Math 157 Fall 2013 **Problem Set 1**

1 Step functions

Problem 1. Compute $\int_{-3}^{3} [t^2] dt$.

Problem 2. Suppose that $s : [a, b] \to \mathbb{R}$ and $t : [a, b] \to \mathbb{R}$ are step functions. Prove that

$$\int_a^b s + t = \int_a^b s + \int_a^b t.$$

Problem 3. Let $s : [a, b] \to \mathbb{R}$ be a step function and let $P = \{x_0, \ldots, x_n\}$ be a partition of the interval $[a, b]$ for which s is constant on each subinterval $[x_{k-1}, x_k]$. Let s_k denote the value of s on the k-th subinterval; i.e., $s(x) = s_k$ for all $x_{k-1} < x < s_k$.. Define

$$\oint_a^b s = \sum_{k=1}^{n} s_k^3 (x_k - x_{k-1}).$$

For this new theory of integration, which of the following properties hold?

(a) $\oint_a^b s + t = \oint_a^b s + \oint_a^b t$

(b) $\oint_a^b cs = c \oint_a^b s$

(c) $\oint_a^b s + \oint_b^c s = \oint_a^c s$

(d) $\oint_{a+c}^{b+c} s(x)dx = \oint_a^b s(x + c)dx.$

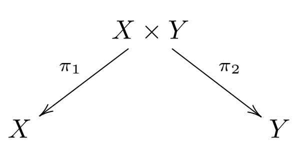
2 Products and unions

2.1 The Cartesian product

Recall, for two sets X and Y we define the product $X \times Y$ to be the set

$$X \times Y = \{(x, y) : x \in X, y \in Y\}.$$

There are two important functions $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$. Here's a diagram:

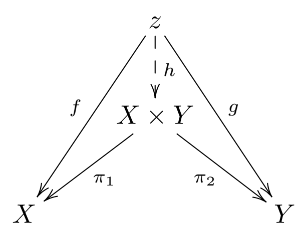


The functions $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are called *projections* and are defined by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$.

The product $X \times Y$ and the projections π_1 and π_2 satisfy the following important property.

For any set Z and any functions $f : Z \to X$ and $g : Z \to Y$, there exists a unique function $h : X \times Y \to Z$ satisfying $\pi_1 h = f$ and $\pi_2 h = g$.

Figure 1: The important property for Cartesian product of sets

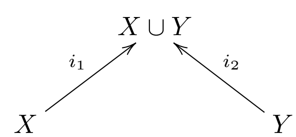


2.2 The union

Recall, for two sets X and Y we define the union $X \cup Y$ to be the set

$$X \cup Y = \{a : a \in X \text{ or } a \in Y\}.$$

There are two important functions $i_1 : X \to X \cup Y$ and $i_2 : Y \to X \cup Y$. Here's a diagram:



The functions $i_1 : X \to X \cup Y$ and $i_2 : Y \to X \cup Y$ are called *inclusions* and are defined by $i_1(x) = x$ and $i_2(y) = y$. The union $X \cup Y$ and the inclusions i_1 and i_2 satisfy the following important property:

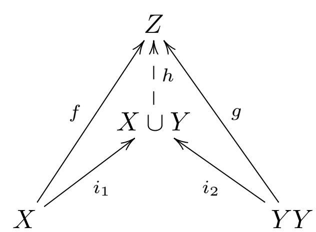
For any set Z and any functions $f : X \to Z$ and $g : Y \to Z$ such that $f(x) = g(y)$ if $x = y$ there exists a unique function $h : X \cup Y \to Z$ so that $hi_1 = f$ and $hi_2 = g$.

Problem 4. Let $X = \{1, 2\}$ and $Y = \{2, 3\}$ and $Z = \{\heartsuit, \diamondsuit\}$.

(a) Give an example of functions $f : Z \to X$, $g : Z \to Y$, and $h : Z \to X \times Y$ making the diagram in Figure 2.1 commute.

(b) Give an example of functions $f : X \to Z$, $g : Y \to Z$, and $h : X \cup Y \to Z$ making the diagram in Figure 2.2 commute.

Figure 2: The important property for the union of sets



(c) Are there any functions $X \to X \times Y$ and $X \cup Y \to X$?

Definition 1. For any sets A and B, let $\text{Functions}(A, B)$ denote the set of functions $A \to B$.

Problem 5. True or False:

(a) For any sets X, Y, the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are surjective.

(b) For any sets X, Y, the inclusions $i_1 : X \to X \cup Y$ and $i_2 : Y \to X \cup Y$ are injective.

(c) For all sets X, Y, and Z, there is a bijection between sets

$$\text{Functions}(Z, X) \times \text{Functions}(Z, Y) \to \text{Functions}(Z, X \times Y)$$

(d) For all sets X, Y, and Z, there is a bijection between sets

$$\text{Functions}(X, Z) \times \text{Functions}(Y, Z) \to \text{Functions}(X \times Y, Z)$$

(e) For all sets X, Y, and Z, there is a bijection between sets

$$\text{Functions}(X, Z) \cup \text{Functions}(Y, Z) \to \text{Functions}(X \cup Y, Z)$$

(f) For all sets X, Y, and Z with $X \cap Y = \emptyset$, there is a bijection between sets

$$\text{Functions}(X, Z) \cup \text{Functions}(Y, Z) \to \text{Functions}(X \cup Y, Z)$$

Problem 6. Prove or disprove: for all sets X, Y, Z, we have $X \times (Y \cup B) = (X \times Y) \cup (X \times Z)$.