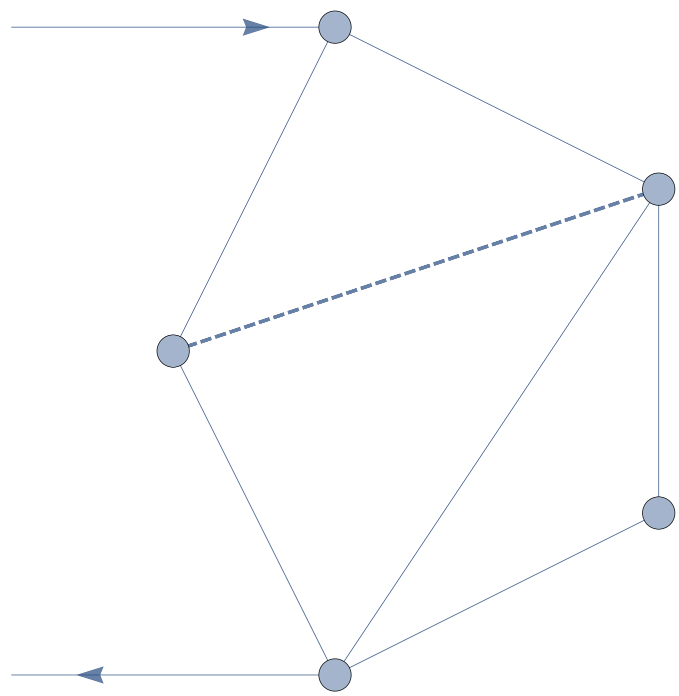
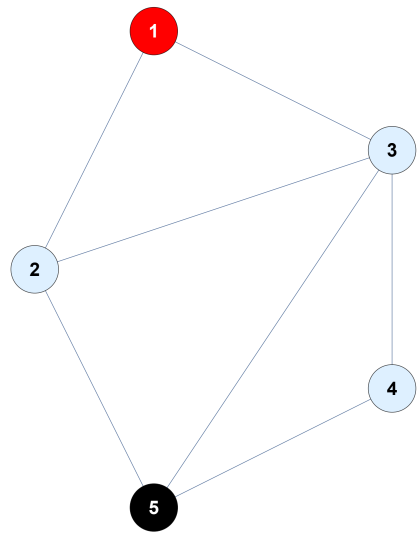
Problem 1. Consider the graph below as an electrical circuit where each edge has resistance 1 and a current of size 1 enters the graph at the source at the top and exits the graph at the sink at the bottom.



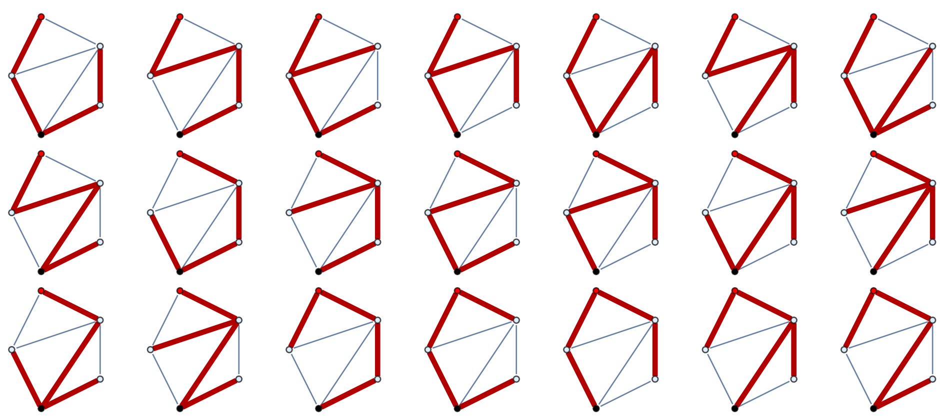
(a) Compute the number of spanning trees. Do it several ways: divide the product of the nonzero eigenvalues of the Laplacian by the number of vertices, compute the determinant of the reduced Laplacian, draw pictures of all of them.

Answer. See the mathematica notebook FinalAnswer1.nb for details. The answer is 21. I labelled the vertices as indicated below, and obtained the laplacian as indicated below:



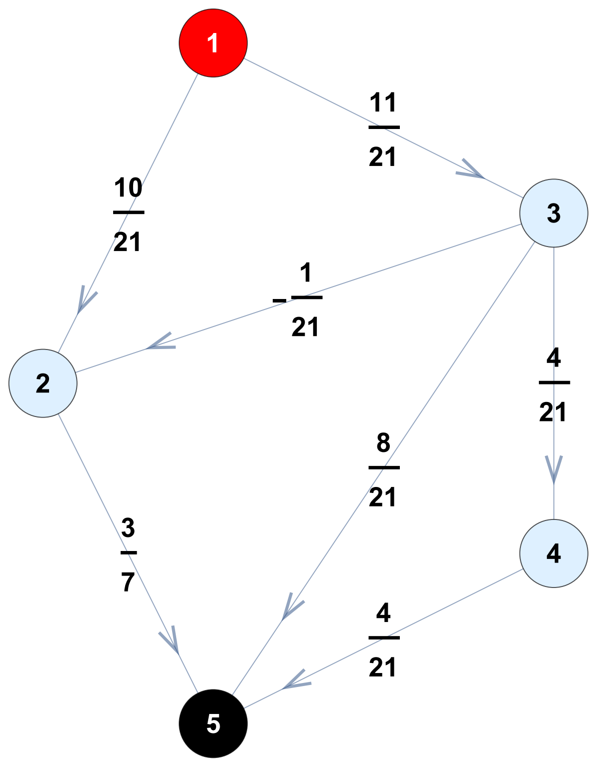
$$\begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & -1 & 3 \end{pmatrix}$$

Here's a picture of all 21 spanning trees:



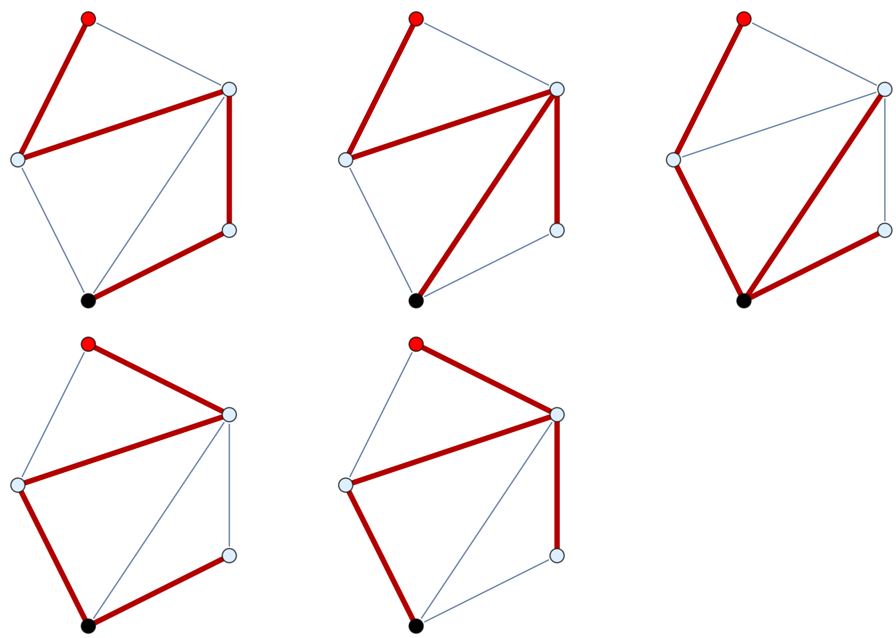
(b) Determine the current flowing in each edge by solving the equations in Kirchoff's laws.

Answer. Again, see the notebook for the computation. The result is:

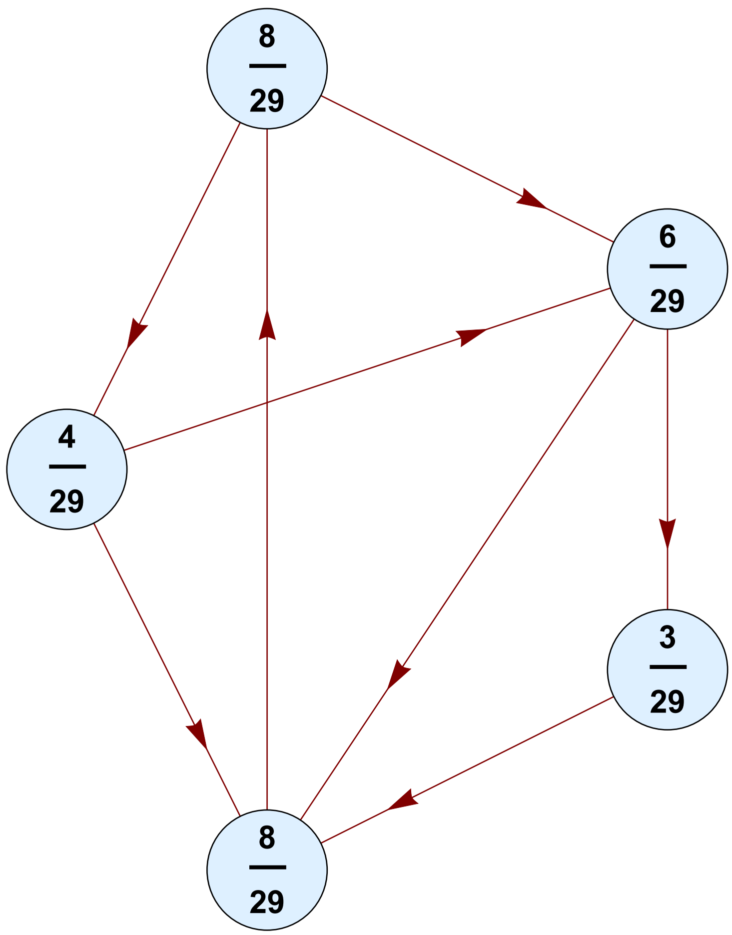


(c) Determine the current flowing in the dotted edge by Theorem 1 in Section II.1 in the book. Your answer here will be a few pictures.

