EXAM

Midterm

Math 157

October 26, 2004

ANSWERS

Mathematical induction

Problem 1. Do either part (a) or part (b)

(a) Let c be any fixed natural number. Prove that $1^c + 2^c + \cdots + n^c > \frac{n^{c+1}}{c+1}$ for every $n \in \mathbb{N}$.

Answer:
First we establish a key inequality. For any real numbers a and b, a quick check will show that

$$a^{c+1} - b^{c+1} = (a-b)(a^c + a^{c-1}b + a^{c-2}b^2 + \cdots + a^2b^{c-2} + ab^{c-1} + b^c).$$

So, for $a = k+1$ and $b = k$, we have

$$\begin{aligned}(k+1)^{c+1} - k^{c+1} &= (k+1-k)\left((k+1)^c + (k+1)^{c-1}k + \cdots + (k+1)k^{c-1} + k^c\right)\\ &= \left((k+1)^c + (k+1)^{c-1}k + \cdots + (k+1)n^{c-1} + k^c\right)\end{aligned}$$

Since $k < k+1$, each term in the sum on the right is less than or equal to $(k+1)^c$ (since $(k+1)^{c-1}k < (k+1)^{c-1}(k+1) = (k+1)^c$ and $(k+1)^{c-2}k^2 < (k+1)^{c-2}(k+1)^2 = (k+1)^c$, and so on). Therefore, we have $(k+1)^{c+1} - k^{c+1} < (c+1)(k+1)^c$ establishing the key inequality

$$k^{c+1} + (c+1)(k+1)^c > (k+1)^{c+1}.$$

Let $P(n)$ be the statement

$$1^c + 2^c + \cdots + n^c > \frac{n^{c+1}}{c+1}$$

and notice that $P(1)$ is the statement $1 > \frac{1}{c+1}$, which is true.

Now, assume that $P(k)$ is true for some $k \in \mathbb{N}$. This means that $1^c + 2^c + \cdots + k^c > \frac{k^{c+1}}{c+1}$.
Now, consider $1^c + 2^c + \cdots + (k+1)^c$:

$$\begin{aligned}1^c + 2^c + \cdots + (k+1)^c &> \frac{k^{c+1}}{c+1} + (k+1)^c\\ &> \frac{1}{c+1}\left(k^{c+1} + (c+1)(k+1)^c\right)\\ &> \frac{1}{c+1}(k+1)^{c+1}.\end{aligned}$$

This proves that $A(k) \Rightarrow A(k+1)$.
Therefore, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Problem 1. Continued.

(b) Let F_k denote the k-th Fibonacci number, and let c be any fixed natural number, $c \geq 2$. Prove that $F_{n+c} = F_c F_{n+1} + F_{c-1} F_n$ for every $n \in \mathbb{N}$.

Answer:
Recall, the Fibonacci numbers $F_0, F_1, F_2, \ldots,$ are defined inductively by $F_1 = 1$, $F_2 = 1$, and for $k \geq 3$

$$F_k = F_{k-1} + F_{k-2}.$$

The first few Fibonacci numbers are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots,$
Now, we begin our proof. Let $P(n)$ be the statement

$$\text{“}F_{k+c} = F_c F_{k+1} + F_{c-1} F_k \text{ for all } k \leq n.\text{”}$$

$P(1)$ is the statement that $F_{k+c} = F_c F_{k+1} + F_{c-1} F_k$ for all $k \leq 1$. There is only one thing to check. Namely that $F_{1+c} = F_c F_2 + F_{c-1} F_1$. Using the fact that $F_1 = F_2 = 1$, we have

$$\begin{aligned} F_c F_2 + F_{c-1} F_1 &= F_c + F_{c-1} \\ &= F_{c+1}. \end{aligned}$$

Therefore, $F_{1+c} = F_c F_2 + F_{c-1} F_1$. This proves that $P(1)$ is true.
Let us consider $P(2)$. This is the statement $F_{k+c} = F_c F_{k+1} + F_{c-1} F_k$ for all $k \leq 2$. Which amounts to checking that

$$F_{1+c} = F_c F_2 + F_{c-1} F_1 \text{ and } F_{2+c} = F_c F_3 + F_{c-1} F_2.$$

We already checked the first statement. To check the second one, look at:

$$\begin{aligned} F_c F_3 + F_{c-1} F_2 &= 2F_c + F_{c-1} \\ &= F_c + F_c + F_{c-1} \\ &= F_c + F_{c+1} \\ &= F_{c+2}. \end{aligned}$$

Therefore, $F_{2+c} = F_c F_3 + F_{c-1} F_2$. So, it is true that $F_{k+c} = F_c F_{k+1} + F_{c-1} F_k$ for all $k \leq 2$.
Now, suppose that $P(x)$ is true This means that we assume that $F_{k+c} = F_c F_{k+1} + F_{c-1} F_k$ for all $k \leq x$. Now consider F_{x+1+c}:

$$\begin{aligned} F_{x+1+c} &= F_{x+c} + F_{x+c-1} && \text{(by the definition of the Fibonacci numbers)} \\ &= (F_c F_{x+1} + F_{c-1} F_x) + (F_c F_x + F_{c-1} F_{x-1}) && \text{(using } P(x) \text{ and } P(x-1)) \\ &= F_c (F_{x+1} + F_x) + F_{c-1} (F_x + F_{x-1}) \\ &= F_c F_{x+2} + F_{c-1} F_{x+1}. \end{aligned}$$

This shows that $F_{x+1+c} = F_c F_{x+2} + F_{c-1} F_{x+1}$, and consequently, that $F_{k+c} = F_c F_{k+1} + F_{c-1} F_k$ for all $k \leq x + 1$.; that is, $P(n+1)$ is true.
Therefore, by the principle of mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$ and the theorem is proved.

