THE MAASS SPACE FOR $\mathrm{U}(2,2)$ AND THE BLOCH-KATO CONJECTURE FOR THE SYMMETRIC SQUARE MOTIVE OF A MODULAR FORM

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ABSTRACT. Let $K = \mathbf{Q}(i\sqrt{D_K})$ be an imaginary quadratic field of discriminant $-D_K$. We introduce a notion of an adelic Maass space $\mathcal{S}^{\mathrm{M}}_{k,-k/2}$ for automorphic forms on the quasi-split unitary group $\mathrm{U}(2,2)$ associated with K and prove that it is stable under the action of all Hecke operators. When D_K is prime we obtain a Hecke-equivariant descent from $\mathcal{S}^{\mathrm{M}}_{k,-k/2}$ to the space of elliptic cusp forms $S_{k-1}(D_K, \chi_K)$, where χ_K is the quadratic character of K. For a given $\phi \in S_{k-1}(D_K, \chi_K)$, a prime $\ell > k$, we then construct (mod ℓ) congruences between the Maass form corresponding to ϕ and hermitian modular forms orthogonal to $\mathcal{S}^{\mathrm{M}}_{k,-k/2}$ whenever $\mathrm{val}_\ell(L^{\mathrm{alg}}(\mathrm{Symm}^2\,\phi, k)) > 0$. This gives a proof of the holomorphic analogue of the unitary version of Harder's conjecture. Finally, we use these congruences to provide evidence for the Bloch-Kato conjecture for the motives $\mathrm{Symm}^2\,\rho_\phi(k-3)$ and $\mathrm{Symm}^2\,\rho_\phi(k)$, where ρ_ϕ denotes the Galois representation attached to ϕ.

1. INTRODUCTION

In 1990 Bloch and Kato [6] formulated a conjecture whose version relates the order of a Selmer group of a motive M to a special value of an L-function of M. This is a very far-reaching conjecture which is currently known only in a handful of cases, mostly concerning the situations when M arises from a one-dimensional Galois representation. However, in 2004 Diamond, Flach and Guo proved a very strong result in a three-dimensional case [12]. Indeed, they proved the Bloch-Kato conjecture for the adjoint motive $\mathrm{ad}^0\,\rho_\phi$ (and its Tate twist $\mathrm{ad}^0\,\rho_\phi(1)$) of the ℓ-adic Galois representation ρ_ϕ attached to a classical modular form ϕ without any restrictions on the weight (≥ 2) or the level of ϕ. Their proof was highly influenced by the ideas that were first applied by Taylor and Wiles in their proof of Fermat's Last Theorem ([43], [48]).

In 2009 the author proved a (weaker) result providing evidence for the conjecture for a different Tate twist of $\mathrm{ad}^0\,\rho_\phi$ (more precisely for $\mathrm{ad}^0\,\rho_\phi(-1)\chi = \mathrm{Symm}^2\,\rho_\phi(k-3)$ and $\mathrm{ad}^0\,\rho_\phi(2)\chi = \mathrm{Symm}^2\,\rho_\phi(k)$) for modular forms ϕ of any weight $k-1$ (with k divisible by 4), level 4 and non-trivial character χ [27]. The method was different from that of [12] (but similar to the one used by Brown [9] who worked with Saito-Kurokawa lifts). It relied on constructing congruences between a certain lift of ϕ (called the *Maass lift*) to the unitary group $\mathrm{U}(2,2)$ defined with respect to the field $\mathbf{Q}(i)$ and hermitian modular forms (i.e., forms on $\mathrm{U}(2,2)$) which were orthogonal

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to the *Maass space* (the span of such lifts). The elements in the relevant Selmer groups were then constructed using ideas of Urban [44]. Unfortunately, some of the methods implemented in [27] relied significantly on the fact that the class number of $\mathbf{Q}(i)$ is one and could not be directly generalized to deal with other imaginary quadratic fields.

In this paper we develop new tools - among them a new notion of an adelic Maass space and a Rankin-Selberg type formula - which work in sufficient generality. As a consequence we are in particular able to extend the results of [27] to all imaginary quadratic fields of prime discriminant $-D_K$, i.e., to all modular forms ϕ of any prime level D_K, of arbitrary weight $k-1$ (with k divisible by the number of roots of unity contained in K), and nebentypus χ_K being the quadratic character associated with the extension $K/\mathbf{Q}$. Our result on the one hand provides evidence for the Bloch-Kato conjecture for the motives $\mathrm{Symm}^2\,\rho_\phi(k-3)$ and $\mathrm{Symm}^2\,\rho_\phi(k)$ for a rather broad family of modular forms ϕ. On the other hand the congruence itself (between a Maass lift and a hermitian modular form orthogonal to the Maass space) provides a proof of a holomorphic analogue to a conjecture recently formulated by Dummigan extending the so-called Harder's conjecture concerning Siegel modular forms [13].

As alluded above, the first difficulty that one encounters in dealing with a general imaginary quadratic field is the lack of a proper notion of the Maass space in this case. The definition introduced by Krieg [29] does not allow one to define the action of the Hecke operators at non-principal primes. The more recent (very elegant) results due to Ikeda [25] while dealing with class number issues, are not quite sufficient for our purposes. So, in this paper we introduce a new adelic version of the Maass space and carefully study its properties, especially its invariance under the action of the Hecke algebras. This provides us with a correct analogue of the classical Maass lift to the space of automorphic forms on $\mathrm{U}(2,2)(\mathbf{A})$ for any imaginary quadratic field of prime discriminant. We then proceed to construct a congruence between the Maass lift and hermitian modular forms orthogonal to the Maass space.

The method of exhibiting elements in Selmer groups of automorphic forms via constructing congruences between automorphic forms on a higher-rank group has been used by several authors. The original idea can be dated back to the influential paper of Ribet [33] on the converse to Herbrand's Theorem, where for a certain family of Dirichlet characters χ elements in the χ-eigenspace of the class group of a cyclotomic field are constructed by first exhibiting a congruence between a certain Eisenstein series (associated with χ) and a cusp form on GL_2. Higher-rank analogues of this method have been applied to provide evidence for (one inequality in) the Bloch-Kato conjecture for several motives by Bellaïche and Chenevier, Brown, Berger, Böcherer, Dummigan, Schulze-Pillot, as well as Agarwal and the author ([2], [9], [4], [27], [7], [1]). An extension of these ideas was also used to prove results towards the Main Conjecture of Iwasawa Theory by Mazur and Wiles, Urban ([30], [44]) and very recently by Skinner and Urban [41].

The general idea is the following. Given an automorphic form ϕ on an algebraic group M (with an associated Galois representation ρ_ϕ) one lifts ϕ to an automorphic form on G in which M can be realized as a subgroup (in our case $M = \mathrm{Res}_{K/\mathbf{Q}}(\mathrm{GL}_{2/K})$ is the Levi subgroup of a Siegel parabolic of $G = \mathrm{U}(2,2)$). The Galois representation attached to the lift is reducible and has irreducible components related to ρ_ϕ. Assuming divisibility (by a uniformizer ϖ in some extension of $\mathbf{Q}_\ell$) of a certain L-value associated with ϕ one shows (this is usually the technically difficult part) that the lift is congruent (mod ϖ) to an automorphic form π on G whose Galois representation ρ_π is irreducible. Because of the congruence, the mod ϖ reduction of ρ_π must be reducible, but (because ρ_π was irreducible) it can be chosen to represent a non-split extension of its irreducible components, thus giving rise to a non-zero element in some Selmer group (related to ρ_ϕ).

