Problem 1: Prove that the quotient topology is characterized by its universal property

Definition 1. Let $X$ be a topological space, let $S$ be a set, and let $\pi : X \rightarrow S$ be surjective. The quotient topology on $S$ is defined to be the finest topology for which $\pi$ is continuous. Equivalently, a set $U$ in $S$ is open in the quotient topology if and only if $\pi^{-1}(U)$ is open in $X$.

Prove that the quotient topology on $S$ is characterized by the universal property stated below. That is, prove (1) that the quotient topology has the universal property and (2) that the quotient topology is the only topology on $S$ that has the universal property.

Universal property for the quotient topology. For every topological space $Z$ and every function $f : S \rightarrow Z$, $f$ is continuous if and only if $f \circ \pi : X \rightarrow Z$ is continuous.

Here is the picture:

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X  \pi
  ↓
S  ↓ f
   ↓
Z
  \rightarrow f \circ \pi
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Problem 2: Give an example, or prove that no such example exists

(a) A continuous surjection $(0, 1) \to [0, 1]$
(b) A continuous surjection $[0, 1] \to (0, 1)$
(c) A path $p : [0, 1] \to X$ connecting $a$ to $b$ in the space $(X, \tau)$ where
   \[ X = \{a, b, c, d\} \text{ and } \tau = \{\emptyset, \{c\}, \{a, c\}, \{b, c, d\}, X\}. \]


Here are the first two pages of an article published in *The American Mathematical Monthly*, Vol. 74, No. 3 (Mar., 1967), pp. 261-266. Parts have been redacted.

Your problem: Prove Theorem 1 and Theorem 2.

**BETWEEN $T_1$ AND $T_2$**

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1. There has been an outbreak recently of discussion in the MONTHLY on the subject of the Hausdorff separation axiom $T_2$. Specifically, it has been noted that various consequences of $T_2$ are or are not sufficient to imply it. There is a considerable amount of duplication in these results. See [3], [5], [6], [8], [10], [12], [13], [14], [15]. The object of this article is to introduce some systemization into the discussion, to point out its importance, and to show some surprising contact (Theorem 5) with a concept, $k$ space, which has been introduced independently and for quite other purposes.

   One possible importance of separation axioms weaker than $T_2$ lies in the fact that if $X$ is a topological space and $X^+$ its one point compactification ($X^+ = X \cup \{\infty\}$, neighborhoods of $\infty$ are complements of compact closed subsets of $X$, see [7] Chapter 5, Theorem 21) then $X^+$ is not $T_2$ unless $X$ is itself $T_2$ and locally compact. But one may ask what degree of separation $X^+$ has if $X$ is $T_2$
and has some other pleasant property. An example of a result of this type is the contact with \( k \) space mentioned above. Another instance occurs in [1], Theorems 5.1, 5.3.

2. We now introduce two separation axioms, making no claim of novelty. (In [8], the name CC is applied to what we here call a compact KC space. In [1], \( J' \) is used for KC.)

**Definition.** A topological space is called a KC space if every compact set is closed, and a US space if every convergent sequence has exactly one limit to which it converges.

In Theorem 3.1 of [1], equivalences are given between KC and separation properties of compact sets.

**Theorem 1.** \( T_2 \Rightarrow \text{KC} \Rightarrow \text{US} \Rightarrow T_1 \); but no converse implication holds even if the space is compact.

For the next result see [3], [5], [6].

**Theorem 2.** For first countable spaces, \( T_2 \Rightarrow \text{KC} \Rightarrow \text{US} \).

**Definition.** A topological space \( X \) is called locally compact if each neighborhood of each point includes a compact neighborhood of that point.