Some local (at p) properties of residual Galois representations

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1 Preliminary results

In this talk we are going to discuss some local properties of (mod p) Galois representations at the prime p.

Let p be a prime number, N a positive integer prime to p, k an integer and $\epsilon : (\mathbf{Z}/N\mathbf{Z})^\times \to \overline{\mathbf{F}}_p^\times$ a character. Let f be a cusp form of type (N, k, ϵ). We also assume that f is an eigenform for all the Hecke operators T_l (any prime l) with eigenvalue $a_l \in \overline{\mathbf{F}}_p$. For the definition of the Hecke algebra we refer to Kirsten's talk. We will denote by $G_{\mathbf{Q}}$ the absolute Galois group of $\mathbf{Q}$, and by χ the mod p cyclotomic character. We work with arithmetic Frobenius elements.

Theorem 1.1 (Deligne). *There exists a unique (up to isomorphism) continuous semi-simple representation $\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}(V)$, where V is a two-dimensional $\overline{\mathbf{F}}_p$-vector space, such that for all primes $l \nmid Np$,*

- *ρ_f is unramified at l,*
- $\operatorname{tr} \rho_f(\mathrm{Frob}_l) = a_l$
- $\det \rho_f(\mathrm{Frob}_l) = \epsilon(l) l^{k-1}$.

Remark 1.2. By choosing a basis of V we can treat ρ_f as having image in $\mathrm{GL}_2(\overline{\mathbf{F}}_p)$. It then follows from continuity of ρ_f that there exists a finite extension κ of $\mathbf{F}_p$ such that ρ_f factors through some map $G_{\mathbf{Q}} \to \mathrm{GL}_2(\kappa) \hookrightarrow \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ taht we may also denote ρ_f.

We fix an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. This determines a prime $\mathfrak{p}$ of the integral closure $\overline{\mathbf{Z}}$ of $\mathbf{Z}$ in $\overline{\mathbf{Q}}$ over p. Let $G_p \subset G_{\mathbf{Q}}$ denote the decomposition group of $\mathfrak{p}$. Our goal is to study $\rho_{f,p} := \rho_f|_{G_p}$. We will denote by I_p the inertia group inside G_p, by I_p^w the wild inertia subgroup, and set $I_p^t = I_p/I_p^w$ to be the tame inertia quotient. We will identify G_p with $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and G_p/I_p with $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$. Moreover, for any $a \in \overline{\mathbf{F}}_p^\times$ we denote by λ_a the unramified character $\phi : G_p \to \overline{\mathbf{F}}_p^\times$ such that $\phi(\mathrm{Frob}_p) = a$.

Theorem 1.3 (Deligne). *Let f be a cusp form of type (N, k, ϵ) with $2 \le k \le p+1$. Assume f is a Hecke eigenform with eigenvalues a_l, and that $a_p \neq 0$. Then $\rho_{f,p}$ is reducible and*

$$\rho_{f,p} \cong \begin{bmatrix} \chi^{k-1}\lambda_{\epsilon(p)/a_p} & * \\ & \lambda_{a_p} \end{bmatrix}.$$

Remark 1.4. Theorem 1.3 was proved by Deligne in a letter to Serre using p-adic etale cohomology and mod p "vanishing cycles" calculations on modular curves in characteristic p; in particular his proof uses his construction of p-adic Galois representations associated to eigenforms.

Proposition 1.5. *Let f be a cusp form of type $(N, 1, \epsilon)$. Assume that f is a Hecke eigenform and $T_p f = a_p f$. Suppose that the polynomial*

$$P(X) := X^2 - a_p X + \epsilon(p) \tag{1}$$

has two distinct roots in $\overline{\mathbf{F}}_p$, or that f is the reduction of a characteristic zero eigenform of weight 1 and level N. Then $\rho_{f,p}$ is unramified.