Integration

Problem 2.

(a) Let $s : [a, b] \to \mathbb{R}$ be a step function. Define $\int_a^b s$.

Answer:

If s is a step function, then there is a partition $P = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$ of $[a, b]$ for which s is constant on the open subintervals (x_{i-1}, x_i). Let us denote the constant value by s_K; that is, $s(x) = s_k$ for $x \in (x_{k-1}, x_k)$. Then, we define

$$\int_a^b s = s_1(x_1 - x_0) + s_2(x_2 - x_1) + \cdots + s_n(x_n - x_{n-1}).$$

(b) Let $f : [a, b] \to \mathbb{R}$ be any bounded function. Define the statement "f is integrable", and the expression $\int_a^b f$.

Answer:

let $f : [a, b] \to \mathbb{R}$ be any bounded function. If there exists one and only one number I satisfying

$$\int_a^b s \leq I \leq \int_a^b t$$

for all step functions $s, t : [a, b] \to \mathbb{R}$ with $s \leq f \leq t$, then we say that f is integrable. Furthermore, we denote this unique number I by

$$\int_a^b f.$$

Problem 3. True or false:

(a) If x is any rational number and y is any irrational number, then $x + y$ is irrational.

Answer:

True If x could be written $x = \frac{a}{b}$ and $x + y = \frac{c}{d}$ for integers a, b, c, d, then we could write $y = (x + y - x) = \frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd}$.

(b) If x is any rational number and y is any irrational number, then xy is irrational.

Answer:

This is false, for example, if $x = 0$ and $y = \sqrt{2}$. Then x is rational, y is irrational, and $xy = 0$ is rational. (However, if we assume that x is a nonzero rational number, then xy is irrational. Try to prove this and see where in your argument it is important that x be nonzero.)

(c) For any function $f : X \to Y$ and any $C \subseteq Y$, $f^{-1}(\overline{C}) = \overline{f^{-1}(C)}$.

Answer:

First a remark about notation. For a set S, the symbol $\overline{S}$ means the complement of S; that is, $s \in \overline{S} \Leftrightarrow s \notin S$. This notation only makes sense when S is understood to be a subset of a given set U, in which case $\overline{S} := U \setminus S$. So, here, for $C \subseteq Y$,

$$\overline{C} = Y \setminus C$$

and for $f^{-1}(C) \subseteq X$,

$$\overline{f^{-1}(C)} = X \setminus \overline{f^{-1}(C)}.$$

Now, we begin our answer:

True. Let $x \in f^{-1}(\overline{C})$. This means that $f(x) \in \overline{C}$. Hence, $f(x) \notin C$. This implies that $x \notin f^{-1}(C)$. Therefore $x \in \overline{f^{-1}(C)}$ and we've proved that $f^{-1}(\overline{C}) \subseteq \overline{f^{-1}(C)}$.

On the other hand, let $x \in \overline{f^{-1}(C)}$. Then, $x \notin f^{-1)}(C)$. This says that $f(x) \notin C$. So, $f(x) \in \overline{C} \Rightarrow x \in f^{-1}(\overline{C})$. This shows that $\overline{f^{-1}(C)} \subseteq f^{-1}(\overline{C})$.

(d) For any function $f : X \to Y$ and any $A \subseteq X$, $f(\overline{A}) = \overline{f(A)}$.

Answer:

False. Let $f : \mathbb{Z} \to \mathbb{N}$ be given by $f(z) = z^2$ and consider $A = \{0, 1, 2\}$. Then $\overline{A} = \{\ldots, -3, -2, -1, 3, 4, \ldots\}$ and

$$f(\overline{A}) = \{1, 4, 9, 16, 25, \ldots\}.$$

On the other hand, $f(A) = \{0, 1, 4, 9, 16\}$ and

$$\overline{f(A)} = \{2, 3, 5, 6, 7, 10, 11, \ldots\}.$$

Problem 3. Continued.

(e) If A is any nonempty subset of real numbers bounded below, then there is an element $z \in A$ with $z \leq a$ for all $a \in A$.

Answer:

This is false. The open interval $(0, 1) \subseteq \mathbb{R}$ is nonempty and bounded above; and there is no element $a \in (0, 1)$ with $a \leq z$ for all $z \in (0, 1)$. (Note, by the completeness axiom, the set $(0, 1)$ has an infemum, but here $\inf(0, 1) = 0$ is not an element of $(0, 1)$.)

