Problem set 2: due October 29

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Problem 1. Evaluation maps: give examples and justify your answers.

(a) Find a space \( X \) and a space \( Y \) for which the evaluation map \( \text{hom}(X, Y) \times X \rightarrow Y \) is not continuous.

(b) Find a space \( X \) and a space \( Y \) for which the evaluation map \( Y^X \times X \rightarrow Y \) is not continuous.

Problem 2. Let \( X \) be a space and let \( I = [0, 1] \) be the interval. Here’s another sort of evaluation map

\[
\text{Eval} : X \rightarrow \mathcal{F}_{\text{hom}(X, I)}
\]

\[
x \mapsto (f \mapsto f(x))
\]

Prove that if \( X \) is completely regular then the map \( X \rightarrow \overline{\text{Eval}(X)} \) is a compactification. This is called the Stone-Čech compactification of \( X \).

*Hint:* Just do this one step at a time. Check that \( \text{Eval} \) is continuous. Check that it is an injection. Check that it is an open mapping, hence an embedding. Check that it’s image has compact closure.

Problem 3. Let \( X \) be a space and consider \( X^* = X \sqcup \{ \infty \} \). Define a collection of sets in \( X^* \) to consist of the open subsets of \( X \) and the complements \( X^* \setminus K \) of compact sets \( K \subset X \).

(a) Identify a convenient condition on \( X \) that makes the collection of sets described above a topology on \( X^* \).

(b) Identify a further convenient condition on \( X \) that makes \( X^* \) compact.

Problem 4. Let \( X \) be a compact space and let \( \{ f_n \} \) be an increasing sequence in \( \text{hom}(X, \mathbb{R}) \). Prove that if \( \{ f_n \} \) converges pointwise then \( \{ f_n \} \) converges uniformly.

References