Proof. We omit the proof of the case when f is the reduction of a characteristic zero eigenform (cf. Theorem 4.1 in [DS74]), so assume $P(X)$ has distinct roots in $\overline{\mathbf{F}}_p$. Let $f(q) = \sum b_n q^n$ be the q-expansion of f at some cusp. In his talk Hui discussed the action of the operator V_p on q-expansions of cusp forms. In particular $V_p f$ is a cusp form of type (N, p, ϵ) and $V_p f(q) = \sum b_n q^{np}$. On the other hand, Bryden defined the Hasse invariant A, which is a modular form of type $(1, p-1, 1)$ and has q-expansion 1 at all cusps. Hence Af is a modular form of type (N, p, ϵ) and Af has q-expansion $\sum b_n q^n$. As discussed in Hui's talk, both Af and $V_p f$ are eigenforms for all T_l with $l \neq p$ and their T_l-eigenvalues are the same as those of f. On the other hand, since $T_p f = a_p f$ and $\langle p \rangle f = \epsilon(p) f$, it follows from general identities that

$$T_p(Af) = a_p Af - \epsilon(p) V_p f \tag{2}$$

and

$$T_p(V_p f) = Af. \tag{3}$$

Let W be the two-dimensional space spanned by Af and $V_p f$. It follows from formulas (2) and (3) that $P(X)$ is the characteristic polynomial for the action of T_p on W. Let $\alpha \neq \alpha'$ be the roots of $P(X)$. Let g, g' be the corresponding normalized eigenforms of type (N, p, ϵ). Since f, g and g' have the same eigenvalues for T_l, $l \neq p$, we have $\operatorname{tr} \rho_f = \operatorname{tr} \rho_g = \operatorname{tr} \rho_{g'}$ by the Tchebotarev density theorem. We also have $\det \rho_f = \det \rho_g = \det \rho_{g'} = \epsilon$. Hence the semi-simple representations ρ_f, ρ_g and $\rho_{g'}$ are isomorphic by the Brauer-Nesbitt theorem. Since $\alpha\alpha' = \epsilon(p) \neq 0$, both α and α' are non-zero. Hence we can apply Theorem 1.3 and conclude that we have

$$\rho_{g,p} \cong \begin{bmatrix} \lambda_{\epsilon(p)/\alpha} & * \\ & \lambda_\alpha \end{bmatrix} = \begin{bmatrix} \lambda_{\alpha'} & * \\ & \lambda_\alpha \end{bmatrix} \tag{4}$$

and

$$\rho_{g',p} \cong \begin{bmatrix} \lambda_{\epsilon(p)/\alpha'} & * \\ & \lambda_{\alpha'} \end{bmatrix} = \begin{bmatrix} \lambda_{\alpha} & * \\ & \lambda_{\alpha'} \end{bmatrix}. \tag{5}$$

Since $\alpha \neq \alpha'$, the representation space $\rho_{f,p}$ contains lines with distinct characters λ_α and $\lambda_{\alpha'}$, so $\rho_{f,p} \cong \lambda_\alpha \oplus \lambda_{\alpha'}$ is unramified. □

Theorem 1.6 (Gross). *Let $f = \sum a_n q^n$ be as in Theorem 1.3, and suppose that $a_p^2 \neq \epsilon(p)$ if $k = p$. Then $\rho_{f,p}|_{I_p^w}$ is trivial if and only if there exists a cuspidal eigenform $f' = \sum a'_n q^n$ of type (N, k', ϵ), where $k' = p + 1 - k$, such that*

$$l a'_l = l^{k'} a_l \tag{6}$$

for all l.

