A NOTE ON HECKE EIGENVALUES OF HERMITIAN SIEGEL EISENSTEIN SERIES

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ABSTRACT. In this note we compute Hecke eigenvalues of Siegel Eisenstein series on the unitary group $\mathrm{U}(2,2)$ induced from a character. We do this for Hecke operators at primes not dividing the level of the Eisenstein series.

1. INTRODUCTION

In a recent paper Walling [Wal12] computed Hecke eigenvalues of Siegel Eisenstein series on GSp_4 induced from a character. Her method involves directly calculating the action of the operators on the Eisenstein series without prior knowledge of their Fourier coefficients. The goal of this note is to compute eigenvalues of Hecke operators acting on hermitian Siegel Eisenstein series (for primes not dividing the level). Our method, which uses some representation theory (but still no knowledge of Fourier coefficients), makes the calculations much less involved than the ones carried out in [Wal12] (however, let us note here that Walling in [loc.cit.] also computes eigenvalues for the operators at primes dividing the level).

More precisely, let K be an imaginary quadratic field, and let G be the quasi-split unitary group $\mathrm{U}(2,2)$ associated with the extension $K/\mathbf{Q}$. Let ψ be a Hecke character of K. We consider the (in general non-holomorphic) Eisenstein series $E := E(g,s,N,m,\psi)$ (studied e.g., by Shimura [Shi97], [Shi00]) of weight m, level N induced from the representation $\psi\circ\det$ of $M_P(\mathbf{A})$, where M_P is the Levi subgroup of the Siegel parabolic of G. We show that E is an eigenform for certain set of Hecke operators generating the local Hecke algebras at primes $p \nmid N$, and compute the corresponding eigenvalues for these generators. The adelic approach taken here not only simplifies calculations, but is in fact natural whenever the class number of K is greater than one as in that case the associated symmetric space has more than one connected component, and some Hecke operators at non-principal primes permute these components.

2. NOTATION

2.1. **Hecke characters.** Let K be an imaginary quadratic field with ring of integers $\mathcal{O}_K$. Denote by h_K the class number of K. For a number field L let $\mathbf{A}_L$ denote the ring of adeles of L, and put $\mathbf{A} := \mathbf{A}_\mathbf{Q}$. Write $\mathbf{A}_{L,\infty}$ and $\mathbf{A}_{L,\mathrm{f}}$ for the infinite part and the finite part of $\mathbf{A}_L$ respectively. For $\alpha = (\alpha_v) \in \mathbf{A}$ set $|\alpha|_\mathbf{A} := \prod_v |\alpha|_{\mathbf{Q}_v}$, where $|\alpha|_{\mathbf{Q}_v}$ denotes the v-adic norm of α. Hereafter the index v will always denote a (finite or infinite) place of $\mathbf{Q}$. In particular if $v = \infty$, then $\mathbf{Q}_v = \mathbf{R}$ and $|\alpha|_{\mathbf{Q}_\infty}$ denotes the usual absolute value. The letter p will be reserved for a finite place of $\mathbf{Q}$.

By a *Hecke character* of $\mathbf{A}_K^\times$ (or of K, for short) we mean a continuous homomorphism

$$\psi : K^\times \setminus \mathbf{A}_K^\times \to \mathbf{C}^\times.$$

We will think of ψ as a character of $(\mathrm{Res}_{K/\mathbf{Q}} \mathrm{GL}_{1/K})(\mathbf{A})$, where $\mathrm{Res}_{K/\mathbf{Q}}$ stands for the Weil restriction of scalars. We have a factorization $\psi = \prod_v \psi_v$ into local characters $\psi_v : \left(\mathrm{Res}_{K/\mathbf{Q}} \mathrm{GL}_{1/K}\right)(\mathbf{Q}_v) \to \mathbf{C}^\times$. For $M \in \mathbf{Z}$, we set $\psi_M := \prod_{p|M} \psi_p$.

2.2. **The unitary group.** For any affine group scheme X over $\mathbf{Z}$ and any $\mathbf{Z}$-algebra A we denote by $x \mapsto \overline{x}$ the automorphism of $(\mathrm{Res}_{\mathcal{O}_K/\mathbf{Z}} X_{\mathcal{O}_K})(A)$ induced by the non-trivial automorphism of $K/\mathbf{Q}$. Note that $(\mathrm{Res}_{\mathcal{O}_K/\mathbf{Z}} X_{\mathcal{O}_K})(A)$ can be identified with a subgroup of $\mathrm{GL}_n(A \otimes \mathcal{O}_K)$ for some n. In what follows we always specify such an identification. Then for $x \in (\mathrm{Res}_{\mathcal{O}_K/\mathbf{Z}} X_{\mathcal{O}_K})(A)$ we write x^t for the transpose of x, and set $x^* := \overline{x}^t$ and $\hat{x} := (\overline{x}^t)^{-1}$. Moreover, we write $\mathrm{diag}(a_1, a_2, \dots, a_n)$ for the $n \times n$-matrix with $a_1, a_2, \dots a_n$ on the diagonal and all the off-diagonal entries equal to zero.