Let $\phi \in S_{k-1}(D_K, \chi_K)$ be a newform. In our case the lifting procedure is the Maass lift, which produces an automorphic form $f_{\phi,\chi}$ on $\mathrm{U}(2,2)(\mathbf{A})$ (which depends on a certain character χ of the class group of K) whose associated automorphic representation is CAP in the sense of Piatetski-Shapiro [32]. Even though the desired congruence is between Hecke eigenvalues of the lift $f_{\phi,\chi}$ and those of π, we first construct a congruence between Fourier coefficients of these forms and only then deduce the Hecke eigenvalue congruence. The former congruence is achieved by first defining a certain hermitian modular form Ξ (essentially a product a Siegel Eisenstein series and a hermitian theta series) and writing it as:

$$\Xi = \frac{\langle \Xi, f_{\phi,\chi}\rangle}{\langle f_{\phi,\chi}, f_{\phi,\chi}\rangle} f_{\phi,\chi} + g',$$

where g' is a hermitian modular form orthogonal to $f_{\phi,\chi}$. The form Ξ has nice arithmetic properties (in particular its Fourier coefficients are ϖ-adically integral) and we show that the inner products can be expressed by certain L-values. In particular the inner product in the denominator is related to $L^{\mathrm{alg}}(\mathrm{Symm}^2 \phi, k)$. Choosing Ξ so that the special L-values contributing to the inner product in the numerator make it a ϖ-adic unit and assuming the ϖ-adic valuation of $L^{\mathrm{alg}}(\mathrm{Symm}^2 \phi, k)$ is positive we get a congruence between $f_{\phi,\chi}$ and a scalar multiple g of g'. To ensure that g itself is not a Maass lift we construct a certain Hecke operator T^{h} that kills the "Maass part" of g.

Let us now briefly elaborate on the technical difficulties that one encounters in the current paper as opposed to the case of $K = \mathbf{Q}(i)$ which was studied in [27]. First of all, as mentioned above, the Maass space and the Maass lift in the case of the field $\mathbf{Q}(i)$ are well-understood thanks to the work of Kojima, Gritsenko and Krieg ([28], [16], [17], [29]) and in [27] we simply invoke the relevant definitions and properties of these objects. In the current paper we introduce a notion of an adelic Maass space for a general imaginary quadratic field and prove that it is a Hecke-stable subspace of the space of hermitian modular forms. This result does not use the assumption that D_K is prime. One of the difficulties in extending the classical notion of Kojima and Krieg is the fact that when the class number of K is greater than one the classical Maass space (which was defined by Krieg for all imaginary quadratic fields) is not stable under the action of the local Hecke algebras at non-principal primes of K. This is one of the reasons why we chose to formulate the theory in an adelic language, even though it would in principle be possible to extend the classical definitions of Krieg and work with several copies of the hermitian upper half-space. However, we think that the action of the Hecke operators as well as the role played by a central character are most transparent in the adelic setting. When D_K is prime we are able to relate our lift to the results of Krieg and Gritsenko and derive explicit formulas for the descent of the Hecke operators. We also prove an L-function identity relating the standard L-function of a Maass lift to the L-function of the base change to K of the modular form ϕ. All this is the content of section 5. A yet another notion of the Maass lift has in the meantime been introduced by Ikeda [25] using a different approach. This notion agrees with ours in the case of a trivial central character, but not all the formulas necessary for our arithmetic applications are present in [25].

On the other hand a lot of attention in [27] was devoted to computing the Petersson norm of a Maass lift ([27], section 4). Here, however, we use a formula due to Sugano (cf. [25]) to tackle the problem. To derive the congruence one also needs to be able to express the inner product $\langle \Xi, f_{\phi,\chi}\rangle$ by certain L-values related to ϕ. This calculation (drawing heavily on the work of Shimura) becomes somewhat involved in the case of class number larger than one. The relevant computations are carried out in section 7. Since it does not add much to the computational complexity we prove all the results for the group $\mathrm{U}(n,n)$ for a general $n>1$, obtaining this way a general Rankin-Selberg type formula that might be of independent interest (Theorem 7.8). In that section we also prove the integrality of the Fourier coefficients of a certain hermitian theta series involved in the definition of Ξ. Finally in the current paper our construction of the Hecke operator T^{h} is somewhat cleaner, because we work more with completed Hecke algebras. This is carried out in section 6.

In section 8 we collect all the results to arrive at the desired congruence, first between the Fourier coefficients of $f_{\phi,\chi}$ and a hermitian modular form orthogonal to the Maass space (Theorem 8.10) and then between the Hecke eigenvalues of $f_{\phi,\chi}$ and those of some hermitian Hecke eigenform f also orthogonal to the Maass space (Corollary 8.17). The latter can only be achieved modulo the first power of ϖ even if $L^{\mathrm{alg}}(\mathrm{Symm}^2\,\phi,k)$ is divisible by a higher power of ϖ. This is not a shortcoming of our method but a consequence of the fact that there may be more than one f congruent to $f_{\phi,\chi}$ and the L-value (conjecturally) controls contributions from all such f. In fact this is precisely what we prove by studying congruences between $f_{\phi,\chi}$ and *all* the possible eigenforms f orthogonal to the Maass space and as a result give a lower bound (in terms of $L^{\mathrm{alg}}(\mathrm{Symm}^2\,\phi,k)$) on the index of an analogue of the classical Eisenstein ideal (which we in our case call the *Maass ideal*) in the appropriate Hecke algebra (section 8.3). Finally, we demonstrate how our results imply the holomorphic analogue of Dummigan's version of Harder's conjecture for the group $\mathrm{U}(2,2)$ (section 8.4).

The congruence can be used to deduce the existence of certain non-zero elements in the Selmer groups $H^1_f(K,\mathrm{ad}^0\,\rho_f(-1))$ and $H^1_f(K,\mathrm{ad}^0\,\rho_f(2))$ and hence get a result towards the Bloch-Kato conjecture for these motives. For this we use a theorem of Urban [44]. The relevant results are stated in section 9. We note that the above Selmer groups are over K, while the conjecture relates $L^{\mathrm{alg}}(\mathrm{Symm}^2\,\phi,k)$ to the order of the corresponding Selmer group over $\mathbf{Q}$. More precisely, under some mild assumptions one has $\mathrm{val}_\ell(L^{\mathrm{alg}}(\mathrm{Symm}^2\,\phi,k)) = \mathrm{val}_\ell(L^{\mathrm{alg}}(\mathrm{Symm}^2\,\phi,k-3)) = \mathrm{val}_\ell(L^{\mathrm{alg}}(\mathrm{ad}\,\phi,-1,\chi_K))$, so $L^{\mathrm{alg}}(\mathrm{Symm}^2\,\phi,k)$ should control the order of $H^1_f(\mathbf{Q},\mathrm{Symm}^2\,\rho_\phi(k-3)) = H^1_f(\mathbf{Q},\mathrm{ad}^0\,\rho_\phi(-1)\chi_K)$ (we are grateful to Neil Dummigan for pointing out an error in an earlier version of the article).

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2. Notation and terminology

In this section we introduce some basic concepts and establish notation which will be used throughout this paper unless explicitly indicated otherwise.

2.1. **Number fields and Hecke characters.** Throughout this paper ℓ will always denote a fixed odd prime. We write i for $\sqrt{-1}$. Let $K = \mathbf{Q}(i\sqrt{D_K})$ be a fixed imaginary quadratic extension of $\mathbf{Q}$ of discriminant $-D_K$, and let $\mathcal{O}_K$ be the ring of integers of K. We will write Cl_K for the class group of K and h_K for $\#\,\mathrm{Cl}_K$. For $\alpha \in K$, denote by $\overline{\alpha}$ the image of α under the non-trivial automorphism of K. Set $N\alpha := N(\alpha) := \alpha\overline{\alpha}$, and for an ideal $\mathfrak{n}$ of $\mathcal{O}_K$, set $N\mathfrak{n} := \#(\mathcal{O}_K/\mathfrak{n})$. As remarked below we will always view K as a subfield of $\mathbf{C}$. For $\alpha \in \mathbf{C}$, $\overline{\alpha}$ will denote the complex conjugate of α and we set $|\alpha| := \sqrt{\alpha\overline{\alpha}}$.

Let L be a number field with ring of integers $\mathcal{O}_L$. For a place v of L, denote by L_v the completion of L at v and by $\mathcal{O}_{L,v}$ the valuation ring of L_v. If p is a place of $\mathbf{Q}$, we set $L_p := \mathbf{Q}_p \otimes_{\mathbf{Q}} L$ and $\mathcal{O}_{L,p} := \mathbf{Z}_p \otimes_{\mathbf{Z}} \mathcal{O}_L$. We also allow $p = \infty$. Set $\hat{\mathbf{Z}} = \varprojlim_n \mathbf{Z}/n\mathbf{Z} = \prod_{p\nmid\infty} \mathbf{Z}_p$ and similarly $\hat{\mathcal{O}}_K = \prod_{v\nmid\infty} \mathcal{O}_{K,v}$. For a finite p, let val_p denote the p-adic valuation on $\mathbf{Q}_p$. For notational convenience we also define $\mathrm{val}_p(\infty) := \infty$. If $\alpha \in \mathbf{Q}_p$, then $|\alpha|_{\mathbf{Q}_p} := p^{-\mathrm{val}_p(\alpha)}$ denotes the p-adic norm of α. For $p = \infty$, $|\cdot|_{\mathbf{Q}_\infty} = |\cdot|_{\mathbf{R}} = |\cdot|$ is the usual absolute value on $\mathbf{Q}_\infty = \mathbf{R}$.

