Problem set 4: Four problems to turn in.

Solutions

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From Fall 2005, Part II

Problem 1. Construct a space with fundamental group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

Answer. One can construct a space $X$ with fundamental group $\mathbb{Z}/3\mathbb{Z}$ as a CW complex by attaching a one cell to a zero cell (to obtain a circle) and then attach a two cell to the circle by the attaching map $z \mapsto z^3$. Another way to see this space is as a quotient of a three-gon by identifying the edges attaching a disc to by identifying the edges of a triangle as pictured below.

Then $X \times X$ has fundamental group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

From Spring 2013, Part II

Problem 2. Consider $\mathbb{R}^3$ with both the $z$-axis removed and the unit circle in the $xy$-plane removed. Calculate the fundamental group of this space.

Answer. There are several ways to do this. Here’s one way that might not be the first way you’d think of, but contains some good ideas that you might find useful elsewhere. Recall that

$$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \sum_{i=1}^{4} x_i^2 = 1\}$$

and note that $S^3 \setminus \{(0, 0, 0, 1)\} \cong \mathbb{R}^3$, either by stereographic projection from $(0, 0, 0, 1)$ or by thinking of $S^3$ as the one point compactification of $\mathbb{R}^3$. Then, $S^3 \setminus C_1 \cong \mathbb{R}^3 \setminus \{z\text{-axis}\}$ where $C_1$ is the circle $C_1 = \{(x_1, x_2, x_3, x_4) \in S^3 : x_1 = x_2 = 0\}$. To obtain the space $X$ in question, remove the circle $C_2$ in the $xy$-plane which yields $X \cong S^3 \setminus \{C_1 \cup C_2\}$. 
It is a fact that $S^3 \setminus \{C_1 \cup C_2\}$ is homotopy equivalent to a torus, which has fundamental group $\mathbb{Z} \times \mathbb{Z}$. One way to see this explicitly is to use a decomposition of $S^3$ as the union of two solid tori, whose intersection is the torus that is the common boundary of the two solid tori. The circles $C_1$ and $C_2$ are the centers of the two solid tori, hence each solid torus (without its center circle) deformation retracts onto its boundary, defining a deformation retract $S^3 \setminus \{C_1 \cup C_2\} \to T$. See the appendix to these solutions for details about the decomposition of $S^3$ into two solid tori.

**From Spring 2014, Part II**

**Problem 3.** One way to represent a topological surface is as a polygon with pairs of sides identified in some fashion. Describe two topologically different surfaces that can be made in this way from a hexagon and prove that these are topologically distinct.

**Answer.** Consider the two figures below. The figure on the left defines a sphere and the one on the right defines a torus. They are topologically distinct because the fundamental groups, the trivial group and $\mathbb{Z} \times \mathbb{Z}$, are not isomorphic.

![Hexagon Figures](image)

**Problem 4.** Let $\gamma$ be a closed curve on the torus as pictured. Let $S$ be the topological space formed by gluing a Möbius strip to the torus by identifying $\gamma$ with the boundary of the Möbius strip. Compute the fundamental group of $S$.

**Answer.** We will use the Seifert-van Kampen theorem. First, let’s setup some notation. Let $T$ denote the torus and $M$ denote the Möbius band. The space in question is the quotient $X = (T \cup M) / \sim$ where the boundary circle of $B \subset M$ is identified with $\gamma \subset T$ by a homeomorphism.

Let $A_1 = T \cup N_1$ and $A_2 = M \cup N_2$ where $N_1$ is a small open neighborhood of the boundary circle $B$ in $M \subset X$ and $N_2$ is a small neighborhood of $\gamma$ in $T \subset X$. 

Then $A_1, A_2$, and $A_1 \cap A_2 = N$ are open sets with $A_1 \cup A_2 \simeq X$. Choose a basepoint $x_0 \in \gamma$. Then, SvK theorem implies the following diagram is a pushout of groups:

$$
\begin{array}{ccc}
\pi_1(A_1 \cap A_2, x_0) & \xrightarrow{(i_1)_*} & \pi_1(A_1, x_0) \\
& \downarrow^{(i_2)_*} & \downarrow^{(j_1)_*} \\
\pi_1(A_2, x_0) & \xrightarrow{(j_2)_*} & \pi_1(X, x_0)
\end{array}
$$

To get a more explicit description we express the groups in question using generators and relations and then express $\pi_1(X, x_0)$ as the quotient of the free product

$$\pi_1(X, x_0) \simeq \pi_1(A_1, x_0) \ast \pi_1(A_2, x_0)/H$$

where $H$ is the normal subgroup generated by the relations identifying $(i_1)_*(h) = (i_2)_*(h)$ for all $h \in \pi_1(A_1 \cap A_2, x_0)$. Note that

$$\pi_1(A_1 \cap A_2) = F(\alpha), \quad \pi_1(A_1) = F(\beta, \gamma)/\langle \beta \gamma = \gamma \beta \rangle, \quad \pi_1(A_2) = F(\delta)$$

In these descriptions, $\alpha$ is the generator for the fundamental group $A_1 \cap A_2 \simeq S^1$, $\beta$ is the class of the equator of the torus, $\gamma$ is the class of the meridian $\gamma$, and $\delta$ is the generator for the fundamental group of $A_2 \simeq M \simeq S^1$. Then

$$\pi_1(A_1, x_0) \ast \pi_1(A_2, x_0) = F(\beta, \gamma, \delta)/\langle \beta \gamma = \gamma \beta \rangle.$$

Notice that $(i_1)_*(\alpha) = \gamma$ and $(i_2)_*(\alpha) = \delta$, making the description of the normal subgroup $H$ simply $\langle \gamma = \delta^2 \rangle$. Thus, the result is

$$\pi_1(X, x_0) = F(\beta, \gamma, \delta)/\langle \beta \gamma = \gamma \beta, \gamma = \delta^2 \rangle = F(\beta, \delta)/\langle \beta \delta^2 = \delta^2 \beta \rangle.$$

**Appendix: Decomposition of $S^3$ into two solid tori**

It will be convenient to consider $S^3$ as the sphere of radius $\sqrt{2}$ instead of radius 1:

$$S^3 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 2.\}$$

Define subsets $A$ and $B$ by

$$A = \{x \in S^3 : x_1^2 + x_2^2 \leq 1\} = \{x \in S^3 : x_3^2 + x_4^2 \geq 1\}$$

$$B = \{x \in S^3 : x_1^2 + x_2^2 \geq 1\} = \{x \in S^3 : x_3^2 + x_4^2 \leq 1\}$$

Since $x_1^2 + x_2^2 \leq 1$ or $x_1^2 + x_2^2 \geq 1$ the union $A \cup B = S^3$. The intersection

$$A \cap B = \{x \in S^3 : x_1^2 + x_2^2 = 1 \text{ and } x_3^2 + x_4^2 = 1\} = S^1 \times S^1.$$
is a torus. To see that $A$ and $B$ are solid tori, consider the maps
\[(x_1, x_2, x_3, x_4) \mapsto \left( (x_1, x_2), \frac{1}{\sqrt{x_3^2 + x_4^2}} (x_3, x_4) \right)\]
and
\[(x_1, x_2, x_3, x_4) \mapsto \left( (x_3, x_4), \frac{1}{\sqrt{x_1^2 + x_2^2}} (x_1, x_2) \right)\]
which define homeomorphisms $A \xrightarrow{\cong} D^2 \times S^1$ and $B \xrightarrow{\cong} D^2 \times S^1$. 