We will denote by $\mathbf{G}_a$ the additive group and by $\mathbf{G}_m$ the multiplicative group. To the imaginary quadratic extension $K/\mathbf{Q}$ one associates the unitary similitude group scheme over $\mathbf{Z}$:

$$G := \mathrm{GU}(2,2) = \{A \in \mathrm{Res}_{\mathcal{O}_K/\mathbf{Z}} \mathrm{GL}_4 \mid AJ\bar{A}^t = \mu(A)J\}$$

with $J = \begin{bmatrix} & -I_2 \\ I_2 & \end{bmatrix}$, where for any $n \in \mathbf{Z}_+$ we write I_n for the $n \times n$ identity matrix and $\mu(A) \in \mathbf{G}_m$. We will also make use of the group

$$U = \mathrm{U}(2,2) = \{A \in G \mid \mu(A) = 1\}.$$

Let p be a rational prime. Write K_p for $K \otimes_{\mathbf{Q}} \mathbf{Q}_p$, and $\mathcal{O}_{K,p}$ for $\mathcal{O}_K \otimes_{\mathbf{Z}} \mathbf{Z}_p$. Note that if p is inert or ramified in K, then $K_p/\mathbf{Q}_p$ is a degree two extension of local fields, and $a \mapsto \overline{a}$ induces the non-trivial automorphism in $\mathrm{Gal}(K_p/\mathbf{Q}_p)$. If p splits in K, denote by $\iota_{p,1}, \iota_{p,2}$ the two distinct embeddings of K into $\mathbf{Q}_p$. Then the map $a \otimes b \mapsto (\iota_{p,1}(a)b, \iota_{p,2}(a)b)$ induces a $\mathbf{Q}_p$-algebra isomorphism $K_p \cong \mathbf{Q}_p \times \mathbf{Q}_p$, and $a \mapsto \overline{a}$ corresponds on the right-hand side to the automorphism defined by $(a,b) \mapsto (b,a)$. We denote the isomorphism $\mathbf{Q}_p \times \mathbf{Q}_p \xrightarrow{\sim} K_p$ by ι_p. For a matrix $g = (g_{ij})$ with entries in $\mathbf{Q}_p \times \mathbf{Q}_p$ we also set $\iota_p(g) = (\iota_p(g_{ij}))$. For a split prime p the map ι_p^{-1} identifies $G(\mathbf{Q}_p)$ with

$$G_p = \{(g_1, g_2) \in \mathrm{GL}_4(\mathbf{Q}_p) \times \mathrm{GL}_4(\mathbf{Q}_p) \mid g_1 J g_2^t = \alpha J, \alpha \in \mathbf{Q}_p^\times\}.$$

Note that the map $(g_1, g_2) \mapsto (g_1, \alpha)$ gives a (non-canonical) isomorphism $G(\mathbf{Q}_p) \cong \mathrm{GL}_4(\mathbf{Q}_p) \times \mathbf{G}_m(\mathbf{Q}_p)$.

2.3. **Compact subgroups.** For an associative ring R with identity and an R-module N we write $M_n(N)$ for the R-module of $n \times n$-matrices with entries in N. Let $x = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in M_{2n}(N)$ with $A, B, C, D \in M_n(N)$. Define $a_x = A$, $b_x = B$, $c_x = C$, $d_x = D$.

For $N \in \mathbf{Z}$ and a rational prime p, set

$$\mathcal{K}_{0,p}(N) = \{x \in G(\mathbf{Q}_p) \mid a_x, b_x, d_x \in M_2(\mathcal{O}_{K,p}), c_x \in M_2(N\mathcal{O}_{K,p})\}.$$

For any p, the group $\mathcal{K}_{0,p} := \mathcal{K}_{0,p}(1) = G(\mathbf{Z}_p)$ is a maximal (open) compact subgroup of $G(\mathbf{Q}_p)$. Note that if $p \nmid N$, then $\mathcal{K}_{0,p} = \mathcal{K}_{0,p}(N)$. Set

$$\mathcal{K}_{0,\infty}^+ := \left\{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in U(\mathbf{R}) \mid A, B \in \mathrm{GL}_2(\mathbf{C}), AA^* + BB^* = I_2, AB^* = BA^* \right\}.$$

Then $\mathcal{K}_{0,\infty}^{+}$ is a maximal compact subgroup of $U(\mathbf{R})$. We will denote by $\mathcal{K}_{0,\infty}$ the subgroup of $G(\mathbf{R})$ generated by $\mathcal{K}_{0,\infty}^{+}$ and $\operatorname{diag}(1,1,-1,-1)$. Then $\mathcal{K}_{0,\infty}$ is a maximal compact subgroup of $G(\mathbf{R})$. Finally, set $\mathcal{K}_0(N) := \mathcal{K}_{0,\infty}^{+}\prod_p \mathcal{K}_{0,p}(N)$. Similarly, we define $\mathcal{K}_1(N) = \mathcal{K}_{0,\infty}^{+}\prod_p \mathcal{K}_{1,p}(N)$ with

$$\mathcal{K}_{1,p}(N) = \{x \in \mathcal{K}_{0,p}(N) \mid a_x - I_2 \in M_2(N\mathcal{O}_{K,p})\}.$$

3. Siegel Eisenstein series

We adopt the following notation. If H is an algebraic group over $\mathbf{Q}$ and $g \in H(\mathbf{A})$, we will write $g_\infty \in H(\mathbf{R})$ for the infinity component of g and g_{f} for the finite component of g, i.e., $g = (g_\infty, g_{\mathrm{f}})$.