Answer. Here are the relevant spanning trees. Using the notation from Theorem 1 in the book, $N(1,5,2,3) = 3$ and $N(1,5,3,2) = 2$ so the current on the $2-3$ edge is $\frac{3-2}{21}$



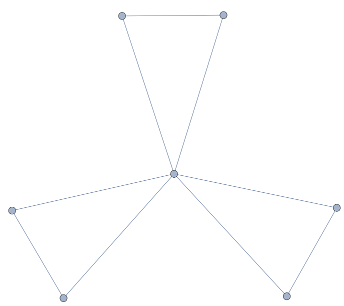
(d) Compute the page ranks for the directed graph obtained by connecting the sink to the source and considering the edges to be directed according to the flow of the current. Again, see the notebook for the computation. The result is:



Problem 2. For each of the following statements, decide whether it is true or false. If the statement is true give a brief proof. If it's false, give a counterexample.

(a) If 3 is an eigenvalue of the adjacency matrix of a graph, then the graph has a vertex of degree 3.

Answer. False. Consider the following graph and its adjacency matrix:



$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

This graph has no vertex of degree three and the vector $\begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue 3.

(b) The adjacency matrix of a bipartite graph with an odd number of vertices is singular.

Answer. True. If λ is an eigenvalue with multiplicity k of the adjacency matrix of a bipartite graph then $-\lambda$ is an eigenvalue of multiplicity k as well. So the specturm is symmetric around the origin. If it has odd length, then it must include 0 and therefore be singular.

(c) Let G be a graph. For any $k = 1, \ldots, \text{size}(G)$, let L_k be the Laplacian of G with the k-the row and k-th column removed. Then L_k is invertible if and only if G is connected.

Answer. Since $\det(L_k)$ is the number of spanning trees, if G is connected then $\det(L_k) > 1$ so L_k is invertible. Conversely, if G is not connected, then it has no spanning trees so $\det(L_k) = 0$ and so L_k is not invertible.

(d) Let $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$. Suppose one entry of v is $+1$, one entry is -1, and all the rest are zero. Then v is an eigenvector with eigenvalue n of the Laplacian of the complete graph K_n.

Answer. This is true. What's more interesting is to observe that the set

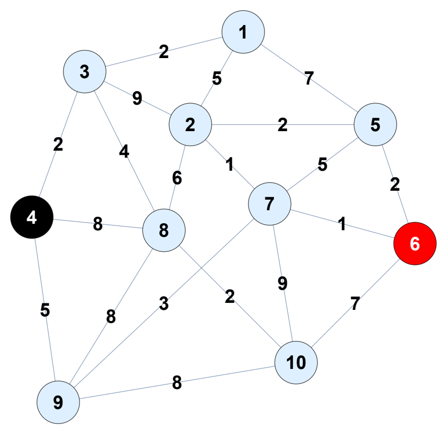
$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ -1 \end{pmatrix}$$

is a linearly independent set of $n-1$ such vectors. Since 0 is always an eigenvalue of the Laplacian (corresponding to the vector consisting of all 1's), we know all the eigenvalues of the Laplacian matrix of the complete graph K_n. Namely,

$$0, n, n, \ldots, n.$$

Therefore, by the spanning tree matrix theorem, the number of spanning trees of K_n is $\frac{1}{n}(n \cdot n \cdots n) = n^{n-2}$. This proves Cayley's amazing formula for the number of labelled trees on n vertices is n^{n-2}.

Problem 3. [Bonus] Consider the electrical network pictured below. The weights on the edges are conductances.



Use absolute voltage potentials $v_6 = 1$ and $v_4 = 0$ (red is the source, black is the sink) to solve for the absolute voltage potentials at every other vertex. Determine the total current of the network.

You'll probably want to use a computer to solve the problem. Here are some options, depending on your programming skills / appetite:

- Set up the equations by hand and ask a computer algebra system to solve the system.
- Modify the class notes from May 12 to include variable conductance and solve the matrix equation corresponding to Kirchoff's current law.
- Write a general program that inputs the adjacency matrix, weighted by conductance, of an electrical network and outputs the absolute voltage potentials and total current.
- Program a computer to approximate the absolute voltage potentials and total current using random walks weighted by conductance as described in Theorem 8 in the book. For this, you can also modify the class notes from May 12 to include variable conductance.

Answer. Here's my solution. The currents are of course rational numbers, but I approximated them to three decimal places to make a more readable picture. See my Mathematica Notebook for the precise currents and the code that produced them.