In this paper we fix once and for all an algebraic closure $\overline{\mathbf{Q}}$ of the rationals and algebraic closures $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$, as well as compatible embeddings $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$ for all finite places p of $\mathbf{Q}$. We extend val_p to a function from $\overline{\mathbf{Q}}_p$ into $\mathbf{Q}$. Let L be a number field. We write G_L for $\mathrm{Gal}(\overline{\mathbf{Q}}/L)$. If $\mathfrak{p}$ is a prime of L, we also write $D_{\mathfrak{p}} \subset G_L$ for the decomposition group of $\mathfrak{p}$ and $I_{\mathfrak{p}} \subset D_{\mathfrak{p}}$ for the inertia group of $\mathfrak{p}$. The chosen embeddings allow us to identify $D_{\mathfrak{p}}$ with $\mathrm{Gal}(\overline{L}_{\mathfrak{p}}/L_{\mathfrak{p}})$. We will always write $\mathrm{Frob}_{\mathfrak{p}} \in D_{\mathfrak{p}}/I_{\mathfrak{p}}$ to denote the *arithmetic* Frobenius.

For a local field E (which for us will always be a finite extension of $\mathbf{Q}_p$ for some prime p) and a choice of a uniformizer $\varpi \in E$, we will write $\mathrm{val}_\varpi : E \to \mathbf{Z}$ for the ϖ-adic valuation on E.

For a number field L let $\mathbf{A}_L$ denote the ring of adeles of L and put $\mathbf{A} := \mathbf{A}_{\mathbf{Q}}$. Write $\mathbf{A}_{L,\infty}$ and $\mathbf{A}_{L,\mathrm{f}}$ for the infinite part and the finite part of $\mathbf{A}_L$ respectively. For $\alpha = (\alpha_p) \in \mathbf{A}$ set $|\alpha|_{\mathbf{A}} := \prod_p |\alpha|_{\mathbf{Q}_p}$. By a *Hecke character* of $\mathbf{A}_L^\times$ (or of L, for short) we mean a continuous homomorphism

$$\psi : L^\times \setminus \mathbf{A}_L^\times \to \mathbf{C}^\times.$$

The trivial Hecke character will be denoted by $\mathbf{1}$. The character ψ factors into a product of local characters $\psi = \prod_v \psi_v$, where v runs over all places of L. If $\mathfrak{n}$ is the ideal of the ring of integers $\mathcal{O}_L$ of L such that

- $\psi_v(x_v) = 1$ if v is a finite place of L, $x_v \in \mathcal{O}_{L,v}^\times$ and $x - 1 \in \mathfrak{n}\mathcal{O}_{L,v}$
- no ideal $\mathfrak{m}$ strictly containing $\mathfrak{n}$ has the above property,

then $\mathfrak{n}$ will be called the *conductor of* ψ. If $\mathfrak{m}$ is an ideal of $\mathcal{O}_L$, then we set $\psi_{\mathfrak{m}} := \prod \psi_v$, where the product runs over all the finite places v of L such that

$v \mid \mathfrak{m}$. For a Hecke character ψ of $\mathbf{A}_L^\times$, denote by ψ^* the associated ideal character. Let ψ be a Hecke character of $\mathbf{A}_K^\times$. We will sometimes think of ψ as a character of $(\mathrm{Res}_{K/\mathbf{Q}}\ \mathrm{GL}_{1/K})(\mathbf{A})$, where $\mathrm{Res}_{K/\mathbf{Q}}$ stands for the Weil restriction of scalars. We have a factorization $\psi = \prod_p \psi_p$ into local characters $\psi_p : \left(\mathrm{Res}_{K/\mathbf{Q}}\ \mathrm{GL}_{1/K}\right)(\mathbf{Q}_p) \to \mathbf{C}^\times$. For $M \in \mathbf{Z}$, we set $\psi_M := \prod_{p\neq\infty,\, p\mid M} \psi_p$. If ψ is a Hecke character of $\mathbf{A}_K^\times$, we set $\psi_{\mathbf{Q}} = \psi|_{\mathbf{A}^\times}$.

2.2. **The unitary group.** For any affine group scheme X over $\mathbf{Z}$ and any $\mathbf{Z}$-algebra A, we denote by $x \mapsto \overline{x}$ the automorphism of $(\mathrm{Res}_{\mathcal{O}_K/\mathbf{Z}}\, X_{\mathcal{O}_K})(A)$ induced by the non-trivial automorphism of $K/\mathbf{Q}$. Note that $(\mathrm{Res}_{\mathcal{O}_K/\mathbf{Z}}\, X_{\mathcal{O}_K})(A)$ can be identified with a subgroup of $\mathrm{GL}_n(A \otimes \mathcal{O}_K)$ for some n. In what follows we always specify such an identification. Then for $x \in (\mathrm{Res}_{\mathcal{O}_K/\mathbf{Z}}\, X_{\mathcal{O}_K})(A)$ we write x^t for the transpose of x, and set $x^* := \overline{x}^t$ and $\hat{x} := (\overline{x}^t)^{-1}$. Moreover, we write $\mathrm{diag}(a_1, a_2, \dots, a_n)$ for the $n \times n$-matrix with $a_1, a_2, \dots a_n$ on the diagonal and all the off-diagonal entries equal to zero.

We will denote by $\mathbf{G}_a$ the additive group and by $\mathbf{G}_m$ the multiplicative group. To the imaginary quadratic extension $K/\mathbf{Q}$ one associates the unitary similitude group scheme over $\mathbf{Z}$:

$$G_n := \mathrm{GU}(n,n) = \{A \in \mathrm{Res}_{\mathcal{O}_K/\mathbf{Z}}\, \mathrm{GL}_{2n} \mid AJ\bar{A}^t = \mu(A)J\},$$

where $J = \begin{bmatrix} & -I_n \\ I_n & \end{bmatrix}$, with I_n denoting the $n \times n$ identity matrix and $\mu(A) \in \mathbf{G}_m$. We will also make use of the groups

$$U_n = \mathrm{U}(n,n) = \{A \in \mathrm{GU}(n,n) \mid \mu(A) = 1\},$$

and

$$\mathrm{SU}(n,n) = \{A \in U_n \mid \det A = 1\}.$$

For $x \in \mathrm{Res}_{\mathcal{O}_K/\mathbf{Z}}(\mathrm{GL}_n)$, we write p_x for $\begin{bmatrix} x & \\ & \hat{x} \end{bmatrix} \in U_n$. Since the case $n = 2$ will be of particular interest to us we set $G = G_2$, $U = U_2$.

Note that if p is inert or ramified in K, then $K_p/\mathbf{Q}_p$ is a degree two extension of local fields and $a \mapsto \overline{a}$ induces the non-trivial automorphism in $\mathrm{Gal}(K_p/\mathbf{Q}_p)$. If p splits in K, denote by $\iota_{p,1}, \iota_{p,2}$ the two distinct embeddings of K into $\mathbf{Q}_p$. Then the map $a \otimes b \mapsto (\iota_{p,1}(a)b, \iota_{p,2}(a)b)$, induces a $\mathbf{Q}_p$-algebra isomorphism $K_p \cong \mathbf{Q}_p \times \mathbf{Q}_p$, and $a \mapsto \overline{a}$ corresponds on the right-hand side to the automorphism defined by $(a,b) \mapsto (b,a)$. We denote the isomorphism $\mathbf{Q}_p \times \mathbf{Q}_p \xrightarrow{\sim} K_p$ by ι_p. For a matrix $g = (g_{ij})$ with entries in $\mathbf{Q}_p \times \mathbf{Q}_p$ we also set $\iota_p(g) = (\iota_p(g_{ij}))$. For a split prime p the map ι_p^{-1} identifies $U_n(\mathbf{Q}_p)$ with

$$U_{n,p} = \{(g_1, g_2) \in \mathrm{GL}_{2n}(\mathbf{Q}_p) \times \mathrm{GL}_{2n}(\mathbf{Q}_p) \mid g_1 J g_2^t = J\}.$$

Note that the map $(g_1, g_2) \mapsto g_1$ gives a (non-canonical) isomorphism $U_n(\mathbf{Q}_p) \cong \mathrm{GL}_{2n}(\mathbf{Q}_p)$. Similarly, one has $G_n(\mathbf{Q}_p) \cong \mathrm{GL}_{2n}(\mathbf{Q}_p) \times \mathbf{G}_m(\mathbf{Q}_p)$.