Proof. We are only going to prove that the existence of f' implies that $\rho_{f,p}$ is tamely ramified. For the converse (which is the hard part of this theorem), see Theorem 13.10 in [Gro90]. Let f' be as in the statement of the theorem and let a'_l denote the eigenvalue of T_l corresponding to f'. Formula (6) implies that $\operatorname{tr} \rho_{f'}(\mathrm{Frob}_l) = \operatorname{tr} (\rho_f \otimes \chi^{k'-1})(\mathrm{Frob}_l)$ for all $l \nmid Np$, where χ denotes the mod p cyclotomic character. Hence by the Tchebotarev density theorem together with the Brauer-Nesbitt theorem as before, we get

$$\rho_{f'} \cong \rho_f \otimes \chi^{k'-1}. \tag{7}$$

First assume $k \neq p$. We begin by show that $a'_p \neq 0$. Note that for all primes $l \neq p$, we have $a'_l = l^{k'-1} a_l$ by assumption. If $a'_p = 0$, we can also write $a'_p = p^{k'-1} a_p$ as $k' \neq 1$. Hence f' of weight k' and $\theta^{k'-1} f$ of weight pk' have the same q-expansions. Here θ is the operator whose properties were discussed in Kirsten's talk. Hence $\theta^{k'-1} f = A^{k'} f'$, which means that $\theta^{k'-1} f$ has filtration k'. However, it was proved in Kirsten's talk that $\theta^{k'-1} f$ has filtration pk', since f has weight at most $p-1$, which yields a contradiction. Hence $a'_p \neq 0$ if $k \neq p$.

Applying Theorem 1.3 to the representation $\rho_{f'}$ (still assuming $k \neq p$), we see that there is a line $L_{f'}$ in the space V of $\rho_{f'}$ such that G_p acts on $L_{f'}$ via the character $\chi^{k'-1} \lambda_{\epsilon(p)/a'_p}$. On the other hand applying Theorem 1.3 to the representation ρ_f, we get a line $\tilde{L}_f$ in its representation space on which G_p acts via the character $\chi^{k-1} \lambda_{\epsilon(p)/a_p}$ and hence a line $L_f \subset V$ on which G_p acts via the character $\chi^{k'-1}(\chi^{k-1} \lambda_{\epsilon(p)/a_p})$.

In particular, the action of I_p on L_f is via $\chi^{k'-1+k-1} = \chi^{p-1} = 1$ and on $L_{f'}$ via $\chi^{k'-1}$. Since $k \neq p$ by assumption, so $\chi^{k'} \neq 1$, $L_f \neq L_{f'}$ and so we conclude that $V = L_f \oplus L_{f'}$. Thus $\rho_{f',p}$ and hence also $\rho_{f,p}$ is diagonalizable; i.e., $\rho_f(\sigma)$ has order prime to p for every $\sigma \in G_p$. Thus $\rho_f|_{I_p^w}$ is trivial.

Now assume $k = p$. Then $f' = \sum a'_n q^n$ is of weight 1. Note that f is in the span W of $V_p f'$ and Af'. Indeed, by looking at q-expansions, we see that $\theta((a_p - a'_p)V_p f' + Af' - f) = 0$, and so there exists a form $h = \sum c_n q^n$ of weight 1 such that

$$(a_p - a'_p)V_p f' + Af' - f = V_p h,$$

and h is an eigenform for T_l with eigenvalue a_l if $l \neq p$ since V_p commutes with such T_l in weights 1 and p (and V_p is injective). Moreover, the p-th coefficient of the q-expansion of $V_p h$ is zero, so $c_1 = 0$. Since $c_l = a_l c_1 = 0$ for all $l \neq p$, we have $\theta(h) = 0$. As θ is injective on forms of weight prime to p, we get $h = 0$, and so we conclude that $f \in W$, as claimed. As $a_p \neq \epsilon(p)/a_p$ by assumption, and the characteristic polynomial for T_p on W has constant term $\epsilon(p)$ (by matrix calculation) with a_p as a root (since $f \in W$), there is a unique Hecke eigenform $g \in W$ whose eigenvalue for T_p is $b_p := \epsilon(p)/a_p$. Then by applying Theorem 1.3 to ρ_f and $\rho_g \cong \rho_f$, we get a line L_f in the space W of ρ_f on which G_p acts via the unramified character $\lambda_{\epsilon(p)/a_p}$ and a line L_g on which G_p acts via $\lambda_{\epsilon(p)/b_p}$. Since $a_p^2 \neq \epsilon(p)$, we again get $V = L_f \oplus L_g$ and conclude that $\rho_f|_{I_p^w}$ is trivial. Note that in this case $\rho_{f,p}$ is in fact unramified. □