Definition 3.1. Let $\mathcal{K}$ be an open compact subgroup of $G(\mathbf{A}_{\mathrm{f}})$. Write Z for the center of G. Let $\mathcal{M}_m(\mathcal{K})$ denote the $\mathbf{C}$-space consisting of functions $f : G(\mathbf{A}) \to \mathbf{C}$ satisfying the following conditions:

- $f(\gamma g) = f(g)$ for all $\gamma \in G(\mathbf{Q})$, $g \in G(\mathbf{A})$,
- $f(gk) = f(g)$ for all $k \in \mathcal{K}$, $g \in G(\mathbf{A})$,
- $f(gu) = \det(c_u i + d_u)^{-m} f(g)$ for all $g \in G(\mathbf{A})$, $u \in \mathcal{K}_\infty = \mathcal{K}_{0,\infty}$ (see (10.7.4) in [Shi97]; also see section 2.3 above for the notation c_u and d_u),
- $f(ag) = a^{-2m} f(g)$ for all $g \in G(\mathbf{A})$ and all $a \in \mathbf{C}^\times = Z(\mathbf{R}) \subset G(\mathbf{R})$.

As before, let $N > 1$ be an integer, and ψ a Hecke character of K satisfying

$$\psi_\infty(x) = x^m |x|^{-m} \tag{3.1}$$

for a positive integer m, and

$$\psi_p(x) = 1 \text{ if } p \neq \infty,\ x \in \mathcal{O}_{K,p}^\times \text{ and } x - 1 \in N\mathcal{O}_{K,p}. \tag{3.2}$$

Set

$$\begin{aligned}(3.3)\quad & \mathcal{M}_m(N,\psi) := \{f \in \mathcal{M}_m(\mathcal{K}_1(N)) \mid f(\gamma g (k_\infty, k_{\mathrm{f}})) = \\ & = \psi_N(\det(a_{k_{\mathrm{f}}}))^{-1} \det(c_{k_\infty} i + d_{k_\infty})^{-m} f(g), g \in G(\mathbf{A}), \gamma \in G(\mathbf{Q}), (k_\infty, k_{\mathrm{f}}) \in \mathcal{K}_0(N)\}.\end{aligned}$$

Let P be the Siegel parabolic of G, i.e., $P = M_P U_P$ where M_P is the subgroup

$$M_P = \left\{ \begin{bmatrix} A & \\ & \alpha \hat{A} \end{bmatrix} \mid A \in \operatorname{Res}_{K/\mathbf{Q}} \mathrm{GL}_{2/K}, \alpha \in \mathbf{G}_m \right\},$$

and U_P is the (abelian) unipotent radical

$$U_P = \left\{ \begin{bmatrix} 1 & & b_1 & b_2 \\ & 1 & \bar{b}_2 & b_4 \\ & & 1 & \\ & & & 1 \end{bmatrix} \mid b_1, b_4 \in \mathbf{G}_a,\ \ b_2 \in \operatorname{Res}_{K/\mathbf{Q}} \mathbf{G}_{a/K} \right\}.$$

Define $\mu_P : M_P(\mathbf{Q}) U_P(\mathbf{A}) \setminus G(\mathbf{A}) \to \mathbf{C}$ by setting

$$\mu_P(g) = \begin{cases} 0 & g \notin P(\mathbf{A})\mathcal{K}_0(N), \\ \psi(\det d_q)^{-1} \psi_N(\det d_k)^{-1} (c_{k_\infty} i + d_{k_\infty})^{-m} & g = qk \in P(\mathbf{A})\mathcal{K}_0(N). \end{cases}$$

Note that μ_P has a local decomposition $\mu_P = \prod_v \mu_{P,v}$, where

$$\mu_{P,v}(q_v k_v) = \begin{cases} \psi_v(\det d_{q_v})^{-1} & \text{if } v \nmid N, v \neq \infty, \\ \psi_v(\det d_{q_v})^{-1} \psi_v(\det d_{k_v}) & \text{if } v \mid N, v \neq \infty, \\ \psi_\infty(\det d_{q_\infty})^{-1} (c_{k_\infty} i + d_{k_\infty})^{-m} & \text{if } v = \infty. \end{cases} \tag{3.4}$$

Let δ_P denote the modulus character of $P(\mathbf{A})$. We extend δ_P to a function on $G(\mathbf{A})$ using the Iwasawa decomposition $G(\mathbf{A}) = P(\mathbf{A})\mathcal{K}_0(N)$ and declaring it to be trivial on $\mathcal{K}_0(N)$. Note that this is well-defined, and that δ_P has a local decomposition $\delta_P = \prod_v \delta_{P,v}$ with

$$\delta_{P,v}\left(\begin{bmatrix} A & \\ & D \end{bmatrix} uk\right) = |\det A \det D^{-1}|^2_{\mathbf{Q}_v}, \tag{3.5}$$

where $\begin{bmatrix} A & \\ & D \end{bmatrix} \in M_P(\mathbf{Q}_v)$, $u \in U_P(\mathbf{Q}_v)$ and $k \in \mathcal{K}_{0,v}(N)$ if $v \neq \infty$, or $k \in \mathcal{K}^+_{0,\infty}$ if $v = \infty$.

Definition 3.2. The series

$$E(g, s, N, m, \psi) := \sum_{\gamma \in P(\mathbf{Q})\backslash G(\mathbf{Q})} \mu_P(\gamma g)\delta_P(\gamma g)^{s/2}$$

is called the (*hermitian*) *Siegel Eisenstein series of weight* m*, level* N *and character* ψ.

For any fixed $g \in G(\mathbf{A})$ the Siegel Eisenstein series converges absolutely for $\mathrm{Re}(s)$ sufficiently large. It admits a meromorphic continuation in s to the entire complex plane with finitely many simple poles (cf. Proposition 19.1 in [Shi97]). It is easily checked that $E(g, s, N, m, \psi) \in \mathcal{M}_m(N, (\psi^c)^{-1})$, where the character ψ^c is defined by $\psi^c(a) := \psi(\overline{a})$.