In $U = U_2$ we choose a maximal torus

$$T = \left\{ \begin{bmatrix} a &&& \\ & b && \\ && \hat{a} & \\ &&& \hat{b} \end{bmatrix} \mid a, b \in \mathrm{Res}_{K/\mathbf{Q}}\, \mathbf{G}_{m/K} \right\},$$

and a Borel subgroup $B = TU_B$ with unipotent radical

$$U_B = \left\{ \begin{bmatrix} 1 & \alpha & \beta & \gamma \\ & 1 & \bar{\gamma} - \bar{\alpha}\phi & \phi \\ & & 1 & \\ & & -\bar{\alpha} & 1 \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathrm{Res}_{K/\mathbf{Q}} \mathbf{G}_{a/K}, \ \ \phi \in \mathbf{G}_a, \ \ \beta + \gamma\bar{\alpha} \in \mathbf{G}_a \right\}.$$

Let

$$T_{\mathbf{Q}} = \left\{ \begin{bmatrix} a & & & \\ & b & & \\ & & a^{-1} & \\ & & & b^{-1} \end{bmatrix} \mid a, b \in \mathbf{G}_m \right\}$$

denote the maximal $\mathbf{Q}$-split torus contained in T. Let $R(U)$ be the set of roots of $T_{\mathbf{Q}}$, and denote by e_j, $j = 1, 2$, the root defined by

$$e_j : \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & a_1^{-1} & \\ & & & a_2^{-1} \end{bmatrix} \mapsto a_j.$$

The choice of B determines a subset $R^+(U) \subset R(U)$ of positive roots. We have

$$R^+(U) = \{e_1 + e_2, e_1 - e_2, 2e_1, 2e_2\}.$$

We fix a set $\Delta(U) \subset R^+(U)$ of simple roots

$$\Delta(U) := \{e_1 - e_2, 2e_2\}.$$

If $\theta \subset \Delta(U)$, denote the parabolic subgroup corresponding to θ by P_θ. We have $P_{\Delta(U)} = U$ and $P_\emptyset = B$. The other two possible subsets of $\Delta(U)$ correspond to maximal $\mathbf{Q}$-parabolics of U:

- the Siegel parabolic $P := P_{\{e_1 - e_2\}} = M_P U_P$ with Levi subgroup

  $$M_P = \left\{ \begin{bmatrix} A & \\ & \hat{A} \end{bmatrix} \mid A \in \mathrm{Res}_{K/\mathbf{Q}} \ \mathrm{GL}_{2/K} \right\},$$

  and (abelian) unipotent radical

  $$U_P = \left\{ \begin{bmatrix} 1 & & b_1 & b_2 \\ & 1 & \bar{b}_2 & b_4 \\ & & 1 & \\ & & & 1 \end{bmatrix} \mid b_1, b_4 \in \mathbf{G}_a, \ \ b_2 \in \mathrm{Res}_{K/\mathbf{Q}} \mathbf{G}_{a/K} \right\}$$

- the Klingen parabolic $Q := P_{\{2e_2\}} = M_Q U_Q$ with Levi subgroup

  $$M_Q = \left\{ \begin{bmatrix} x & & & \\ & a & & b \\ & & \hat{x} & \\ & c & & d \end{bmatrix} \mid x \in \mathrm{Res}_{K/\mathbf{Q}} \mathbf{G}_{m/K}, \ \ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{U}(1,1) \right\},$$

  and (non-abelian) unipotent radical

  $$U_Q = \left\{ \begin{bmatrix} 1 & \alpha & \beta & \gamma \\ & 1 & \bar{\gamma} & \\ & & 1 & \\ & & -\bar{\alpha} & 1 \end{bmatrix} \mid \alpha, \beta, \gamma \in \mathrm{Res}_{K/\mathbf{Q}} \mathbf{G}_{a/K}, \ \ \beta + \gamma\bar{\alpha} \in \mathbf{G}_a \right\}$$

Similarly in U_n we denote by T_n the diagonal torus and by P_n the Siegel parabolic (with Levi isomorphic to $\mathrm{Res}_{K/\mathbf{Q}} \mathrm{GL}_{n/K}$).
For an associative ring R with identity and an R-module N we write $M_n(N)$ for the R-module of $n \times n$-matrices with entries in N. Let $x = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in M_{2n}(N)$ with $A, B, C, D \in M_n(N)$. Define $a_x = A$, $b_x = B$, $c_x = C$, $d_x = D$.

For $M \in \mathbf{Q}$, $N \in \mathbf{Z}$ such that $MN \in \mathbf{Z}$ we will denote by $D_n(M, N)$ the group $U_n(\mathbf{R}) \prod_{p \nmid \infty} \mathcal{K}_{0,n,p}(M, N) \subset U_n(\mathbf{A})$, where

$$\mathcal{K}_{0,n,p}(M, N) = \{x \in U_n(\mathbf{Q}_p) \mid a_x, d_x \in M_n(\mathcal{O}_{K,p}) , \quad b_x \in M_n(M^{-1}\mathcal{O}_{K,p}),\ \ c_x \in M_n(MN\mathcal{O}_{K,p})\} . \tag{2.1}$$

If $M = 1$, denote $D_n(M, N)$ simply by $D_n(N)$ and $\mathcal{K}_{0,n,p}(M, N)$ by $\mathcal{K}_{0,n,p}(N)$. For any finite p, the group $\mathcal{K}_{0,n,p} := \mathcal{K}_{0,n,p}(1) = U_n(\mathbf{Z}_p)$ is a maximal (open) compact subgroup of $U_n(\mathbf{Q}_p)$. Note that if $p \nmid N$, then $\mathcal{K}_{0,n,p} = \mathcal{K}_{0,n,p}(N)$. We write $\mathcal{K}_{0,n,\mathrm{f}}(N) := \prod_{p\nmid\infty} \mathcal{K}_{0,n,p}(N)$ and $\mathcal{K}_{0,n,\mathrm{f}} := \mathcal{K}_{0,\mathrm{f}}(1)$. Note that $\mathcal{K}_{0,n,\mathrm{f}}$ is a maximal (open) compact subgroup of $U_n(\mathbf{A}_\mathrm{f})$. Set

$$\mathcal{K}^+_{0,n,\infty} := \left\{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in U_n(\mathbf{R}) \mid A, B \in \mathrm{GL}_n(\mathbf{C}), AA^* + BB^* = I_n, AB^* = BA^* \right\}.$$

Then $\mathcal{K}^+_{0,n,\infty}$ is a maximal compact subgroup of $U_n(\mathbf{R})$. We will denote by $\mathcal{K}_{0,n,\infty}$ the subgroup of $G_n(\mathbf{R})$ generated by $\mathcal{K}^+_{0,n,\infty}$ and J. Then $\mathcal{K}_{0,n,\infty}$ is a maximal compact subgroup of $G_n(\mathbf{R})$. Let

$$U(m) := \{A \in \mathrm{GL}_m(\mathbf{C}) \mid AA^* = I_m\} .$$

We have

$$\mathcal{K}^+_{0,n,\infty} = U_n(\mathbf{R}) \cap U(2n) \xrightarrow{\sim} U(n) \times U(n),$$

where the last isomorphism is given by

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix} \mapsto (A + iB, A - iB) \in U(n) \times U(n).$$

Finally, set $\mathcal{K}_{0,n}(N) := \mathcal{K}^+_{0,n,\infty}\mathcal{K}_{0,n,\mathrm{f}}(N)$ and $\mathcal{K}_{0,n} := \mathcal{K}_{0,n}(1)$. The last group is a maximal compact subgroup of $U_n(\mathbf{A})$.

