Sequences and nets: separation, closure, and continuity

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1 Three theorems about sequences

Sequences are useful tools that can be used to detect certain properties of spaces, subsets of spaces, and functions between spaces. For example, the following theorems about sequences were covered in previous notes:

Theorem 1. The limits of convergent sequences in a Hausdorff space are unique.

Theorem 2. If \( \{x_n\} \) is a sequence in \( A \) that converges to \( x \), then \( x \in \overline{A} \).

Theorem 3. If \( f : X \to Y \) is continuous then for all sequences \( \{x_n\} \to x \) in \( X \), the sequence \( \{f(x_n)\} \to f(x) \).

2 Three theorems about first countable spaces

In first countable spaces, such as metric spaces, sequences often suffice to characterize properties of spaces, subsets, and functions. For example, here are some theorems stated in proved in previous notes:

Theorem 4. Let \( X \) be a first countable space. Then \( X \) is Hausdorff if and only if the limits of convergent sequences are unique.
**Theorem 5.** Let $X$ be a first countable space and let $A \subset X$. A point $x \in \overline{A}$ if and only if there exists a sequence $\{x_n\}$ in $A$ with $\{x_n\} \to x$.

**Theorem 6.** Suppose $X$ and $Y$ are first countable and $f : X \to Y$ is a function. The function $f : X \to Y$ is continuous if and only if for every sequence $\{x_n\}$ in $X$ with $\{x_n\} \to x$, the sequence $\{f(x_n)\} \to f(x)$.

### 3 Three problems involving non-first countable spaces

For topological spaces that are not first countable, sequences don’t suffice to characterize topological properties.

**Problem 1.** Prove that $\mathbb{R}$ with the co-countable topology (sets with countable complement are open) is a non-Hausdorff space in which convergent sequences have unique limits.

For the next two problems, use the following example [4].

**Example.** Let $X = [0, 1]^{[0, 1]} := \{\text{functions } f : [0, 1] \to [0, 1]\}$ with the product topology and let $A$ be the subset of $X$ consisting of functions whose graphs are “sawtooths” with vertices on the $x$ axis at a finite number of points $\{0, r_1, \ldots, r_n, 1\}$ and spikes of height 1:

**Problem 2.** Show that the zero function $f$ defined by $f(x) = 0$ for all $x \in [0, 1]$ is a limit point of $A$ for which there is no sequence $\{f_n\}$ in $A$ converging to it.

**Problem 3.** Let $Y$ be the subspace of $X$ consisting of integrable functions. Show that the function $I : Y \to \mathbb{R}$ defined by $I(f) = \int_0^1 f$ satisfies $\{I(f_n)\} \to I(f)$ whenever $\{f_n\} \to f$ but that $I$ is not continuous.
4 Nets and three theorems about them

Because sequences fail to characterize closure and continuity in arbitrary topological spaces, we introduce the concept of a net.

Definition 1. A directed set is a pair \((S, \leq)\) where \(S\) is a set and \(\leq\) is a relation on \(S\) satisfying:

- for all \(s \in S\), \(s \leq s\)
- for all \(s, t, u \in S\), \(s \leq t\) and \(t \leq u\) imply \(s \leq u\)
- for all \(s, t \in S\), there exists a \(u \in S\) with \(s \leq u\) and \(t \leq u\).

So a directed set is a set equipped with a reflexive, transitive, directed relation. Note that we are not assuming that the relation define a directed poset—we don’t assume that the relation be anti-symmetric (see Example 3) and this is a difference between the definition here and others (say the definition in Munkres [3]).

Example 1. The pair \((\mathbb{N}, \leq)\) where \(\leq\) means “less than or equal to” defines a directed set.

Example 2. Fix an interval \([a, b]\) in \(\mathbb{R}\) with \(a < b\). Then \((\mathcal{P}, \leq)\) defines a directed set where \(\mathcal{P}\) is the set of partitions of \([a, b]\) and \(\leq\) is defined by \(P \leq Q\) iff \(Q\) is a refinement of \(P\). One might say that the partitions of an interval can be directed by refinement.

