f(A))$. □

Note that $f^{-1}(f(A))$ and $A$ need not be equal. For example, let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c, d\}$ and define

$$\begin{aligned} f : X &\to Y \\ 1 &\mapsto a \\ 2 &\mapsto b \\ 3 &\mapsto b \\ 4 &\mapsto c \end{aligned}$$

Let $A = \{1, 2\}$. Then

$$f^{-1}(f(A)) = f^{-1}(\{a, b\}) = \{1, 2, 3\} \neq A.$$

**Claim 8.** *If $f$ is injective, then for any $A \subseteq X$, we have $A = f^{-1}(f(A))$.*

*Proof.* We only need to show that if $f$ is injective, then $f^{-1}(f(A)) \subseteq A$. So assume $f$ is injective and let $x \in f^{-1}(f(A))$. This means that $f(x) \in f(A)$. Therefore, there exists an element $z \in A$ with $f(z) = f(x)$. Since $f$ is injective, $z = x$ and we see that $x \in A$. □

(h) For any $C \subseteq Y$, compare $f(f^{-1}(C))$ and $C$.

Here "compare" means decide whether $\subseteq$, $\supseteq$, $=$, or none apply.

**Definition 3.** Suppose that $f : X \to Y$ is a function.

(a) We say that $f$ is *injective* or *one to one* if and only if

$$\forall x \in X \forall z \in X\,(f(x) = f(z) \Rightarrow x = z).$$

(b) We say that $f$ is *surjective* or *onto* if and only if

$$\forall y \in Y \exists x \in X\,(f(x) = y).$$

(c) We say that $f$ is *bijective* if $f$ is both injective and surjective.

We may call an injective function an *injection*, a surjective function a *surjection*, and a bijective function a *bijection*.

**Problem 3.** Which apply: injective, surjective, or bijective?

(a) Define $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = 2n$ for every $n \in \mathbb{N}$.

**Answer.** The function $f$ is injective but not surjective. To see that $f$ is injective, suppose $f(n) = f(k)$. This means $2n = 2k$, which implies $n = k$. To see that $f$ is not surjective, note that $1 \notin f(\mathbb{N})$ since $1 \neq 2n$ for any $n \in \mathbb{N}$.

(b) Define $g : \mathbb{N} \setminus \{0, 1\} \to \mathbb{N}$ by $g(n) = n - 1$ for every $n \in \mathbb{N}$.

**Answer.** Here, $g$ is bijective. To see that $g$ is injective, assume $g(n) = g(k)$. This means $n - 1 = k - 1$ which implies $n = k$. To see that $g$ is surjective, let $y \in \mathbb{N}$, then $n = y + 1 \in \mathbb{N} \setminus \{0, 1\}$ and $g(n) = g(y + 1) = y + 1 - 1 = y$.

(c) Let $X = \{\text{functions } \phi : \mathbb{N} \to \mathbb{N}\}$. Define a function $G : X \to \mathbb{N}$ by $G(\phi) = \phi(3)$ for all $\phi \in X$.

**Answer.** This function $G$ is surjective but not injective. To see that it is surjective, let $y \in \mathbb{N}$. Then for the constant function $\phi : \mathbb{N} \to \mathbb{N}$ defined by $\phi(n) = y$, we have $G(\phi) = \phi(3) = y$.

To see that $G$ is not surjective consider the constant function the constant function $\phi : \mathbb{N} \to \mathbb{N}$ defined by $\phi(n) = 6$ and the function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = 2n$. Here,

$$\phi \neq f \text{ and } G(\phi) = \phi(3) = 6 \text{ and } G(f) = f(3) = 6.$$

(d) Let $X = \{\text{functions } \phi : \mathbb{N} \to \{0, 1\}\}$ and let $Y = \{\text{subsets of } \mathbb{N}\}$. Define a function $H : X \to Y$ by $H(f) = f^{-1}(\{1\})$.

**Answer.** The function $H$ is bijective. To see that $H$ is injective, assume that $H(f) = H(g)$ for functions $f, g : \mathbb{N} \to \{0, 1\}$. The assumption that $H(f) = H(g)$ means that $f^{-1}(\{1\}) = g^{-1}(\{1\})$. So, for all $n \in f^{-1}(\{1\})$, we have $f(n) = 1 = g(n)$. If $n \notin f^{-1}(\{1\})$, we must have $f(n) = 0$ and $g(n) = 0$. Therefore, for all $n \in \mathbb{N}$, $f(n) = g(n)$. That is, $f = g$.

To see that $H$ is surjective, let $Y \subset \mathbb{N}$. Define a function $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = 1$ if $n \in Y$ and $f(n) = 0$ if $n \notin Y$. Then $H(f) = f^{-1}(\{1\}) = Y$.