Similarly, we define $\mathcal{K}_{1,n}(N) = \mathcal{K}^+_{0,n,\infty}\mathcal{K}_{1,n,\mathrm{f}}(N)$, where $\mathcal{K}_{1,n,\mathrm{f}}(N) = \prod_{p\nmid\infty} \mathcal{K}_{1,n,p}(N)$,

$$\mathcal{K}_{1,n,p}(N) = \{x \in \mathcal{K}_{0,n,p}(N) \mid a_x - I_n \in M_n(N\mathcal{O}_{K,p})\}.$$

Let $M \in \mathbf{Q}$, $N \in \mathbf{Z}$ be such that $MN \in \mathbf{Z}$. We define the following congruence subgroups of $U_n(\mathbf{Q})$:

$$\begin{aligned} \Gamma^{\mathrm{h}}_{0,n}(M, N) &:= U_n(\mathbf{Q}) \cap D_n(M, N), \\ \Gamma^{\mathrm{h}}_{1,n}(M, N) &:= \{\alpha \in \Gamma^{\mathrm{h}}_{0,n}(M, N) \mid a_\alpha - 1 \in M_n(N\mathcal{O}_K)\},, \\ \Gamma^{\mathrm{h}}_{n}(M, N) &:= \{\alpha \in \Gamma^{\mathrm{h}}_{1,n}(M, N) \mid b_\alpha \in M_n(M^{-1}N\mathcal{O}_K)\} \end{aligned} \tag{2.2}$$

and set $\Gamma^{\mathrm{h}}_{0,n}(N) := \Gamma^{\mathrm{h}}_{0,n}(1, N)$, $\Gamma^{\mathrm{h}}_{1,n}(N) := \Gamma^{\mathrm{h}}_{1,n}(1, N)$ and $\Gamma^{\mathrm{h}}_{n}(N) := \Gamma^{\mathrm{h}}_{n}(1, N)$. When $n = 2$ we drop it from notation. Note that the groups $\Gamma^{\mathrm{h}}_0(N)$, $\Gamma^{\mathrm{h}}_1(N)$ and $\Gamma^{\mathrm{h}}(N)$ are U_n-analogues of the standard congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$ and $\Gamma(N)$ of $\mathrm{SL}_2(\mathbf{Z})$. In general the superscript 'h' will indicate that an object is in some way related to the group U_n. The letter 'h' stands for 'hermitian', as this is the standard name of modular forms on U_n.

2.3. **Modular forms.** In this paper we will make use of the theory of modular forms on congruence subgroups of two different groups: $\mathrm{SL}_2(\mathbf{Z})$ and $U(\mathbf{Z})$. We will use both the classical and the adelic formulation of the theories. In the adelic framework one usually speaks of automorphic forms rather than modular forms and in this case SL_2 is usually replaced with GL_2. For more details see e.g. [15], chapter 3. In the classical setting the modular forms on congruence subgroups of $\mathrm{SL}_2(\mathbf{Z})$ will be referred to as *elliptic modular forms*, and those on congruence subgroups of $\Gamma_{\mathbf{Z}}$ as *hermitian modular forms*. Since the theory of elliptic modular forms is well-known we will only summarize the main facts below. Section 3 will be devoted to hermitian modular forms.

2.3.1. *Elliptic modular forms.* The theory of elliptic modular forms is well-known, so we omit most of the definitions and refer the reader to standard sources, e.g. [31]. Let

$$\mathbf{H} := \{z \in \mathbf{C} \mid \mathrm{Im}\,(z) > 0\}$$

denote the complex upper half-plane. In the case of elliptic modular forms we will denote by $\Gamma_0(N)$ the subgroup of $\mathrm{SL}_2(\mathbf{Z})$ consisting of matrices whose lower-left entries are divisible by N, and by $\Gamma_1(N)$ the subgroup of $\Gamma_0(N)$ consisting of matrices whose upper left entries are congruent to 1 modulo N. Let $\Gamma \subset \mathrm{SL}_2(\mathbf{Z})$ be a congruence subgroup. Set $M_m(\Gamma)$ (resp. $S_m(\Gamma)$) to denote the $\mathbf{C}$-space of elliptic modular forms (resp. cusp forms) of weight m and level Γ. We also denote by $M_m(N, \psi)$ (resp. $S_m(N, \psi)$) the space of elliptic modular forms (resp. cusp forms) of weight m, level N and character ψ. For $f, g \in M_m(\Gamma)$ with either f or g a cusp form, and $\Gamma' \subset \Gamma$ a finite index subgroup, we define the Petersson inner product

$$\langle f, g\rangle_{\Gamma'} := \int_{\Gamma'\backslash\mathbf{H}} f(z)\overline{g(z)}(\mathrm{Im}\, z)^{m-2}\, dx\, dy,$$

and set

$$\langle f, g\rangle := \frac{1}{[\overline{\mathrm{SL}_2(\mathbf{Z})} : \overline{\Gamma}']} \langle f, g\rangle_{\Gamma'},$$

where $\overline{\mathrm{SL}_2(\mathbf{Z})} := \mathrm{SL}_2(\mathbf{Z})/\langle -I_2\rangle$ and $\overline{\Gamma}'$ is the image of Γ' in $\overline{\mathrm{SL}_2(\mathbf{Z})}$. The value $\langle f, g\rangle$ is independent of Γ'.

Every elliptic modular form $f \in M_m(N, \psi)$ possesses a Fourier expansion $f(z) = \sum_{n=0}^{\infty} a(n)q^n$, where throughout this paper in such series q will denote $e(z) := e^{2\pi i z}$. For $\gamma = \left[\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right] \in \mathrm{GL}_2^+(\mathbf{R})$, set $j(\gamma, z) = cz + d$.

Let $D = D_K$ be a prime. In this paper we will be particularly interested in the space $S_m(D_K, \chi_K)$, where χ_K is the quadratic character of $(\mathbf{Z}/D\mathbf{Z})^{\times}$ associated with the extension $K = \mathbf{Q}(\sqrt{-D})$. Regarded as a function $\mathbf{Z} \to \{1, -1\}$, it assigns the value 1 to all prime numbers p such that (p) splits in K and the value -1 to all prime numbers p such that (p) is inert in K. Note that since the character χ_K is primitive, the space $S_m(D_K, \chi_K)$ has a basis consisting of primitive normalized eigenforms. We will denote this (unique) basis by $\mathcal{N}$. For $f = \sum_{n=1}^{\infty} a(n)q^n \in \mathcal{N}$, set $f^{\rho} := \sum_{n=1}^{\infty} \overline{a(n)}q^n \in \mathcal{N}$.

Fact 2.1. ([31], section 4.6) *One has $a(p) = \chi_K(p)\overline{a(p)}$ for any rational prime $p \nmid D_K$.*

This implies that $a(p) = \overline{a(p)}$ if (p) splits in K and $a(p) = -\overline{a(p)}$ if (p) is inert in K.

For $f \in \mathcal{N}$ and E a finite extension of $\mathbf{Q}_\ell$ containing the eigenvalues of T_n, $n = 1, 2, \dots$ we will denote by $\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(E)$ the Galois representation attached to f by Deligne (cf. e.g., [11], section 3.1). We will write $\overline{\rho}_f$ for the reduction of ρ_f modulo a uniformizer of E with respect to some lattice Λ in E^2. In general $\overline{\rho}_f$ depends on the lattice Λ, however the isomorphism class of its semisimplification $\overline{\rho}_f^{\mathrm{ss}}$ is independent of Λ. Thus, if $\overline{\rho}_f$ is irreducible (which we will assume), it is well-defined.

3. Hermitian modular forms

3.1. Classical theory. For $n > 1$, set $\mathbf{i}_n := iI_n$ and define

$$\mathbf{H}_n := \{Z \in M_n(\mathbf{C}) \mid -\mathbf{i}_n(Z - Z^*) > 0\}.$$

We call $\mathbf{H}_n$ the *hermitian upper half-plane of degree* n. The group $G_n^+(\mathbf{R}) = \{x \in G_n(\mathbf{R}) \mid \mu(x) > 0\}$ acts transitively on $\mathbf{H}_n$ via

$$gZ := (a_g Z + b_g)(c_g Z + d_g)^{-1}.$$

Definition 3.1. We say that a subgroup $\Gamma \subset G_n^+(\mathbf{R})$ is a *congruence subgroup* if

- Γ is commensurable with $U_n(\mathbf{Z})$, and
- there exists $N \in \mathbf{Z}_{>0}$ such that $\Gamma \supset \Gamma^{\mathrm{h}}(N) := \{g \in U_n(\mathbf{Z}) \mid g \equiv I_{2n} \pmod{N}\}$.