(f) If A is any nonempty subset of natural numbers, then there is an element $z \in A$ with $z \leq a$ for all $a \in A$.

Answer:

True. This is the well ordering principle of the natural numbers.

(g) Let $g : [0, 1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} x & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 1 & \text{otherwise.} \end{cases}$$

Then g is integrable and $\int_0^1 g = 1$.

Answer:

This is true. Notice that for any step function t with $g < t$, $\int_0^1 t \geq 1$, with equality attained when t is the constant function given by $t(x) = 1$. Therefore, $\overline{I}(g) = 1$.

Now, consider the function $s_k : [0, 1] \to \mathbb{R}$ defined by

$$s_k = \begin{cases} 0 & \text{if } 0 \leq x \leq \frac{1}{k}, \\ 0 & \text{if } x = \frac{1}{n} \text{ for some } n \leq k, \\ 1 & \text{otherwise .} \end{cases}$$

Then, s_k is a step function and $\int_0^1 s_k = 1 - \frac{1}{k}$. Therefore,

$$\underline{I}(g) \geq \sup\left\{1 - \frac{1}{k}\right\} = 1.$$

Since, one has $\underline{I}(g) \leq \overline{I}(g)$ for any function g, we conclude in this case that $\underline{I}(g) = \overline{I}(g) = 1$. This proves that g is integrable, and in fact, that $\int_0^1 g = 1$.

Problem 3. Continued.

(h) Let $g : [0,1] \to \mathbb{R}$ defined by as above. There are step function s and t with $\int_0^1 s = \underline{I}(g)$ and $\int_0^1 t = \overline{I}(g)$.

Answer:

No, there is no step function $s < g$ with $\underline{I}(g) = 1$, since g takes on the value 0 for infinitely many x, any step function $s < g$, must be ≤ 0 for at least a small open interval. This implies that $\int_0^1 s \leq 1$ for any step function $s < g$.

(i) Let $h(x) = [\sqrt{x}]$. Then h is a step function and $\int_1^{16} h = 34$.

Answer:

True. Here,

$$h(x) = \begin{cases} 1 & \text{if } 1 \leq x < 4, \\ 2 & \text{if } 4 \leq x < 9, \\ 3 & \text{if } 9 \leq x < 16, \\ 4 & \text{if } x = 16. \end{cases}$$

So, $\int_1^{16} h = 1(3) + 2(5) + 3(7) = 34$.

(j) If f is decreasing on $[a,b]$ then f is integrable and $(b-a)f(b) \leq \int_a^b f \leq (b-a)f(a)$.

Answer:

True. This is a consequence of the montonic function theorem, and the fact that the constant functions $s = f(b)$ and $t = f(a)$ are step functions satisfying $s \leq f \leq t$.

Problem 4. Bonus. Suppose that instead of the usual definition, we had defined the integral of a step function s by the formula

$$\int_a^b s = \sum_{i=1}^{n} \left(s_i^2\right)(x_i - x_{i-1})$$

where $\{x_i\}$ is a partition of $[a, b]$ for which s takes the constant value s_i on the open subinterval (x_{i-1}, x_i). Then, a different theory of integration would result, with possibly different properties.
Of the following two familiar properties of the integral, only one is true under the modified definition. For two bonus points, prove the claim that is true and disprove the claim that is false:

Claim. *For all step functions s and all constants $c \in \mathbb{R}$, $\int_a^b cs = c\int_a^b s$.*

Answer:
This is false. Consider the step function $s : [0, 1] \to \mathbb{R}$ defined by $s(x) = 3$ and let $c = 2$. Then, according to the modified definition,

$$\int_0^1 (2s) = \int_0^1 6 = 6^2(1-0) = 36$$

while

$$2\int_0^1 (s) = 2\int_0^1 3 = 2(3^2(1-0)) = 2(9) = 18.$$

(The correct formula, by the way, would be $\int_a^b cs = c^2 \int_a^b s$.)

Claim. *For all step functions s and all $a, b, c \in \mathbb{R}$ with $a < b < c$, $\int_a^c s = \int_a^b s + \int_b^c s$.*

Answer:
This is true. To prove it, let P be a partition of $[a, c]$ that includes the number b and for which s takes constant values s_k on the k-th subinterval. Write P as

$$\{a = x_0 < x_1 < \cdots < x_{t-1} < x_t = b < x_{t+1} < \cdots < x_n = c.\}$$

So, $\{x_0 < x_1 < \cdots < x_{t-1} < x_t\}$ is a partition of $[a, b]$ and $\{x_t < x_{t+1} < \cdots < x_n\}$ is a partition of $[b, c]$. We have

$$\begin{aligned}\int_a^c s &= s_1^2(x_1 - x_0) + \cdots + s_t^2(x_t - x_{t-1}) + s_{t+1}^2(x_{t+1} - x_t) + \cdots + s_n^2(x_n - x_{n-1}) \\ &= \int_a^b s + \int_b^c s.\end{aligned}$$