2 A theorem of Mazur

Let p be an odd prime and ℓ a prime with $\ell \not\equiv 1 \bmod p$. (We allow $\ell = p$.) Let f be a normalized cuspidal eigenform of weight 2, level N, and trivial charater. We drop the assumption that $p \nmid N$. Suppose $N = M\ell$ with $\ell \nmid M$. Here is a basic "level lowering" theorem.

Theorem 2.1 (Mazur). *Suppose the Galois representation $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ attached to f is irreducible. If $\ell \neq p$, then assume that $\rho|_{G_\ell}$ is unramified, and if $\ell = p$, then assume that ρ is finite at p. Then ρ is modular of type $(M, 2, 1)$.*

Proof. We will suppress some of the technical details. Let $\mathfrak{X}_0(N)$ over $\mathbf{Z}_\ell$ be the coarse moduli space for the $\Gamma_0(N)$-moduli problem on generalized elliptic curves over $\mathbf{Z}_{(\ell)}$-schemes. Let $J_0(N)$ be the jacobian of $X_0(N)$ over $\mathbf{Q}$. Let $\mathfrak{J}_0(N)$ be the Neron model of $J_0(N)$ over $\mathbf{Z}_{(\ell)}$. It follows from the work of Deligne-Rapoport that $\mathfrak{X}_0(N)_{\mathbf{F}_\ell}$ is the union of two copies of $\mathfrak{X}_0(M)_{\mathbf{F}_\ell}$, which intersect transversally at geometric points corresponding to precisely supersingular elliptic curves. Their theory shows that $\mathfrak{X}_0(M)_{\mathbf{F}_\ell}$ is smooth, as $\ell \nmid M$.

Lemma 2.2. *The normalization map $\mathfrak{X}_0(M)_{\mathbf{F}_\ell} \sqcup \mathfrak{X}_0(M)_{\mathbf{F}_\ell} \to \mathfrak{X}_0(N)_{\mathbf{F}_\ell}$ induces a short exact sequence*

$$0 \to T_0(N)_{\mathbf{F}_\ell} \to \mathfrak{J}_0(N)^0_{\mathbf{F}_\ell} \to \mathfrak{J}_0(M)_{\mathbf{F}_\ell} \times \mathfrak{J}_0(M)_{\mathbf{F}_\ell} \to 0, \tag{8}$$

where $T_0(N)_{\mathbf{F}_\ell}$ is the torus whose character group $X := \mathrm{Hom}_{\overline{\mathbf{F}}_\ell}(T_0(N)_{\overline{\mathbf{F}}_\ell}, (\mathbf{G}_m)_{\overline{\mathbf{F}}_\ell})$ as a $\mathbf{Z}[\mathrm{Gal}(\overline{\mathbf{F}}_\ell/\mathbf{F}_\ell)]$-module is the group of degree zero divisors on $\mathfrak{X}_0(M)_{\overline{\mathbf{F}}_\ell}$ with support in the supersingular points.

Proof. The existence of exact sequence (8) is a general fact about semi-stable curves, cf. Example 8 in section 9.2 of [BLR90]. To any semi-stable curve Y over an algebraically closed field one associates the graph $\Gamma(Y)$ whose vertices are the irreducible components of Y and whose edges are the singular points of Y: an edge corresponding to a point P joins vertices V_1 and V_2 if and only if the irreducible components corresponding to V_1 and V_2 intersect at P. (We get a "loop" at a vertex if it lies on only one irreducible component, corresponding to "formal self-crossing".) The character group X can be canonically identified with $H_1(\Gamma(\mathfrak{X}_0(N)_{\overline{\mathbf{F}}_\ell}), \mathbf{Z})$ as a $\mathbf{Z}[\mathrm{Gal}(\overline{\mathbf{F}}_\ell/\mathbf{F}_\ell)]$-module. Using this identification, one proves that X has the alternative description in terms of the supersingular geometric points. □

