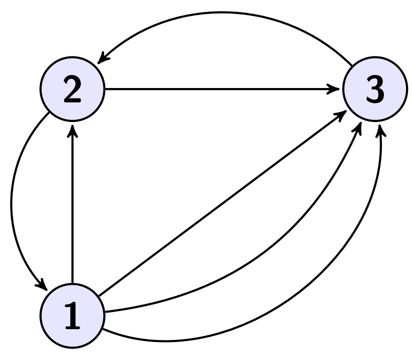
A matrix-tree theorem for directed multigraphs

In class, we stated and proved the matrix-tree theorem for ordinary graphs. There's a version for *directed multigraphs* as well. First, you need to know what the Laplacian of a directed multigraph is.

Definition. The *Laplacian* of a directed multigraph G with vertices $\{v_1, \ldots, v_n\}$ is defined to be the $n \times n$ matrix whose (i, j) entry l_{ij} is defined by

$$l_{ij} = \begin{cases} \text{the outdegree } d^+(v_i) \text{ of } v_i & \text{if } i = j, \\ -s & \text{if } i \neq j \text{ and there are } s \text{ edges from } v_i \text{ to } v_j. \end{cases}$$

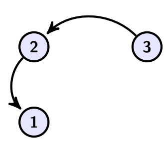
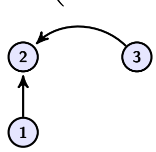
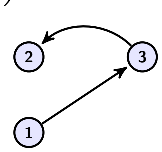
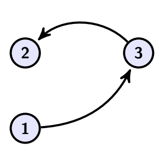
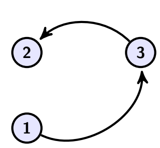
Example. Here's a picture of a directed multigraph and its Laplacian.



$$\begin{pmatrix} 4 & -1 & -3 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

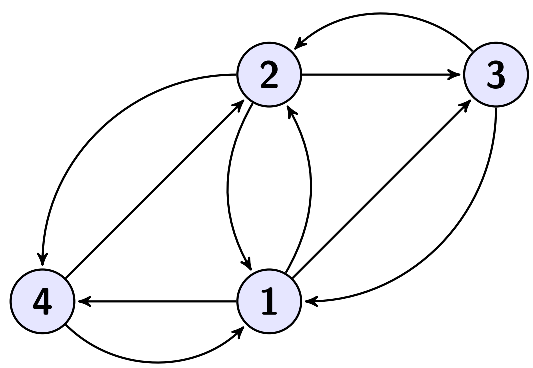
And now here is a directed multigraph version of the matrix tree theorem:

Directed Multigraph Matrix Tree Theorem. *Let G be a directed multigraph, let L be its Laplacian matrix, and let L_i be the Laplacian matrix with the i-th row and i-th column removed. Then the number of spanning trees oriented towards the vertex i is equal to* $\det(L_i)$.

Example. In the example above,  is the only spanning tree oriented towards the first vertex and $\det(L_1) = \det \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = 1$. Also, there are four spanning trees oriented towards the second vertex:     and $\det(L_2) = \det \begin{pmatrix} 4 & -3 \\ 0 & 1 \end{pmatrix} = 4$.

Your Problems

Problem 1. [10 points] Let G be the directed graph pictured below.



Use the BEST Theorem and the directed-multigraph version of the matrix-tree theorem to count the Euler circuits in G. *Hint*: there are more than five and less than one hundred.

Answer. It's worth noticing that the outdegree equals the indegree of every vertex, so the graph has Euler circuits. The BEST theorem states that the number c of Euler circuits in G is given by

$$c = t_i \prod_{j=1}^{n} (d^+(v_j) - 1)!$$

where t_i is the number of spanning trees oriented towards v_i (the number is the same for any i). By the multigraph tree-matrix theorem $t_i = \det(L_i)$ of the Laplacian reduced by removing row i and column i. For the given graph, we have the laplacian and the reduced laplacian

$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix} \text{ and } L_1 = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

A quick computation shows $\det(L_1) = 8$, so there are $t_1 = 8$ spanning trees oriented towards the first vertex. Therefore, the graph has

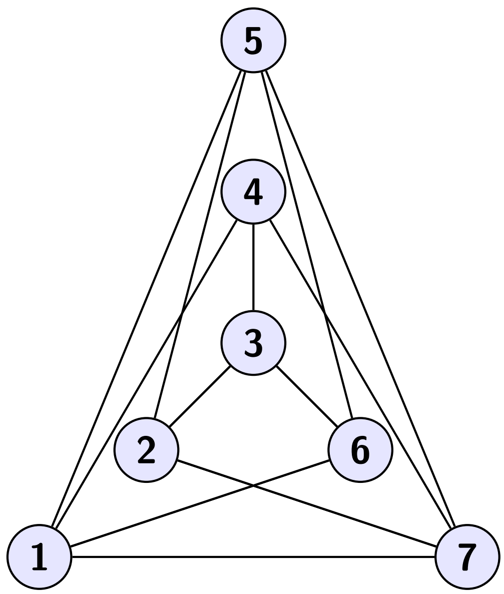
$$c = 8 \prod_{j=1}^{4} (d^+(v_j) - 1)! = 8(2)!(2)!(1)!(1)! = 32$$

Euler circuits.

Problem 2. Planar graphs.

(a) **[5 points]** Prove that $K_{3,3}$ is nonplanar. $K_{3,3}$ has $e = 9$ edges and $v = 6$ vertices. In a planar embedding of any graph with f faces, we have $v - e + f = 2$. For $K_{3,3}$, this means a planar embedding would have $f = 5$ faces. Supposing we have a planar embedding, let $f_1, \ldots, f_5$ be the number of edges in the cycles that bound each face. Since each edge belongs to two faces, we have $2e = \sum_{i=1}^{5} f_i$. Since in $K_{3,3}$, the smallest cycle has length four, and $e = 9$, we have $18 = f_1 + f_2 + f_3 + f_4 + f_5 \leq 20$ a contradiction.

(b) **[5 points]** Is the graph pictured below is planar? If yes, prove it with a picture. If no, identify a K_5 or $K_{3,3}$ minor.



Answer. This graph is planar. Here's a picture.