4. The Hecke algebra

4.1. **Generalities.** Let p be a rational prime and (V, π) a smooth representation of $G(\mathbf{Q}_p)$. For a summary of the theory of such representations see e.g., [Bum97], section 4.2. Recall that $\mathcal{K}_{0,p} := G(\mathbf{Z}_p)$ is a maximal (open) compact subgroup of $G(\mathbf{Q}_p)$. Fix a left-invariant Haar measure dg on $G(\mathbf{Q}_p)$ such that $\int_{\mathcal{K}_{0,p}} dg = 1$. We denote by $\mathcal{H}_{\mathcal{K}_{0,p}}$ the algebra of compactly supported, smooth (i.e., locally constant), bi-$\mathcal{K}_{0,p}$-invariant functions from $G(\mathbf{Q}_p)$ into $\mathbf{C}$. The multiplication in $\mathcal{H}_{\mathcal{K}_{0,p}}$ is defined to be the convolution

$$(\phi_1 \star \phi_2)(g) = \int_{G(\mathbf{Q}_p)} \phi_1(gh^{-1})\phi_2(h)dh.$$

For any subset H of $G(\mathbf{Q}_p)$ denote by $[H]$ the characteristic function of H. The function $[\mathcal{K}_{0,p}]$ is the identity element of $\mathcal{H}_{\mathcal{K}_{0,p}}$. The algebra $\mathcal{H}_{\mathcal{K}_{0,p}}$ is called the spherical Hecke algebra of $G(\mathbf{Q}_p)$.

Lemma 4.1. *The algebra* $\mathcal{H}_{\mathcal{K}_{0,p}}$ *is commutative.*

Proof. The proof is the same as the proof of Theorem 4.6.1 in [Bum97]. □

The representation π defines an action $v \mapsto \pi(\phi)v$ of $\mathcal{H}_{\mathcal{K}_{0,p}}$ on V by

$$\pi(\phi)(v) := \int_{G(\mathbf{Q}_p)} \phi(g)\pi(g)v\, dg.$$

Before we state our results let us begin with the following lemma which explains how everything we are about to state can be easily expressed in terms of right cosets as opposed to left cosets.

Lemma 4.2. *Let $a \in G(\mathbf{Q}_p)$ be a diagonal matrix, and assume that the double coset space $\mathcal{K}_{0,p}a\mathcal{K}_{0,p}$ decomposes as $\mathcal{K}_{0,p}a\mathcal{K}_{0,p} = \bigsqcup_{i=1}^n a_i\mathcal{K}_{0,p}$. Then $\mathcal{K}_{0,p}a\mathcal{K}_{0,p} = \bigsqcup_{i=1}^n \mathcal{K}_{0,p}\mu(a_i)\overline{a}_i^{-1}$.*

Proof. The map $\varphi : G \to G$ defined by $\varphi(x) = \mu(x)\overline{x}^{-1}$ is an anti-involution on G. If $a \in G(\mathbf{Q}_p)$ is diagonal, then $\varphi(a) = JaJ^{-1}$. Since $J \in \mathcal{K}_{0,p}$, we get $\mathcal{K}_{0,p}a\mathcal{K}_{0,p} = \mathcal{K}_{0,p}\varphi(a)\mathcal{K}_{0,p}$. Applying φ to the equality $\mathcal{K}_{0,p}a\mathcal{K}_{0,p} = \bigsqcup_{i=1}^n a_i\mathcal{K}_{0,p}$ we obtain the lemma. □

For any subgroup H of $G(\mathbf{Q}_p)$ we denote by V^H the subspace of V consisting of vectors fixed by H.

Lemma 4.3. *Let $a \in G(\mathbf{Q}_p)$, and assume that the double coset space $\mathcal{K}_{0,p}a\mathcal{K}_{0,p}$ decomposes as $\mathcal{K}_{0,p}a\mathcal{K}_{0,p} = \bigsqcup_{i=1}^n a_i\mathcal{K}_{0,p}$ with $a_i \in G(\mathbf{Q}_p)$. Then for $v \in V^{\mathcal{K}_{0,p}}$ we have*

$$\pi([\mathcal{K}_{0,p}a\mathcal{K}_{0,p}])v = \sum_{i=1}^n \pi(a_i)v.$$

Proof. We have

$$\pi([\mathcal{K}_{0,p}a\mathcal{K}_{0,p}])v = \int_{\mathcal{K}_{0,p}a\mathcal{K}_{0,p}} \pi(g)v\,dg = \sum_{i=1}^n \int_{a_i\mathcal{K}_{0,p}} \pi(g)v\,dg.$$

Making a change of variable $g \mapsto a_i g$, and using the fact that dg is a left-invariant measure we get

$$\pi([\mathcal{K}_{0,p}a\mathcal{K}_{0,p}])v = \sum_{i=1}^n \int_{\mathcal{K}_{0,p}} \pi(a_i g)v\,dg = \sum_{i=1}^n \pi(a_i)v.$$

□

Lemma 4.4. *Let $\phi \in \mathcal{H}_{\mathcal{K}_{0,p}}$ and $v \in V^{\mathcal{K}_{0,p}}$. Then $\pi(\phi)v \in V^{\mathcal{K}_{0,p}}$.*