There is yet another tautological short exact sequence

$$0 \to \mathfrak{J}_0(N)^0_{\mathbf{F}_\ell} \to \mathfrak{J}_0(N)_{\mathbf{F}_\ell} \to \Phi_0(N)_{\mathbf{F}_\ell} \to 0, \tag{9}$$

where $\Phi_0(N)_{\mathbf{F}_\ell}$ is the finite etale group scheme of components of $\mathfrak{J}_0(N)_{\mathbf{F}_\ell}$.

The Hecke algebra $\mathbf{T}$ of level N acts on $J_0(N)$ and by the Neron mapping property also on $\mathfrak{J}_0(N)$. This action respects the exact sequences (8) and (9) due to functoriality considerations with identity components and maximal toric parts.

Since ρ is absolutely irreducible and modular of type $(N, 2, 1)$, by the Brauer-Nesbitt theorem and the triviality of Brauer groups of finite fields there exists a maximal ideal $\mathfrak{m}$ of $\mathbf{T}$ with an embedding $\kappa := \mathbf{T}/\mathfrak{m} \hookrightarrow \overline{\mathbf{F}}_p$ and a two-dimensional κ-vector space V with $G_{\mathbf{Q}}$-action such that the induced action on $\overline{\mathbf{F}}_p \otimes_\kappa V$ is via ρ and such that $J_0(N)(\overline{\mathbf{Q}})[\mathfrak{m}]$ is isomorphic to the direct sum of a finite number of copies of V as $\kappa[G_{\mathbf{Q}}]$-modules.

First assume that $\ell = p$, as this is the case that will be of interest to us. Let W be the finite etale κ-vector space scheme over $\mathbf{Q}$ such that $W(\overline{\mathbf{Q}}) = V$ as $\kappa[G_{\mathbf{Q}}]$-modules. By assumption ρ is finite at p, so (by oddness of p and Raynaud's theorems when "$e < p - 1$") there exists a finite flat κ-vector space scheme $\mathfrak{W}$ over $\mathbf{Z}_{(\ell)}$ extending W. By the remarks above we can choose an injection of V into $J_0(N)(\overline{\mathbf{Q}})$ which gives rise to a closed immersion of W into $J_0(N)$.

Lemma 2.3. *There exists a closed immersion $\mathfrak{W} \hookrightarrow \mathfrak{J}_0(N)$ prolonging the closed immersion $W \hookrightarrow J_0(N)$.*

Proof. We are going to prove the following more general statement:

Claim 2.4. *Let R denote a complete dvr of mixed characteristic $(0, p)$ with absolute ramification index $e < p - 1$. Denote by K the fraction field of R. Let $\mathfrak{G}$ be a finite flat commutative p-group over R, and A an abelian variety over K with semi-stable reduction. Let $\mathfrak{A}$ be the Neron model (over R) of A (so $\mathfrak{A}$ has closed fiber with semi-abelian identity component, whence $\mathfrak{A}[n]$ is quasi-finite, flat, and separated over R for every nonzero n in $\mathbf{Z}$). Suppose there exists a closed immersion $\iota_K : G \hookrightarrow A$, where $G = \mathfrak{G}_K$. Furthermore, assume that the Galois module $(\mathfrak{A}[p]/\mathfrak{A}[p]_{\mathrm{f}})(\overline{K})$ is unramified, where $\mathfrak{A}[p]_{\mathrm{f}}$ denotes the "finite" part of the quasi-finite flat separated R-group $\mathfrak{A}[p]$. Then ι_K extends to a closed immersion $\iota : \mathfrak{G} \hookrightarrow \mathfrak{A}$.*

We remark here that the hypotheses of semi-stable reduction and unramifiedness are satisfied by $\mathfrak{A} = \mathfrak{J}_0(N)$ over $\mathbf{Z}_{(\ell)}$. This is a consequence of [Gro72] and uses the fact that $\mathfrak{X}_0(N)$ has semi-stable reduction over $\mathbf{Z}_{(\ell)}$ (as $\ell \nmid M$).

