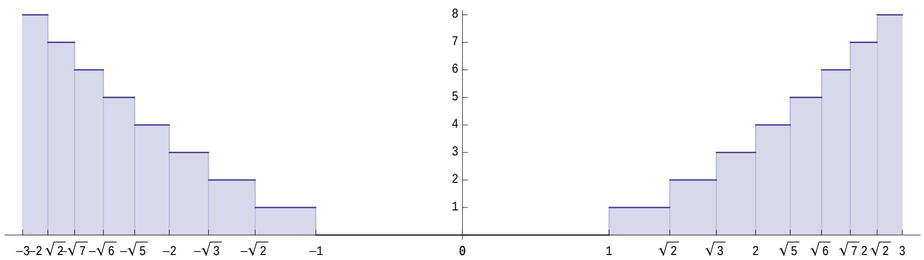
1 Step functions

Problem 1. Compute $\int_{-3}^{3} [t^2] dt$.

Answer. The function $g : [-3, 3] \to \mathbb{R}$ defined by $g(t) = [t^2]$ is constant on the open subintervals of the partition

$$\mathcal{P} = \{-3, -\sqrt{8}, -\sqrt{7}, -\sqrt{6}, -\sqrt{5}, -2, -1, 0, 1, \sqrt{2}, \sqrt{3}, 2, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, 3\}$$

Here's a picture:



We compute

$$\begin{aligned}
\int_{-3}^{3} [t^2] dt &= \sum_{k=1}^{16} g_k (x_k - x_{k-1}) \\
&= 8\left(-\sqrt{8} - (-3)\right) + 7\left(-\sqrt{7} + (-\sqrt{8})\right) + \cdots + 8\left(3 - \sqrt{8}\right) \\
&= 42 - 6\sqrt{2} - 2\sqrt{3} - 2\sqrt{5} - 2\sqrt{6} - 2\sqrt{7}
\end{aligned}$$

Problem 2. Suppose that $s : [a, b] \to \mathbb{R}$ and $t : [a, b] \to \mathbb{R}$ are step functions. Prove that

$$\int_a^b s + t = \int_a^b s + \int_a^b t.$$

Proof. Let $\mathcal{P} = \{x_0, \ldots, x_n\}$ be a common partition for which s and t are constant and take the values s_k and t_k respectively on the open subintervals (x_{k-1}, x_k) for $k = 1, \ldots, n$. Then

$$\begin{aligned}
\int_a^b s + t &= \sum_{k=1}^{n} (s_k + t_k)(x_k - x_{k-1}) \\
&= \sum_{k=1}^{n} \left((s_k)(x_k - x_{k-1}) + (t_k)(x_k - x_{k-1})\right) \\
&= \sum_{k=1}^{n} (s_k)(x_k - x_{k-1}) + \sum_{k=1}^{n} (t_k)(x_k - x_{k-1}) \\
&= \int_a^b s + \int_a^b t.
\end{aligned}$$

□

Problem 3. Let $s : [a, b] \to \mathbb{R}$ be a step function and let $P = \{x_0, \ldots, x_n\}$ be a partition of the interval $[a, b]$ for which s is constant on each subinterval $[x_{k-1}, x_k]$. Let s_k denote the value of s on the k-th subinterval; i.e., $s(x) = s_k$ for all $x_{k-1} < x < s_k$.. Define

$$\oint_a^b s = \sum_{k=1}^{n} s_k^3 (x_k - x_{k-1}).$$

For this new theory of integration, which of the following properties hold?

(a) $\oint_a^b s + t = \oint_a^b s + \oint_a^b t$

Answer. This property fails to hold. For example, let $s(x) = 1$ and $t(x) = 2$ for $x \in [0, 1]$. Then

$$\oint_0^1 s+t = (1+2)^3(1-0) = 27 \text{ and } \oint_a^b s + \oint_a^b t = 1^3(1-0) + 2^3(1-0) = 9.$$

(b) $\oint_a^b cs = c \oint_a^b s$

Answer. This property fails to hold. For example, let $s(x) = 1$ for $x \in [0, 1]$. Then, checking the property for $c = 2$, we have

$$\oint_0^1 2s = (2(1))^3(1-0) = 8 \text{ and } 2\oint_a^b s = 2\left(1^3(1-0)\right) = 2.$$