Proof. We need to show that $\pi(k)\pi(\phi)v = \pi(\phi)v$ for $k \in \mathcal{K}_{0,p}$. We have

$$\pi(k)\pi(\phi)v = \pi(k)\int_{G(\mathbf{Q}_p)} \phi(g)\pi(g)v\,dg = \pi(k)\pi(\phi)v = \int_{G(\mathbf{Q}_p)} \phi(g)\pi(kg)v\,dg.$$

Making a change of variable $g \mapsto k^{-1}g$, and using the fact that dg is left-invariant we see that the last integral equals $\int_{G(\mathbf{Q}_p)} \phi(k^{-1}g)\pi(g)v\,dg$. Since ϕ is left $\mathcal{K}_{0,p}$-invariant, the lemma follows. □

Corollary 4.5. *Let $a \in G(\mathbf{Q}_p)$. If $\mathcal{K}_{0,p}a\mathcal{K}_{0,p} = \bigsqcup_{i=1}^n a_i\mathcal{K}_{0,p}$ with $a_i \in G(\mathbf{Q}_p)$ and $v \in V^{\mathcal{K}_{0,p}}$, then $\sum_{i=1}^n \pi(a_i)v \in V^{\mathcal{K}_{0,p}}$.*

Proof. By Lemma 4.3 we have $\sum_{i=1}^n \pi(a_i)v = \pi([\mathcal{K}_{0,p}a\mathcal{K}_{0,p}])v$. Hence the claim follows by Lemma 4.4. □

4.2. **Double coset decompositions.**

4.2.1. *The case of a split prime.* Let p be a prime which splits in K. Write $(p) = \mathfrak{p}\overline{\mathfrak{p}}$. Recall that $G(\mathbf{Q}_p) \cong \mathrm{GL}_4(\mathbf{Q}_p) \times \mathbf{G}_m(\mathbf{Q}_p)$. An element g of $G(\mathbf{Q}_p)$ can be written as $g = (g_1, g_2) \in \mathrm{GL}_4(\mathbf{Q}_p) \times \mathrm{GL}_4(\mathbf{Q}_p)$ with $g_2 = -\mu(g)J(g_1^t)^{-1}J$. Set

- $T_{\mathfrak{p}} := \mathcal{K}_{0,p}(\mathrm{diag}(1,p,p,p), \mathrm{diag}(1,1,p,1))\mathcal{K}_{0,p}$,
- $T_p := \mathcal{K}_{0,p}(\mathrm{diag}(1,1,p,p), \mathrm{diag}(1,1,p,p))\mathcal{K}_{0,p}$,
- $\Delta_{\mathfrak{p}} := \mathcal{K}_{0,p}(pI_4, I_4)\mathcal{K}_{0,p}$.

It is a standard fact that the $\mathbf{C}$-algebra $\mathcal{H}_p$ is generated by the operators $T_{\mathfrak{p}}$, $T_{\overline{\mathfrak{p}}}$, T_p, $\Delta_{\mathfrak{p}}$, $\Delta_{\overline{\mathfrak{p}}}$ and their inverses.

Proposition 4.6. *We have the following decompositions*

$$\begin{aligned} T_{\mathfrak{p}} = & \bigsqcup_{a,b,c \in \mathbf{Z}/p\mathbf{Z}} \left(\begin{bmatrix} p & & b & \\ & p & c & \\ & & 1 & \\ & & -a & p \end{bmatrix}, \begin{bmatrix} p & a & b & c \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) \mathcal{K}_{0,p} \sqcup \\ & \bigsqcup_{d,e \in \mathbf{Z}/p\mathbf{Z}} \left(\begin{bmatrix} p & & & d \\ & p & & e \\ & & p & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & p & d & e \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) \mathcal{K}_{0,p} \sqcup \\ & \bigsqcup_{f \in \mathbf{Z}/p\mathbf{Z}} \left(\begin{bmatrix} 1 & & & \\ -f & p & & \\ & & p & \\ & & & p \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & p & f \\ & & & 1 \end{bmatrix} \right) \mathcal{K}_{0,p} \sqcup \\ & \left(\begin{bmatrix} p & & & \\ & 1 & & \\ & & p & \\ & & & p \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & p \end{bmatrix} \right) \mathcal{K}_{0,p}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} T_p = & \bigsqcup_{b,c,d,e \in \mathbf{Z}/p\mathbf{Z}} \left(\begin{bmatrix} p & & b & c \\ & p & d & e \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} p & & b & d \\ & p & c & e \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) \mathcal{K}_{0,p} \sqcup \\ & \bigsqcup_{a,c,f \in \mathbf{Z}/p\mathbf{Z}} \left(\begin{bmatrix} p & a & & c \\ & 1 & & \\ & & p & f \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & \\ -f & p & & c \\ & & 1 & \\ & & -a & p \end{bmatrix} \right) \mathcal{K}_{0,p} \sqcup \\ & \bigsqcup_{e,f \in \mathbf{Z}/p\mathbf{Z}} \left(\begin{bmatrix} 1 & & & \\ & p & & e \\ & & p & f \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & \\ -f & p & & e \\ & & p & \\ & & & 1 \end{bmatrix} \right) \mathcal{K}_{0,p} \sqcup \\ & \bigsqcup_{a,b \in \mathbf{Z}/p\mathbf{Z}} \left(\begin{bmatrix} p & a & b & \\ & 1 & & \\ & & 1 & \\ & & & p \end{bmatrix}, \begin{bmatrix} p & & b & \\ & 1 & & \\ & & 1 & \\ & & -a & p \end{bmatrix} \right) \mathcal{K}_{0,p} \sqcup \\ & \bigsqcup_{d \in \mathbf{Z}/p\mathbf{Z}} \left(\begin{bmatrix} 1 & & & \\ & p & d & \\ & & 1 & \\ & & & p \end{bmatrix}, \begin{bmatrix} p & & & d \\ & 1 & & \\ & & p & \\ & & & 1 \end{bmatrix} \right) \mathcal{K}_{0,p} \sqcup \\ & \left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{bmatrix} \right) \mathcal{K}_{0,p}. \end{aligned} \tag{4.2}$$