Proof of the Claim. We first show that $\iota_K|_{\mathfrak{G}^0_K}$ extends. The composite

$$\mathfrak{G}^0_K \hookrightarrow G \hookrightarrow \mathfrak{A}[p^n]_K \to (\mathfrak{A}[p^n]/\mathfrak{A}[p^n]_{\mathrm{f}})_K$$

is the zero map since, as discussed by Nick in his talk, Raynaud's results on finite flat group schemes imply that a finite flat connected R-group has no nonzero unramified quotients of its K-fiber. Hence we have a commutative diagram

$$\begin{array}{ccc} G & \longrightarrow & \mathfrak{A}[p]_K \\ \uparrow & & \uparrow \\ \mathfrak{G}^0_K & \longrightarrow & (\mathfrak{A}[p]_{\mathrm{f}})_K \end{array}$$

where all the maps are closed immersions. Now, using the fact that $\mathfrak{A}[p]_{\mathrm{f}}$ is finite flat (as opposed to $\mathfrak{A}[p]$, which is generally only quasi-finite and flat), the lower horizontal arrow extends to a closed immersion $\mathfrak{G}^0 \hookrightarrow \mathfrak{A}[p]_{\mathrm{f}}$ by Raynaud's Theorem ([Ray74], Corollary 3.3.6). As $\mathfrak{A}[p]^0_{\mathrm{f}} \subset \mathfrak{A}^0$, we have proved that $\iota_K|_{\mathfrak{G}^0_K}$ extends.

We will now show that ι extends. Since $\mathfrak{G}^0$ is a finite flat commutative group, it makes sense to talk about the quotient $(\mathfrak{G} \times \mathfrak{A}^0)/\mathfrak{G}^0$ with $\mathfrak{G}^0$ embedded by the "twisted diagonal" $x \mapsto (x, -x)$. We have a commutative diagram

$$\begin{array}{ccc} \mathfrak{G} & \longrightarrow & (\mathfrak{G} \times \mathfrak{A}^0)/\mathfrak{G}^0\ . \\ \uparrow & & \uparrow \\ \mathfrak{G}^0 & \longrightarrow & \mathfrak{A}^0 \end{array} \tag{10}$$

Our goal is to show that the closed embedding $\mathfrak{G}^0 \hookrightarrow \mathfrak{A}$ factors through the left vertical arrow in diagram (10). First note that it suffices to show that $(\mathfrak{G} \times \mathfrak{A}^0)/\mathfrak{G}^0$ is smooth. Indeed, the natural addition map $G \times A \to A$ factors through $(G \times A)/(\mathfrak{G}^0)_K \to A$, so if $(\mathfrak{G} \times \mathfrak{A}^0)/\mathfrak{G}^0$ is smooth, then $(G \times A)/(\mathfrak{G}^0)_K \to A$ extends to $\phi : (\mathfrak{G} \times \mathfrak{A}^0)/\mathfrak{G}^0 \to \mathfrak{A}$ by the Neron mapping property. Diagram (10) gets amalgamated to a diagram

$$\begin{array}{ccccc} \mathfrak{G} & \longrightarrow & (\mathfrak{G} \times \mathfrak{A}^0)/\mathfrak{G}^0 & & \\ \uparrow & & \uparrow & \overset{\phi}{\searrow} & \\ \mathfrak{G}^0 & \longrightarrow & \mathfrak{A}^0 & \longrightarrow & \mathfrak{A} \end{array} \quad . \tag{11}$$

To establish commutativity of diagram (11) it is enough (by separatedness and flatness over R) to check commutativity of the right triangle on generic fibers. Since $(\mathfrak{A}^0)_K = A$, the corresponding triangle on generic fibers clearly commutes, so we get the desired result concerning (10).