(c) $\oint_a^b s + \oint_b^c s = \oint_a^c s$

Answer. This property holds. Let $\mathcal{P} = \{x_0, \ldots, x_n\}$ be a partition of $[a, c]$ for which s is constant on the open subintervals (x_{k-1}, x_k) and assume $b = x_j \in \mathcal{P}$. So that $\{x_0, \ldots, x_j\}$ is a partition of $[a, b]$ and $\{x_j, \ldots, x_n\}$ is a partition of $[b, c]$.

$$\begin{aligned}\oint_a^b s + \oint_b^c s &= \sum_{k=1}^{j} s_k^3(x_k - x_{k-1}) + \sum_{k=j}^{n} s_k^3(x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} s_k^3(x_k - x_{k-1}) \\ &= \oint_a^c s\end{aligned}$$

(d) $\oint_{a+c}^{b+c} s(x)dx = \oint_a^b s(x+c)dx.$

Answer. This property holds. It's convenient to let $t : [a,b] \to \mathbb{R}$ be the function defined by $t(x) = s(x+c)$. Note that $s(x) = t(x-c)$. Now, let $\mathcal{P} = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$ for which t is constant on the open subintervals (x_{k-1}, x_k). Say

$$t(x) = t_k \text{ for } x_{k-1} < x < x_k.$$

So,

$$\oint_a^b s(x+c)dx = \int_a^b t = \sum_{k=1}^n t_k(x_k - x_{k-1}).$$

Consider the partition $\mathcal{Q} = \{x_0+c, \ldots, x_n+c\}$ of $[a+c, b+c]$. Note that for any $x \in (x_{k-1}+c, x_k+c)$, we have $x-c \in (x_{k-1}, x_k)$ so $s(x) = t(x-c) = t_k$. Thus s is constant on the subinterval $(x_{k-1}+c, x_k+c)$. Therefore,

$$\oint_{a+c}^{b+c} s(x)dx = \sum_{k=1}^n t_k((x_k+c) - (x_{k-1}+c)) = \sum_{k=1}^n t_k(x_k - x_{k-1}).$$

Thus, $\oint_{a+c}^{b+c} s(x)dx = \oint_a^b s(x+c)dx.$

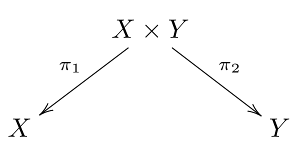
2 Products and unions

2.1 The Cartesian product

Recall, for two sets X and Y we define the product $X \times Y$ to be the set

$$X \times Y = \{(x,y) : x \in X, y \in Y\}.$$

There are two important functions $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$. Here's a diagram:

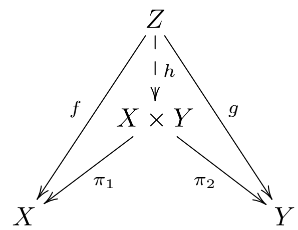


The functions $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are called *projections* and are defined by $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$.

The product $X \times Y$ and the projections π_1 and π_2 satisfy the following important property.

> *For any set Z and any functions $f : Z \to X$ and $g : Z \to Y$, there exists a unique function $h : Z \to X \times Y$ satisfying $\pi_1 h = f$ and $\pi_2 h = g$.*

Figure 1: The important property for Cartesian product of sets

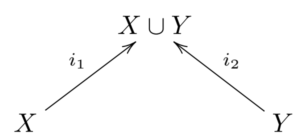


2.2 The union

Recall, for two sets X and Y we define the union $X \cup Y$ to be the set

$$X \cup Y = \{a : a \in X \text{ or } a \in Y\}.$$

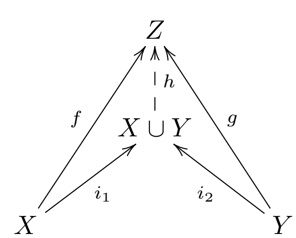
There are two important functions $i_1 : X \to X \cup Y$ and $i_2 : Y \to X \cup Y$. Here's a diagram:



The functions $i_1 : X \to X \cup Y$ and $i_2 : Y \to X \cup Y$ are called *inclusions* and are defined by $i_1(x) = x$ and $i_2(y) = y$. The union $X \cup Y$ and the inclusions i_1 and i_2 satisfy the following important property:

> *For any set Z and any functions $f : X \to Z$ and $g : Y \to Z$ such that $f(x) = g(y)$ if $x = y$ there exists a unique function $h : X \cup Y \to Z$ so that $hi_1 = f$ and $hi_2 = g$.*

Figure 2: The important property for the union of sets



Problem 4. Let $X = \{1, 2\}$ and $Y = \{2, 3\}$ and $Z = \{\heartsuit, \diamondsuit\}$.

(a) Give an example of functions $f : Z \to X$, $g : Z \to Y$, and $h : Z \to X \times Y$ making the diagram in Figure 2.1 commute.

Answer. Let

$$\begin{array}{ccc} Z \xrightarrow{f} X & Z \xrightarrow{g} Y & Z \xrightarrow{h} X \times Y \\ \heartsuit \longmapsto 1 & \heartsuit \longmapsto 3 & \heartsuit \longmapsto (1,3) \\ \diamondsuit \longmapsto 2 & \diamondsuit \longmapsto 3 & \diamondsuit \longmapsto (2,3) \end{array}$$

Note that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$:

$$\begin{array}{cc} Z \xrightarrow{h} X \times Y \xrightarrow{\pi_1} X & Z \xrightarrow{h} X \times Y \xrightarrow{\pi_2} Y \\ \heartsuit \longmapsto (1,3) \longmapsto 1 & \heartsuit \longmapsto (1,3) \longmapsto 3 \\ \diamondsuit \longmapsto (2,3) \longmapsto 2 & \diamondsuit \longmapsto (2,3) \longmapsto 3 \end{array}$$

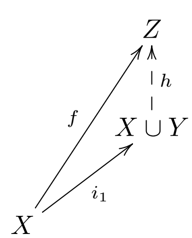
(b) Give an example of functions $f : X \to Z$, $g : Y \to Z$, and $h : X \cup Y \to Z$ making the diagram in Figure 2.2 commute.

Answer. Let

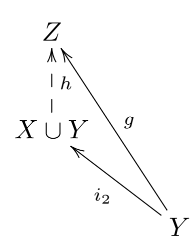
$$\begin{array}{ccc} X \xrightarrow{f} Z & Y \xrightarrow{g} Z & X \cup Y \xrightarrow{h} Z \\ 1 \longmapsto \diamondsuit & 2 \longmapsto \diamondsuit & 1 \longmapsto \diamondsuit \\ 2 \longmapsto \diamondsuit & 3 \longmapsto \heartsuit & 2 \longmapsto \diamondsuit \\ & & 3 \longmapsto \heartsuit \end{array}$$

Note that $h \circ i_1 = f$ and $hi\circ_2 = g$:

$$\begin{array}{c} X \xrightarrow{i_1} X \cup Y \xrightarrow{h} Z \\ 1 \longmapsto 1 \longmapsto \diamondsuit \\ 2 \longmapsto 2 \longmapsto \diamondsuit \end{array}$$



$$\begin{array}{c} Y \xrightarrow{i_2} X \cup Y \xrightarrow{h} Z \\ 2 \longmapsto 2 \longmapsto \diamondsuit \\ 3 \longmapsto 3 \longmapsto \heartsuit \end{array}$$



(c) Are there any functions $X \to X \times Y$ and $X \cup Y \to X$?

Answer. As long as the sets X and Y are nonempty, then sure, there will exists functions $X \to X \times Y$ and $X \cup Y \to X$. However, in contrast to the projection $X \times Y \to X$ and the inclusion $X \to X \cup Y$, there's no natural way to define such functions for arbitrary sets X and Y.

Definition 1. For any sets A and B, let Functions(A, B) denote the set of functions $A \to B$.

Problem 5. True or False:

(a) For any sets X, Y, the projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are surjective.

Answer. True.