Note that every congruence subgroup Γ must be contained in $U_n(\mathbf{Q})$, because commensurability with $U_n(\mathbf{Z})$ and the fact that $\Gamma \subset G_n^+(\mathbf{R})$ force $\Gamma \subset G_n(\mathbf{Q})$ and $\mu(\Gamma)$ to be a finite subgroup of $\mathbf{R}_+^\times$ hence to be trivial.

For $g \in G_n^+(\mathbf{R})$ and $Z \in \mathbf{H}_n$ set

$$j(g, Z) := \det(c_\gamma Z + d_\gamma),$$

and for a positive integer k, a non-negative integer ν and a function $F : \mathbf{H}_n \to \mathbf{C}$ define

$$F|_{k,\nu} g(Z) := \det(g)^{-\nu} j(g, Z)^{-k} F(gZ).$$

When $\nu = 0$, we will usually drop it from notation and simply write $F|_k g(Z)$.

Definition 3.2. Let $\Gamma \subset G_n^+(\mathbf{R})$ be a congruence subgroup. We say that a function $F : \mathbf{H}_n \to \mathbf{C}$ is a *hermitian semi-modular form of weight* (k, ν) *and level* Γ if $F|_{k,\nu}\gamma = F$ for every $\gamma \in \Gamma$. If in addition F is holomorphic, we call it a *hermitian modular form of weight* (k, ν) *and level* Γ. The space of hermitian semi-modular (resp. modular) forms of weight (k, ν) and level Γ will be denoted by $M_{n,k,\nu}^{\mathrm{sh}}(\Gamma)$ (resp. $M_{n,k,\nu}^{\mathrm{h}}(\Gamma)$). We also set $M_{n,k,\nu}^{\mathrm{h}} = M_{n,k,\nu}^{\mathrm{h}}(U_n(\mathbf{Z}))$. If $n = 2$ or $\nu = 0$ we drop them from notation.

Set $\mathcal{J}(K) = \frac{1}{2}\#\mathcal{O}_K^\times$. Note that $\mathcal{J}(K) = 1$ when $D_K > 12$.

Remark 3.3. Suppose $\Gamma \subset U_n(\mathbf{Z})$. It is a Theorem of Hel Braun ([8], Theorem I on p. 143) that $\det U_n(\mathbf{Z}) = \{u^2 \mid u \in \mathcal{O}_K^\times\}$. This in particular implies that $(\det U_n(\mathbf{Z}))^{-\nu} = \{1\}$ if $\mathcal{J}(K) \mid \nu$. In such case, we have $M_{n,k,\nu}^{\mathrm{sh}}(\Gamma) = M_{n,k}^{\mathrm{sh}}(\Gamma)$.

If $\Gamma = \Gamma_{0,n}^{\mathrm{h}}(N)$ for some $N \in \mathbf{Z}$, then we say that F is of level N. Forms of level 1 will sometimes be referred to as forms of *full level*. One can also define hermitian semi-modular forms with a character. Let $\Gamma = \Gamma_{0,n}^{\mathrm{h}}(N)$ and let $\psi : \mathbf{A}_K^\times \to \mathbf{C}^\times$

be a Hecke character such that for all finite p, $\psi_p(a) = 1$ for every $a \in \mathcal{O}_{K,p}^{\times}$ with $a - 1 \in N\mathcal{O}_{K,p}$. We say that F is *of level* N *and character* ψ if

$$F|_m\gamma = \psi_N(\det a_\gamma)F \quad \text{for every } \gamma \in \Gamma_{0,n}^{\mathrm{h}}(N).$$

Denote by $M_{n,k}^{\mathrm{sh}}(N,\psi)$ (resp. $M_{n,k}^{\mathrm{h}}(N,\psi)$) the $\mathbf{C}$-space of hermitian semi-modular (resp. modular) forms of weight k, level N and character ψ. If $n = 2$ we drop it from notation.

Write $Z \in \mathbf{H}_n$ as $Z = X + \mathbf{i}_n Y$, where $X = \mathrm{Re}\,(Z)$ and $Y = \mathrm{Im}\,(Z)$. Let $M_n \cong \mathbf{G}_a^{n^2}$ denote the additive group of $n \times n$ matrices. A hermitian semi-modular form of level $\Gamma_n^{\mathrm{h}}(M,N)$ possesses a Fourier expansion

$$F(Z) = \sum_{\tau \in \mathcal{S}_n(M)(\mathbf{Z})} c_F(\tau, Y) e(\mathrm{tr}\,\tau X),$$

where $\mathcal{S}_n(M)(\mathbf{Z}) = \{x \in S_n(\mathbf{Z}) \mid \mathrm{tr}\, xL(M) \subset \mathbf{Z}\}$ with $S_n = \{h \in \mathrm{Res}_{\mathcal{O}_K/\mathbf{Z}} M_{n/\mathcal{O}_K} \mid h^* = h\}$ and $L(M) = S_n(\mathbf{Z}) \cap M_n(M\mathcal{O}_K)$. As usually when $n = 2$, we drop it from notation. As we will be particularly interested in the case when $M = 1$, we set

$$\mathcal{S} := \mathcal{S}(1) = \left\{ \begin{bmatrix} t_1 & t_2 \\ \overline{t_2} & t_3 \end{bmatrix} \in M_2(K) \mid t_1, t_3 \in \mathbf{Z}, t_2 \in \frac{1}{2}\mathcal{O}_K \right\}.$$

If F is holomorphic the dependence of $c(h, Y)$ on Y is explicit:

$$c_F(h, Y) = e(\mathrm{tr}\,(\mathbf{i}_n hY)) c_F(h),$$

where $c_F(h)$ depends only on h. Then one can write

$$F(Z) = \sum_{h \in S_n(\mathbf{Q})} c_F(h) e(\mathrm{tr}\,(hZ)).$$

For $F \in M_{n,k}^{\mathrm{h}}(\Gamma)$ and $\alpha \in G_n^+(\mathbf{R})$ one has $F|_k\alpha \in M_{n,k}^{\mathrm{h}}(\alpha^{-1}\Gamma\alpha)$ and there is an expansion

$$F|_k\alpha = \sum_{\tau \in S_n(\mathbf{Q})} c_\alpha(\tau) e(\mathrm{tr}\,\tau Z).$$

We call F a *cusp form* if for all $\alpha \in G_n^+(\mathbf{R})$, $c_\alpha(\tau) = 0$ for every τ such that $\det \tau = 0$. Denote by $S_{n,k}^{\mathrm{h}}(\Gamma)$ (resp. $S_{n,k}^{\mathrm{h}}(N,\psi)$) the subspace of cusp forms inside $M_{n,k}^{\mathrm{h}}(\Gamma)$ (resp. $M_{n,k}^{\mathrm{h}}(N,\psi)$). If $\psi = \mathbf{1}$, set $M_{n,k}^{\mathrm{sh}}(N) := M_{n,k}^{\mathrm{sh}}(N,\mathbf{1})$ and $S_{n,k}^{\mathrm{h}}(N) := S_{n,k}^{\mathrm{h}}(N,\mathbf{1})$. If $n = 2$ we drop it from notation.

Theorem 3.4 (q-expansion principle, [23], section 8.4)**.** *Let ℓ be a rational prime and N a positive integer with $\ell \nmid N$. Suppose all Fourier coefficients of $F \in M_{n,k}^{\mathrm{h}}(N,\psi)$ lie inside the valuation ring $\mathcal{O}$ of a finite extension E of $\mathbf{Q}_\ell$. If $\gamma \in U_n(\mathbf{Z})$, then all Fourier coefficients of $F|_m\gamma$ also lie in $\mathcal{O}$.*

If F and F' are two hermitian modular forms of weight k, level Γ and character ψ, and either F or F' is a cusp form, we define for any finite index subgroup Γ' of Γ, the Petersson inner product

$$\langle F, F' \rangle_{\Gamma'} := \int_{\Gamma' \backslash \mathbf{H}_n} F(Z)\overline{F'(Z)}(\det Y)^{m-4} dX dY,$$

where $X = \mathrm{Re}\, Z$ and $Y = \mathrm{Im}\, Z$, and

$$\langle F, F' \rangle = [\overline{U_n(\mathbf{Z})} : \overline{\Gamma}']^{-1} \langle F, F' \rangle_{\Gamma'},$$

where $\overline{U_n(\mathbf{Z})} := U_n(\mathbf{Z})/\langle \mathbf{i}_{2n}\rangle$ and $\overline{\Gamma}'$ is the image of Γ' in $\overline{U_n(\mathbf{Z})}$. The value $\langle F, F'\rangle$ is independent of Γ'.