Proof. This follows easily from the corresponding decompositions for the group $\mathrm{GL}_4(\mathbf{Q}_p)$. $\square$

4.2.2. *The case of an inert prime.* Let p be a prime which is inert in K. Set

- $T_p := \mathcal{K}_{0,p} \,\mathrm{diag}(1,1,p,p)\mathcal{K}_{0,p}$,
- $T_{1,p} := \mathcal{K}_{0,p} \,\mathrm{diag}(1,p,p^2,p)\mathcal{K}_{0,p}$,
- $\Delta_p := \mathcal{K}_{0,p} pI_4 \mathcal{K}_{0,p}$.

The operators T_p, $T_{1,p}$, Δ_p and their inverses generate the $\mathbf{C}$-algebra $\mathcal{H}_p$.

Proposition 4.7. *We have the following decompositions*

$$
\begin{aligned}
T_p = & \bigsqcup_{\substack{b,d\in\mathbf{Z}/p\mathbf{Z}\\ c\in\mathcal{O}_K/p\mathcal{O}_K}} \begin{bmatrix} p & & b & c \\ & p & \overline{c} & d \\ & & 1 & \\ & & & 1 \end{bmatrix} \mathcal{K}_{0,p} \sqcup \bigsqcup_{e\in\mathbf{Z}/p\mathbf{Z}} \begin{bmatrix} 1 & & & \\ & p & & e \\ & & p & \\ & & & 1 \end{bmatrix} \mathcal{K}_{0,p} \sqcup \\
& \bigsqcup_{\substack{a\in\mathcal{O}_K/p\mathcal{O}_K\\ b\in\mathbf{Z}/p\mathbf{Z}}} \begin{bmatrix} p & a & b & \\ & 1 & & \\ & & 1 & \\ & & -\overline{a} & p \end{bmatrix} \mathcal{K}_{0,p} \sqcup \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & p & \\ & & & p \end{bmatrix} \mathcal{K}_{0,p}.
\end{aligned} \tag{4.3}
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$$
\begin{aligned}
T_{1,p} = & \bigsqcup_{\substack{a,c\in\mathcal{O}_K/p\mathcal{O}_K\\ b\in\mathbf{Z}/p^2\mathbf{Z}}} \begin{bmatrix} p^2 & \overline{a}p & b+a\overline{c} & \overline{c}p \\ & p & c & \\ & & 1 & \\ & & -a & p \end{bmatrix} \mathcal{K}_{0,p} \sqcup \bigsqcup_{\substack{c\in\mathcal{O}_K/p\mathcal{O}_K\\ d\in\mathbf{Z}/p^2\mathbf{Z}}} \begin{bmatrix} p & & & \overline{c} \\ & p^2 & cp & d \\ & & p & \\ & & & 1 \end{bmatrix} \mathcal{K}_{0,p} \\
& \bigsqcup_{a\in\mathcal{O}_K/p\mathcal{O}_K} \begin{bmatrix} p & \overline{a} & & \\ & 1 & & \\ & & p & \\ & & -ap & p^2 \end{bmatrix} \mathcal{K}_{0,p} \sqcup \begin{bmatrix} 1 & & & \\ & p & & \\ & & p^2 & \\ & & & p \end{bmatrix} \mathcal{K}_{0,p} \sqcup \\
& \bigsqcup_{\substack{b,d\in\mathbf{Z}/p\mathbf{Z}\\ bd\equiv 0 \pmod{p}\\ (b,d)\neq(0,0)}} \begin{bmatrix} p & & b & \\ & p & & d \\ & & p & \\ & & & p \end{bmatrix} \mathcal{K}_{0,p} \sqcup \bigsqcup_{\substack{b\in(\mathbf{Z}/p\mathbf{Z})^\times\\ c\in(\mathcal{O}_K/p\mathcal{O}_K)^\times}} \begin{bmatrix} p & & b & c \\ & p & \overline{c} & |c|^2 b^{-1} \\ & & p & \\ & & & p \end{bmatrix} \mathcal{K}_{0,p}
\end{aligned} \tag{4.4}
$$

Proof. See the proof of Lemma 5.3 in [Klo09] and references cited there. $\square$

5. Eigenvalues

Fix a rational prime $p \nmid N$. For ψ' a Hecke character of K as in section 3 we let the local Hecke algebra $\mathcal{H}_{\mathcal{K}_{0,p}}$ act on functions in $\mathcal{M}_m(N,\psi')$ in the usual way, i.e., by means of the canonical embedding $G(\mathbf{Q}_p) \hookrightarrow G(\mathbf{A})$ and the right regular action on the p-component. More specifically, let $a \in G(\mathbf{Q}_p)$, and let $A \subset G(\mathbf{Q}_p)$ be such that $\mathcal{K}_{0,p} a \mathcal{K}_{0,p} = \bigsqcup_{\alpha\in A} \alpha\mathcal{K}_{0,p}$. Then for any $f \in \mathcal{M}_m(N,\psi')$ we have

$$[\mathcal{K}_{0,p} a \mathcal{K}_{0,p}] f(g) = \sum_{\alpha\in A} f(g\alpha).$$

It is a standard fact that $[\mathcal{K}_{0,p} a \mathcal{K}_{0,p}] f \in \mathcal{M}_m(N,\psi')$ (see e.g., [Shi00], section 20.3), but we will not use it.