(b) For any sets X, Y, the inclusions $i_1 : X \to X \cup Y$ and $i_2 : Y \to X \cup Y$ are injective.

Answer. True.

(c) For all sets X, Y, and Z, there is a bijection between sets

$$\text{Functions}(Z, X) \times \text{Functions}(Z, Y) \to \text{Functions}(Z, X \times Y)$$

Answer. True. The followin is a bijection

$$\begin{array}{ccc} \text{Functions}(Z, X) \times \text{Functions}(Z, Y) & \longleftarrow & \text{Functions}(Z, X \times Y) \\ (\pi_1 \circ h, \pi_2 \circ h) & \longleftarrow\!\!\dashv & h \end{array}$$

(d) For all sets X, Y, and Z, there is a bijection between sets

$$\text{Functions}(X, Z) \times \text{Functions}(Y, Z) \to \text{Functions}(X \times Y, Z)$$

Answer. False. For example if $X = \{1\}$, $Y = \{a, b, c\}$, and $Z = \{\heartsuit, \diamondsuit\}$, then $X \times Y$ has 3 elements so there are 9 functions $X \times Y \to Z$. On the other hand, there are two function from $X \to Z$ and 9 functions from $Y \to Z$, so the set Functions$(X, Z) \times$ Functions(Y, Z) has 18 elements. There's no bijection between a set with 9 elements and a set with 18 elements.

(e) For all sets X, Y, and Z, there is a bijection between sets

$$\text{Functions}(X, Z) \cup \text{Functions}(Y, Z) \to \text{Functions}(X \times Y, Z)$$

Answer. False. For example if $X = \{1, 2\}$, $Y = \{2, 3\}$, and $Z = \{\heartsuit, \diamondsuit\}$, then $X \cup Y$ has 3 elements so there are 9 functions $X \times Y \to Z$. On the other hand, there are 4 function from $X \to Z$ and 9 functions from $Y \to Z$, none are equal, so the set $\mathrm{Functions}(X, Z) \cup \mathrm{Functions}(Y, Z)$ has 13 elements. There's no bijection between a set with 9 elements and a set with 13 elements.

(f) For all sets X, Y, and Z with $X \cap Y = \emptyset$, there is a bijection between sets

$$\mathrm{Functions}(X, Z) \times \mathrm{Functions}(Y, Z) \to \mathrm{Functions}(X \cup Y, Z)$$

Answer. True. The followin is a bijection

$$\begin{array}{ccc} \mathrm{Functions}(X, Z) \times \mathrm{Functions}(Y, Z) & \longleftarrow & \mathrm{Functions}(X \cup Y, Z) \\ (h \circ i_1, h \circ i_2) & \longmapsfrom & h \end{array}$$

Problem 6. Prove or disprove: for all sets X, Y, Z, we have $X \times (Y \cup Z) = (X \times Y) \cup (X \times Z)$.

Answer. This is true. Here's a proof. Let $p \in Z \times (Y \cup Z)$. So, $p = (x, w)$ for some $x \in X$ and $w \in Y \cup Z$. If $w \in Y$, then $(x, w) \in X \times Y \subseteq X \times Y \cup X \times Z$. if $w \in Z$, then $(x, w) \in X \times Z \subseteq X \times Y \cup X \times Z$. This proves that $X \times (Y \cup Z) \subseteq (X \times Y) \cup (X \times Z)$.

Now suppose $p \in (X \times Y) \cup (X \times Z)$. This means $p \in (X \times Y)$ or $p \in (X \times Y)$. If $p \in (X \times Y)$, then $p = (x, y)$ for some $x \in X$ and some $y \in Y$. Since $Y \subseteq Y \cup Z$, we have $x \in X$ and $y \in Y \cup Z$ and we see that $p = (x, y) \in X \times (Y \cup Z)$. If, on the other hand, $p \in (X \times Z)$, then $p = (x, z)$ for some $x \in X$ and some $z \in Z$. Since $Z \subseteq Y \cup Z$, we have $x \in X$ and $z \in Y \cup Z$ and we see that $p = (x, z) \in X \times (Y \cup Z)$. This proves that $(X \times Y) \cup (X \times Z) \subseteq X \times (Y \cup Z)$.