It remains to prove that $(\mathfrak{G} \times \mathfrak{A}^0)/\mathfrak{G}^0$ is smooth over R. Note that $\mathfrak{G} \times \mathfrak{A}^0$ is of finite type and flat as are $\mathfrak{G}$ and $\mathfrak{A}^0$. As $\mathfrak{G}^0$ is finite and flat over R, the map $\mathfrak{G} \times \mathfrak{A}^0 \to (\mathfrak{G} \times \mathfrak{A}^0)/\mathfrak{G}^0$ is finite, flat and surjective. Hence $(\mathfrak{G} \times \mathfrak{A}^0)/\mathfrak{G}^0$ is of finite type and flat over R. Thus to verify smoothness of $(\mathfrak{G} \times \mathfrak{A}^0)/\mathfrak{G}^0$, it is enough to check smoothness of fibers. As the geometric generic fiber is clearly smooth (it is just a product of copies of A), it remains to check smoothness of the closed fiber. This follows from the following Fact.

Fact 2.5. *Let k be a field, G a finite commutative k-group and H a smooth commutative k-group. Suppose we are given a closed immersion of k groups $G^0 \hookrightarrow H$. Then $(G \times H)/G^0$ is k-smooth, where $G^0 \hookrightarrow G \times H$ is $g^0 \mapsto (g^0, (g^0)^{-1})$.*

To prove Fact 2.5, we may assume that k is algebraically closed. Then $G = G^0 \times G^{\text{et}}$, hence $(G \times H)/G^0 = G^{\text{et}} \times H$, which is smooth. □

Lemma 2.6. *The image of $\mathfrak{W}_{\mathbf{F}_p}$ in $\Phi_0(N)_{\mathbf{F}_p}$ is zero.*

Proof. It is a theorem of Ribet (cf. Theorem 3.12 in [Rib90]) that $\Phi_0(N)_{\mathbf{F}_p}$ is Eisenstein in the sense that for all q prime to p, the operator T_q acts on it via multiplication by $q + 1$. Thus the action of $\mathbf{T}$ on $\Phi_0(N)_{\mathbf{F}_p}$ factors through $\mathbf{T}/I$, where I is the ideal of $\mathbf{T}$ generated by the elements $T_q - q - 1$ for q prime to p. Note that since ρ is irreducible, I is relatively prime to $\mathfrak{m}$. Indeed, if $\mathfrak{m}$ and I were not relatively prime, then $T_q \equiv q + 1 \bmod \mathfrak{m}$ for almost all q. By Tchebotarev Denisty Theorem we have then $\operatorname{tr} \rho = 1 + \chi$ and $\det \rho = \chi$. Brauer-Nesbitt Theorem implies then that $\rho^{ss} \cong \left[\begin{smallmatrix} 1 & \\ & \chi \end{smallmatrix}\right]$, which contradicts the irreducibility of ρ. The Lemma follows. □

Lemma 2.6 implies that $\mathfrak{W}_{\mathbf{F}_p}$ lands inside $\mathfrak{J}_0(N)^0_{\mathbf{F}_p}$. From now on assume that ρ is not modular of type $(M, 2, 1)$. Then it follows (since ρ is irreducible) that the image of $\mathfrak{W}_{\mathbf{F}_p}$ inside $\mathfrak{J}_0(M)_{\mathbf{F}_p} \times \mathfrak{J}_0(M)_{\mathbf{F}_p}$ must be trivial ([Rib90], Thoerem 3.11). Hence, $\mathfrak{W}_{\mathbf{F}_p}$ lands inside the torus $T_0(N)_{\mathbf{F}_p}$.

The Hecke algebra acts on the character group X, and Ribet showed in [Rib90] (Lemma 6.3) that $V \subset \operatorname{Hom}(X/\mathfrak{m}X, \mu_p)$, where μ_p denotes the Galois modules of p-th roots of 1 over $\mathbf{Q}$.