Theorem 5.1. *Let ψ be a Hecke character of K as in section 3. The Eisenstein series $E(g,s,N,m,\psi)$ is an eigenfunction of the operator $[\mathcal{K}_{0,p} a \mathcal{K}_{0,p}]$ with eigenvalue*

$$\lambda([\mathcal{K}_{0,p} a \mathcal{K}_{0,p}]) = \sum_{\alpha\in A} \mu_{P,p}(\alpha)\delta_{P,p}(\alpha)^{s/2} = \sum_{\alpha\in A} \psi_p(\det d_\alpha)^{-1}\delta_{P,p}(\alpha)^{s/2}.$$

Proof. In this proof we fix $s \in \mathbf{C}$ in the range of absolute convergence of $E(g,s,N,m,\psi)$. For any place v of $\mathbf{Q}$ and any $g \in G(\mathbf{Q}_v)$ set

$$f_v(g,s) = \mu_{P,v}(g)\delta_{P,v}(g)^{s/2}$$

with $\mu_{P,v}$ as in (3.4) and $\delta_{P,v}$ as in (3.5). Let $x \in G(\mathbf{Q}_p)$. As $\mathcal{K}_{0,p}(N) = \mathcal{K}_{0,p}$, a maximal compact subgroup of $G(\mathbf{Q}_p)$, we conclude that $f_p(\cdot,s)$ is right-$\mathcal{K}_{0,p}$-invariant. Let I be the $\mathbf{C}$-space of all functions $h : G(\mathbf{Q}_p) \to \mathbf{C}$ such that

(i) there exists an open subgroup $U_h \subset G(\mathbf{Q}_p)$ with the property that $h(gu) = h(g)$ for all $u \in U_h$, $g \in G(\mathbf{Q}_p)$ and

(ii) $h(qg) = f_p(q,s)h(g)$ for all $q \in P(\mathbf{Q}_p)$, $g \in G(\mathbf{Q}_p)$.

First note that $f_p(qg,s) = f_p(q,s)f_p(g,s)$ for $q \in P(\mathbf{Q}_p)$ and $g \in G(\mathbf{Q}_p)$. To see this, one uses the Iwasawa decomposition to write $g = q_g k_g$ with $q_g \in P(\mathbf{Q}_p)$ and $k_g \in \mathcal{K}_{0,p}$ and the fact that $f_p(\cdot,s)$ is a character when restricted to $P(\mathbf{Q}_p)$. Hence we conclude that $f_p(\cdot,s) \in I$. Moreover, $G(\mathbf{Q}_p)$ acts on I by the right regular action, and condition (i) guarantees that this action induces a smooth representation which we denote by π. Note that, indeed, if $h \in I$, then $U_{\pi(g)h} = gU_h g^{-1}$. Hence condition (i) defining the membership of I is satisfied for $\pi(g)h$.

Furthermore, it is clear that $\dim_{\mathbf{C}} I^{\mathcal{K}_{0,p}} = 1$ as $f_p(\cdot,s) \in I^{\mathcal{K}_{0,p}}$, and any right-$\mathcal{K}_{0,p}$-invariant function $h \in I$ is of the form $h(qk) = h(q) = f_p(q,s)h(I_4)$. The fact that $f_p(\cdot,s) \in I^{\mathcal{K}_{0,p}}$ allows us to use Corollary 4.5 to conclude that $\sum_{\alpha\in A}\pi(\alpha)f_p(\cdot,s) \in I^{\mathcal{K}_{0,p}}$. However, since $\dim_{\mathbf{C}} I^{\mathcal{K}_{0,p}} = 1$, there must exist $\lambda \in \mathbf{C}$ (independent of x) such that

$$\sum_{\alpha\in A} f_p(x\alpha,s) = \sum_{\alpha\in A}\pi(\alpha)f_p(x,s) = \lambda f_p(x,s). \tag{5.1}$$

We can find λ by plugging in $x = I_4$, and thus get

$$\lambda = f_p(I_4,s)^{-1}\sum_{\alpha\in A} f_p(\alpha,s) = \sum_{\alpha\in A}\mu_{P,p}(\alpha)\delta_{P,p}(\alpha)^{s/2} = \sum_{\alpha\in A}\psi_p(\det d_\alpha)^{-1}\delta_{P,p}(\alpha)^{s/2}.$$

In other words we have shown that the Hecke operator $[\mathcal{K}_{0,p}a\mathcal{K}_{0,p}]$ acts on the space I, and $f_p(\cdot,s)$ is an eigenvector for this operator with eigenvalue λ as above.

