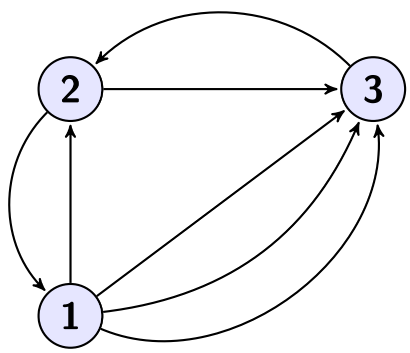
# A matrix-tree theorem for directed multigraphs

In class, we stated and proved the matrix-tree theorem for ordinary graphs. There's a version for *directed multigraphs* as well. First, you need to know what the Laplacian of a directed multigraph is.

**Definition.** The *Laplacian* of a directed multigraph $G$ with vertices $\{v_1, \ldots, v_n\}$ is defined to be the $n \times n$ matrix whose $(i, j)$ entry $l_{ij}$ is defined by

$$l_{ij} = \begin{cases} \text{the outdegree } d^+(v_i) \text{ of } v_i & \text{if } i = j, \\ -s & \text{if } i \neq j \text{ and there are } s \text{ edges from } v_i \text{ to } v_j. \end{cases}$$

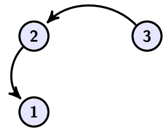
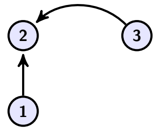
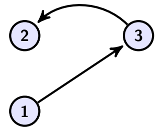
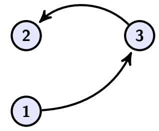
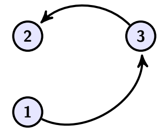
**Example.** Here's a picture of a directed multigraph and its Laplacian.




$$\begin{pmatrix} 4 & -1 & -3 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

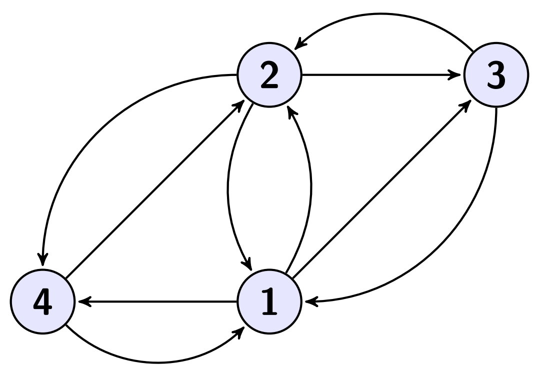
And now here is a directed multigraph version of the matrix tree theorem:

**Directed Multigraph Matrix Tree Theorem.** *Let $G$ be a directed multigraph, let $L$ be its Laplacian matrix, and let $L_i$ be the Laplacian matrix with the $i$-th row and $i$-th column removed. Then the number of spanning trees oriented towards the vertex $i$ is equal to* $\det(L_i)$.

**Example.** In the example above,  is the only spanning tree oriented towards the first vertex and $\det(L_1) = \det \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = 1$. Also, there are four spanning trees oriented towards the second vertex:     and $\det(L_2) = \det \begin{pmatrix} 4 & -3 \\ 0 & 1 \end{pmatrix} = 4$.

# Your Problems

**Problem 1.** **[10 points]** Let $G$ be the directed graph pictured below.




Use the BEST Theorem and the directed-multigraph version of the matrix-tree theorem to count the Euler circuits in $G$. *Hint*: there are more than five and less than one hundred.

**Problem 2.** Planar graphs.

(a) **[5 points]** Prove that $K_{3,3}$ is nonplanar.

(b) **[5 points]** Is the graph pictured below is planar? If yes, prove it with a picture. If no, identify a $K_5$ or $K_{3,3}$ minor.