3.2. **Adelic theory.**

Notation 3.5. We adopt the following notation. If H is an algebraic group over $\mathbf{Q}$, and $g \in H(\mathbf{A})$, we will write $g_\infty \in H(\mathbf{R})$ for the infinity component of g and g_{f} for the finite component of g, i.e., $g = (g_\infty, g_{\mathrm{f}})$.

Definition 3.6. Let $\mathcal{K}$ be an open compact subgroup of $G_n(\mathbf{A}_{\mathrm{f}})$. Write Z_n for the center of G_n. Let $\mathcal{M}'_{k,\nu}(\mathcal{K})$ denote the $\mathbf{C}$-space consisting of functions $f : G_n(\mathbf{A}) \to \mathbf{C}$ satisfying the following conditions:

- $f(\gamma g) = f(g)$ for all $\gamma \in G_n(\mathbf{Q})$, $g \in G_n(\mathbf{A})$,
- $f(g\kappa) = f(g)$ for all $\kappa \in \mathcal{K}$, $g \in G_n(\mathbf{A})$,
- $f(gu) = (\det u)^{-\nu} j(u, \mathbf{i}_n)^{-k} f(g)$ for all $g \in G_n(\mathbf{A})$, $u \in \mathcal{K}_\infty = \mathcal{K}_{0,n,\infty}$ (see (10.7.4) in [38]),
- $f(ag) = a^{-2n\nu - nk} f(g)$ for all $g \in G_n(\mathbf{A})$ and all $a \in \mathbf{C}^\times = Z_n(\mathbf{R}) \subset G_n(\mathbf{R})$.

Let $\psi : \mathbf{A}_K^\times \to \mathbf{C}^\times$ be a Hecke character of conductor dividing N. Set

$$\text{(3.1)} \quad \mathcal{M}'_k(N, \psi) := \{f \in \mathcal{M}'_k(\mathcal{K}_{1,n}(N)) \mid f(\gamma g(\kappa_\infty, \kappa_{\mathrm{f}})) = $$
$$= \psi'_N(\det(a_{\kappa_{\mathrm{f}}}))^{-1} j(\kappa_\infty, \mathbf{i}_n)^{-k} f(g), g \in G_n(\mathbf{A}), \gamma \in G_n(\mathbf{Q}), (\kappa_\infty, \kappa_{\mathrm{f}}) \in K_{0,n}(N)\}.$$

Remark 3.7. Note that the center $Z_n(\mathbf{R})$ of $G_n(\mathbf{R})$ acts via the infinite part of a Hecke character of infinity type $(-2n\nu - nk, 0)$. In particular this action is trivial if $\nu = -k/2$.

It is well-known (see e.g., [10], Theorem 3.3.1) that for any finite subset $\mathcal{B}$ of $\mathrm{GL}_n(\mathbf{A}_{K,\mathrm{f}})$ of cardinality h_K with the property that the canonical projection $c_K : \mathbf{A}_K^\times \twoheadrightarrow \mathrm{Cl}_K$ restricted to $\det \mathcal{B}$ is a bijection, the following decomposition holds

$$\text{(3.2)} \qquad \mathrm{GL}_n(\mathbf{A}_K) = \bigsqcup_{b \in \mathcal{B}} \mathrm{GL}_n(K)\, \mathrm{GL}_n(\mathbf{C}) b\, \mathrm{GL}_n(\hat{\mathcal{O}}_K).$$

We will call any such $\mathcal{B}$ a *base*. We always assume that a base comes with a fixed ordering, so in particular if we consider a tuple $(f_b)_{b\in\mathcal{B}}$ indexed by elements of $\mathcal{B}$, and apply a non-trivial permutation σ to the elements f_b, we do not consider the tuples $(f_b)_{b\in\mathcal{B}}$ and $(f_{\sigma(b)})_{b\in\mathcal{B}}$ to be the same.

Proposition 3.8. *Let $\mathcal{K}$ be a compact subgroup of $G_n(\mathbf{A}_{\mathrm{f}})$ such that $\det \mathcal{K} \supset \hat{\mathcal{O}}_K^\times$. There exists a finite subset $\mathcal{C} \subset U_n(\mathbf{A}_{\mathrm{f}})$ such that the following decomposition holds*

$$\text{(3.3)} \qquad G_n(\mathbf{A}) = \bigsqcup_{c \in \mathcal{C}} G_n(\mathbf{Q}) G_n^+(\mathbf{R}) c \mathcal{K}.$$

Moreover, each element of $\mathcal{C}$ can be taken to be of the form p_b for some b in a fixed base $\mathcal{B}$. The same holds for U_n in place of G_n.

Proof. This is proved like Lemmas 5.11(4) and 8.14 of [38]. □

We will call any set $\mathcal{C}$ of cardinality h_K for which the decomposition (3.3) holds a *unitary base.*

Corollary 3.9. *If $(h_K, 2n) = 1$ a base $\mathcal{B}$ can be chosen so that for all $b \in \mathcal{B}$ the matrices b and p_b are scalar matrices and $bb^* = b^*b = I_n$.*

Proof. It follows from the Tchebotarev Density Theorem, that elements of Cl_K can be represented by prime ideals. Since all the inert ideals are principal, Cl_K can be represented by prime ideals lying over split primes of the form $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$. Let Σ be a representing set consisting of such ideals $\mathfrak{p}$. As $(2n, h_K) = 1$, the set Σ^{2n} consisting of elements of Σ raised to the power $2n$ is also a representing set for Cl_K. Moreover, as $\mathfrak{p}\overline{\mathfrak{p}}$ is a principal ideal, $\overline{\mathfrak{p}} = \mathfrak{p}^{-1}$ as elements of Cl_K, hence $\Sigma' := \{\mathfrak{p}^n\overline{\mathfrak{p}}^{-n}\}_{\mathfrak{p}\in\Sigma}$ also represents all the elements Cl_K. Elements of Σ' can be written adelically as $\alpha_{\mathfrak{p}}^n$, with $\alpha_{\mathfrak{p}} = (1, 1, \dots, 1, p, p^{-1}, 1, \dots) \in \mathbf{A}_{K,\mathrm{f}}$, where p appears on the $\mathfrak{p}$-th place and p^{-1} appears at the $\overline{\mathfrak{p}}$-th place. Set $b_{\mathfrak{p}} = \alpha_{\mathfrak{p}} I_n$. Then we can take $\mathcal{B} = \{b_{\mathfrak{p}}\}_{\mathfrak{p}^n\overline{\mathfrak{p}}^{-n}\in\Sigma'}$ and we have $p_{b_{\mathfrak{p}}} = \alpha_{\mathfrak{p}} I_{2n}$. It is also clear that $bb^* = b^*b = I_n$. □

Proposition 3.10. *Suppose that $\mathcal{K} \subset G_n(\mathbf{A}_\mathrm{f})$ is a compact subgroup such that $\det \mathcal{K} \supset \hat{\mathcal{O}}_K^\times$. Let $\mathcal{N}'_{k,\nu}(\mathcal{K})$ denote the $\mathbf{C}$-space of functions $f : U_n(\mathbf{A}) \to \mathbf{C}$ satisfying the conditions of Definition 3.6, but with $g \in U_n(\mathbf{A})$. Then the map $f \mapsto f|_{U_n(\mathbf{A})}$ gives an isomorphism $\mathcal{M}'_{k,\nu}(\mathcal{K}) \cong \mathcal{N}'_{k,\nu}(\mathcal{K})$.*