Example 3. Let \(X\) be a metric space and fix a point \(x \in X\). Then we can define a directed set by \((X, \leq)\) where \(y \leq z\) iff \(d(y, x) \geq d(z, x)\). Observe that this relation is not in general anti-symmetric. One might say that the points of \(X\) are directed toward \(x\).

Example 4. Let \(X\) be a topological space and let \(x \in X\). We define a directed set \((U, \leq)\) where \(U\) is the set of neighborhoods of \(x\) and for two neighborhoods \(U\) and \(U'\), we say \(U \leq U'\) iff \(U' \subset U\). One might say that the neighborhoods of a point form a directed set under reverse inclusion.

Problem 4. Show that the directed sets defined in the examples above are in fact directed sets.

Definition 2. A net in a set \(X\) is a function \(x : S \rightarrow X\) where \(S\) is a directed set. We usually write \(x_s\) for \(x(s)\) and may denote the net by \(\{x_s\}\). If \(X\) is a topological space and \(z \in X\), we say that a net \(\{x_s\}\) converges to \(z\) and write \(\{x_s\} \rightarrow z\) if and only if for all open sets \(U\) with \(z \in U\) there exists a \(t \in S\) so that for all \(t \leq s, x_s \in U\).

Example 5. If \(\{x_n\}\) is a sequence and as a sequence \(\{x_n\} \rightarrow x\), then \(\{x_n\} \rightarrow x\) as a net.
Example 6. Let \( f : [a, b] \to \mathbb{R} \) be a bounded function and let \( \mathcal{P} \) be the partitions of \([a, b]\) made into a directed set by refinement. We define two nets \( U_P \) and \( L_P \) (for right hand endpoint and left hand endpoint) in \( \mathbb{R} \) as follows: For any partition \( P = \{a = x_0 < x_1 < \ldots < x_n = b\} \) define
\[
U_P = \sum_{i=1}^{n} \left( \sup_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1}) \quad \text{and} \quad L_P = \sum_{i=1}^{n} \left( \inf_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1}).
\]
Then the nets \( U_P \) and \( L_P \) converge to the upper and lower integral of \( f \) respectively. The function \( f \) is Riemann integrable if and only if the nets \( \{U_P\} \) and \( \{L_P\} \) converges to the same value and that value is the called integral \( \int_{a}^{b} f \).

Example 6 was one of the primary motivations for the definition [1, 2] of a net and convergence of nets is sometimes referred to as “Moore-Smith” convergence.

Example 7. Let \( X \) be a topological space and let \( x \in X \). For every neighborhood \( U \) of \( x \), choose a point \( x_U \in U \). Then \( \{x_U\} \) defines a net in \( X \) and \( \{x_U\} \to x \), where the directed set is the one in Example 4.

This last example is the key observation behind the following theorems. I’ll prove one, and leave the others as exercises.

Theorem 7. A space is Hausdorff if and only if limits of convergent nets is unique.

Proof. Exercise.

Theorem 8. A point \( x \in \overline{A} \) if and only if there exists a net \( \{x_s\} \) in \( A \) with \( \{x_s\} \to x \).

Proof. Suppose that \( \{x_s\} \) is a net in \( A \) with \( \{x_s\} \to x \). Then every open set around \( x \) contains \( x_s \) for some \( s \) and hence contains points of \( A \), proving that \( x \in \overline{A} \).

Conversely, if \( x \in \overline{A} \), every neighborhood \( U \) of \( x \) contains a point, call it \( x_U \), of \( A \). This defines a net \( \{x_U\} \) in \( A \) converging to \( x \).

Theorem 9. A function \( f : X \to Y \) is continuous if and only if for every net \( \{x_s\} \) in \( X \) with \( \{x_s\} \to x \), the net \( \{f(x_s)\} \to f(x) \).

Proof. Exercise.

References