Lemma 2.7. *The action of* Frob_p *on $X/\mathfrak{m}X$ coincides with the action of T_p and of $-w_p$, where w_p denotes the level p Atkin-Lehner involution.*

Proof. For the proof that the actions of T_p and of $-w_p$ on $T_0(N)_{\overline{\mathbf{F}}_p}$ (and hence on $X/\mathfrak{m}X$) coincide, see [Rib90], Proposition 3.7. We will show that the action of Frob_p and of $-w_p$ coincide on $X/\mathfrak{m}X$. Since $X = H_1(\Gamma(\mathfrak{X}_0(N)_{\overline{\mathbf{F}}_p}), \mathbf{Z})$, it is enough to consider the action of Frob_p and of $-w_p$ on $\Gamma(\mathfrak{X}_0(N)_{\overline{\mathbf{F}}_p})$. By using the moduli interpretation of $\mathfrak{X}_0(N)_{\mathbf{F}_p}$ one sees that Frob_p and w_p are both involutions and have the same effect on the edges of $\Gamma(\mathfrak{X}_0(N)_{\overline{\mathbf{F}}_p})$, but Frob_p fixes the vertices of $\Gamma(\mathfrak{X}_0(N)_{\overline{\mathbf{F}}_p})$, while w_p swaps them. Hence $\mathrm{Frob}_p \circ w_p$ swaps the vertices of $\Gamma(\mathfrak{X}_0(N)_{\overline{\mathbf{F}}_p})$ and fixes its edges, which on the homology has the same effect as multiplication by -1. □

Since w_p is an involution and κ is a field, Lemma 2.7 implies that G_p acts on $\mathrm{Hom}(X/\mathfrak{m}X, \mu_p)$ via $\psi\chi$, where ψ is an unramified quadratic character. In particular the action of G_p on $\det_\kappa V$ is via $\psi^2\chi^2 = \chi^2$. On the other hand, G_p acts on $\det_\kappa V$ by $\det\rho = \chi$, hence $\chi = \chi^2$, and thus χ must be trivial, which is an obvious contradiction since p is odd. This finishes the proof of Theorem 2.1 in the case when $\ell = p$.

The case $\ell \neq p$ proceeds along the same lines, but is simpler as one does not need an analogue of Lemma 2.3. Instead we note that when $\ell \neq p$, the Galois module V is unramified by assumption. Thus there exists a finite etale κ-vector space scheme $\mathfrak{W}$ over $\mathbf{Z}_{(\ell)}$ such that $V \cong \mathfrak{W}(\overline{\mathbf{Q}})$ as $\kappa[G_{\mathbf{Q}}]$-modules. Hence the injection $V \hookrightarrow J_0(N)(\overline{\mathbf{Q}})[\mathfrak{m}]$ gives rise to the a closed immersion $\mathfrak{W}_{\mathbf{Q}} \hookrightarrow J_0(N)$. As $\ell \neq p$, $\mathfrak{W}$ is smooth over $\mathbf{Z}_{(\ell)}$, hence the extension $\mathfrak{W} \to \mathfrak{J}_0(N)$ exists by the Neron mapping property. We claim that it is a closed immerion. Indeed, let $\overline{\mathfrak{W}}$ denote the schematic closure of the image of $\mathfrak{W}$ in $\mathfrak{J}_0(N)$. Then $\overline{\mathfrak{W}}$ is finite and flat over R, but since it is an ℓ-group with $\ell \neq p$, it is also etale. Hence the fact that $\overline{\mathfrak{W}}_{\mathbf{Q}} \xrightarrow{\sim} \mathfrak{W}_{\mathbf{Q}}$ implies that $\overline{\mathfrak{W}} \xrightarrow{\sim} \mathfrak{W}$. In that case the final conclusion that $\chi = \chi^2$ implies that $\ell \equiv 1 \bmod p$, which was assumed not to be the case. □

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