We now claim that this implies that $E(g,s,N,m,\psi)$ is also an eigenfunction for $[\mathcal{K}_{0,p}a\mathcal{K}_{0,p}]$ with the same eigenvalue. Write $E(g)$ for $E(g,s,N,m,\psi)$ for short. For $g \in G(\mathbf{A})$ write $g_v \in G(\mathbf{Q}_v)$ for its vth component. Note that by definition

$$E(g) = \sum_{\gamma\in P(\mathbf{Q})\backslash G(\mathbf{Q})}\prod_v f_v(\gamma g_v, s).$$

One has

$$[\mathcal{K}_{0,p}a\mathcal{K}_{0,p}]E(g) = \sum_{\alpha\in A}E(g\alpha) = \sum_{\alpha\in A}\sum_{\gamma\in P(\mathbf{Q})\backslash G(\mathbf{Q})} f_p(\gamma g_p\alpha,s)\prod_{v\neq p}f_v(\gamma g_v,s). \tag{5.2}$$

By our assumption on s the sum on the right converges absolutely for every $\alpha \in A$, so we can interchange the order of summation and obtain

$$\begin{aligned}[\mathcal{K}_{0,p}a\mathcal{K}_{0,p}]E(g) &= \sum_{\gamma\in P(\mathbf{Q})\backslash G(\mathbf{Q})}\prod_{v\neq p}f_v(\gamma g_v,s)\sum_{\alpha\in A}f_p(\gamma g_p\alpha,s) = \\ &= \sum_{\gamma\in P(\mathbf{Q})\backslash G(\mathbf{Q})}\prod_{v\neq p}f_v(\gamma g_v,s)\lambda f_p(\gamma g_p,s) = \lambda E(g),\end{aligned} \tag{5.3}$$

where the second equality follows from (5.1) by putting $x = \gamma g_p$. □

Recall that we denote by ψ_N the product $\prod_{v\mid N}\psi_v$. Note that we can regard ψ_N as a Dirichlet character of $(\mathcal{O}_K/N\mathcal{O}_K)^\times$. Hence for $\alpha \in \mathcal{O}_K$ we will simply write $\psi_N(\alpha)$ for $\psi_N(j(\alpha))$, where j denotes the canonical embedding of K into $\prod_{v\mid N}K_v$.

Corollary 5.2. *Let $\lambda(T)$ denote the eigenvalue of the Hecke operator T corresponding to the Eisenstein series $E(g,s,N,m,\psi)$. Then*

$$\lambda(T) = \begin{cases} (p+1)\psi_p(\iota_p((p,1)))^{-1}[p^{2-s}+\psi_N(p)p^s] & T = T_{\mathfrak{p}}, \\ p^{4-2s}+\psi_N(p)p\prod_{\mathfrak{p}|p}(N\mathfrak{p}+1)+\psi_N(p)^2p^{2s} & T = T_p, \\ (p^2+1)\psi_N(p)[p^{4-2s}+\psi_N(p)(p-1)+\psi_N(p)^2p^{2s}] & T = T_{1,p}, \end{cases} \tag{5.4}$$

where the first case holds for a split p with $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$, the middle one for both split and inert p, and the last case holds for inert p.

Proof. This is straightforward using Theorem 5.1 and the double coset decompositions in section 4.2. Let us only note that we made use of the following calculation: Let $\gamma_p \in \mathbf{A} \hookrightarrow \mathbf{A}_K$ be the adele whose pth component equals p and all the other ones equal 1. Then one has

$$\psi_p(\iota((p,p))^{-1}) = \psi(\gamma_p^{-1}) = \psi(p\gamma_p^{-1}) = \psi_\infty(p)\psi_N(p) = \psi_N(p).$$

Indeed, the first equality follows from the definition of the map ι, the second one from the $K^\times$-invariance of ψ, the third one from (3.2) and the assumption that $p \nmid N$, and the last equality follows from (3.1) as $p \in \mathbf{R}$. □

Remark 5.3. The results of this paper can easily be translated to the classical setup, where automorphic forms are regarded as functions on (h_K-many copies of) the symmetric space $\mathbf{H}_2 := \{Z \in M_4(\mathbf{C}) \mid -i(Z-Z^*) > 0\}$. The group $U(\mathbf{R})$ acts on $\mathbf{H}_2$ in the following way: $gZ = (a_gZ+b_g)(c_gZ+d_g)^{-1}$. For $x \in U(\mathbf{A}_\mathrm{f})$, $g \in U(\mathbf{R})$ and $Z = g\,\mathrm{diag}(i,i)$ define ([Shi00], (17.23a))

$$E_x(Z,s,m,\psi,N) = \det(c_gi+d_g)^m E(xg,s,N,m,\psi).$$

If $h_K = 1$, then we can take $x = I_4$ and write $E(Z,s,m,\psi,N)$ for $E_{I_4}(Z,s,m,\psi,N)$. Then it follows from Theorem 5.1 that $E(Z,s,m,\psi,N)$ is an eigenfunction for all the Hecke operators at primes $p \nmid N$, and one can compute its eigenvalues using Corollary 5.2. Moreover, in this case the element $(p,1) \in \mathcal{O}_{K,p}$ represents the principal ideal $\pi\mathcal{O}_K$, where π is a generator of $\mathfrak{p}$, and one can compute

$$\psi_p((p,1)) = \psi_\infty(\pi)^{-1}\psi_N(\pi)^{-1} = \pi^{-m}p^{m/2}\psi_N(\pi)^{-1}.$$

If $h_K > 1$ the series $E(g,\dots)$ corresponds to h_K-many functions $E_x(Z,\dots)$ of $\mathbf{H}_2$, where $\det x$ runs over distinct ideal classes of $\mathcal{O}_K$. Then each $E_x(Z,\dots)$ is an eigenform for all the Hecke operators in Corollary 5.2 except for $T_{\mathfrak{p}}$, where $\mathfrak{p}$ is nonprincipal. However, taken together as an h_K-tuple the Eisenstein series $E_x(Z,\dots)$ can be interpreted as an eigenvector of the operator $T_{\mathfrak{p}}$.

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