Proof. One has the following short exact sequence of group schemes over $\mathbf{Z}$:

$$1 \to U_n \to G_n \xrightarrow{\mu} \mathbf{G}_m \to 1.$$

We first show that the induced map

$$f \mapsto f|_{U_n(\mathbf{A})} : \quad \mathcal{M}'_{k,\nu}(\mathcal{K}) \to \mathcal{N}'_{k,\nu}(\mathcal{K}) \tag{3.4}$$

is injective. Indeed, let $f \in \mathcal{M}'_{k,\nu}(\mathcal{K})$. Using Proposition 3.8 we write any $g \in G_n(\mathbf{A})$ as $g = \gamma g_{\mathbf{R}} c$ with $\gamma \in G_n(\mathbf{Q})$, $g_{\mathbf{R}} \in G_n^+(\mathbf{R})$, $c \in U_n(\mathbf{A}_\mathrm{f})$. Then $f(g) = f(g_{\mathbf{R}} c)$. Let $x = \mathrm{diag}((\mu(g_{\mathbf{R}}))^{1/2}, (\mu(g_{\mathbf{R}}))^{1/2}, \dots)$. Then $\mu(x) = \mu(g_{\mathbf{R}})$ and $x \in Z_n(\mathbf{R})$. Set $y = x^{-1} g_{\mathbf{R}}$. Then $\mu(y) = 1$. Hence by Definition 3.6

$$f(g) = f(xyc) = \mu(g_{\mathbf{R}})^{-n\nu - nk/2} f(yc) = f(yc),$$

so f is completely determined by what it does on $U_n(\mathbf{A})$.

It remains to show the surjectivity of (3.4). To do this we need to show that every $f \in \mathcal{N}'_{k,\nu}(\mathcal{K})$ has an extension to $G_n(\mathbf{A})$. Fix a unitary base $\mathcal{C}$. Let $f \in \mathcal{N}'_{k,\nu}(\mathcal{K})$. Let $g = \gamma g_{\mathbf{R}} p_b \kappa$ with $\gamma \in G_n(\mathbf{Q})$, $g_{\mathbf{R}} \in G_n^+(\mathbf{R})$, $c \in \mathcal{C}$ and $\kappa \in \mathcal{K}$. Write $g_{\mathbf{R}} = \sqrt{\mu(g_{\mathbf{R}})} \cdot y$. Then $y \in U_n(\mathbf{R})$. Set $f(g) := \mu(g_{\mathbf{R}})^{-n\nu - nk/2} f(yc)$. We need to show that this extension of f is well-defined. Let $g = \gamma' g'_{\mathbf{R}} c\kappa'$ be a different decomposition of g with $\gamma' \in G_n(\mathbf{Q})$, $g'_{\mathbf{R}} \in G_n^+(\mathbf{R})$, $\kappa' \in \mathcal{K}$. Then $g = (\gamma g_{\mathbf{R}}, \gamma c\kappa) = (\gamma' g'_{\mathbf{R}}, \gamma' c\kappa')$, where the first component is the ∞-component of g and the second is the finite component of g. Thus $g'_{\mathbf{R}} = (\gamma')^{-1}\gamma g_{\mathbf{R}}$ and $c = (\gamma')^{-1}\gamma c\kappa\kappa'$. The latter equality implies that $\mu((\gamma')^{-1}\gamma) \in \hat{\mathbf{Z}}^\times \cap \mathbf{Q}^\times = \{\pm 1\}$ which combined with the first equality and the fact that $g_{\mathbf{R}}, g'_{\mathbf{R}} \in G_n^+(\mathbf{R})$ implies that $\mu(g'_{\mathbf{R}} g_{\mathbf{R}}^{-1}) = \mu((\gamma')^{-1}\gamma) = 1$, i.e., in particular $\mu(g_{\mathbf{R}}) = \mu(g'_{\mathbf{R}})$ and $(\gamma')^{-1}\gamma \in U_n(\mathbf{Q})$. Write $g'_{\mathbf{R}} = \sqrt{\mu(g'_{\mathbf{R}})} y'$ with $y' \in U_n(\mathbf{R})$. We have

$$\begin{aligned} &\mu(g'_{\mathbf{R}})^{-n\nu - nk/2} f(y'c) = \mu(g'_{\mathbf{R}})^{-n\nu - nk/2} f((\mu(g'_{\mathbf{R}})^{-1/2} (\gamma')^{-1}\gamma g_{\mathbf{R}}, c) \\ &= \mu(g_{\mathbf{R}})^{-n\nu - nk/2} f((\mu(g_{\mathbf{R}})^{-1/2} (\gamma')^{-1}\gamma g_{\mathbf{R}}, (\gamma')^{-1}\gamma c\kappa\kappa') = \mu(g_{\mathbf{R}})^{-n\nu - nk/2} f(yc). \end{aligned} \tag{3.5}$$

□

In view of Proposition 3.10 in what follows we will often not distinguish between automorphic forms defined on $G_n(\mathbf{A})$ and those on $U_n(\mathbf{A})$.

Every $f \in \mathcal{M}_{n,k,\nu}(\mathcal{K})$ possesses a Fourier expansion, i.e., for every $q \in \mathrm{GL}_n(\mathbf{A}_K)$, and every $h \in S(\mathbf{Q})$ there exists a complex number $c_f(h,q)$ such that one has

$$f\left(\begin{bmatrix} I_n & \sigma \\ & I_n \end{bmatrix}\begin{bmatrix} q & \\ & \hat{q} \end{bmatrix}\right) = \sum_{h \in S_n(\mathbf{Q})} c_f(h,q) e_{\mathbf{A}}(\mathrm{tr}\, h\sigma) \tag{3.6}$$

for every $\sigma \in S_n(\mathbf{A})$. Here $e_{\mathbf{A}}$ is defined in the following way. Let $a = (a_v) \in \mathbf{A}$, where v runs over all the places of $\mathbf{Q}$. If $v = \infty$, set $e_v(a_v) = e^{2\pi i a_v}$. If $v = p$, a finite prime, set $e_v(a_v) = e^{-2\pi i y}$, where y is a rational number such that $a_v - y \in \mathbf{Z}_p$. Then we set $e_{\mathbf{A}}(a) = \prod_v e_v(a_v)$.

Suppose $2 \nmid h_K$. For $g \in U_n(\mathbf{A})$, write $g = \gamma g_0 c \in U_n(\mathbf{Q})U_n(\mathbf{R})U_n(\mathbf{A}_{\mathrm{f}})$ with $c \in U_n(\mathbf{A}_{\mathrm{f}})$ and g_0 such that $\det g_0 = |\det g_0| e^{i\varphi}$ satisfies $0 \leq \varphi < 2\pi/\mathcal{J}(K)$. Note that such a g_0 exists and $\det g_0^{\nu}$ is independent of the choice of g_0.

Proposition 3.11. *Let $f \in \mathcal{M}_{n,k,\nu}(\mathcal{K})$. Let $g = (g_\infty, 1) \in U_n(\mathbf{R})U_n(\mathbf{A}_{\mathrm{f}})$. Set $Z := g_\infty \mathbf{i}_n$. Let $\mathcal{C}$ be a unitary base. For $c \in \mathcal{C}$, set $f_c(Z) = (\det g_\infty)^{\nu} j(g_\infty, \mathbf{i}_n)^k f(g_\infty c)$ and write Γ_c for $U_n(\mathbf{Q}) \cap (G_n^+(\mathbf{R}) \times c\mathcal{K}c^{-1})$. The map $f \mapsto (f_c)_{c\in\mathcal{C}}$ defines a $\mathbf{C}$-linear isomorphism $\Phi_{\mathcal{C}} : \mathcal{M}'_{n,k,\nu}(\mathcal{K}) \xrightarrow{\sim} \prod_{c \in \mathcal{C}} M^{\mathrm{sh}}_{n,k,\nu}(\Gamma_c)$.*

Proof. This follows from [38], section 10. □

If h_K is odd, $\mathcal{B}$ is a base and $\mathcal{C} = \{p_b\}_{b \in \mathcal{B}}$, we write Γ_b instead of Γ_{p_b} and f_b instead of f_{p_b} for $b \in \mathcal{B}$, and $\Phi_{\mathcal{B}}$ instead of $\Phi_{\mathcal{C}}$.

Definition 3.12. Let $\mathcal{C}$ be a unitary base. A function $f \in \mathcal{M}'_{n,k,\nu}(\mathcal{K})$ whose image under the isomorphism $\Phi_{\mathcal{C}}$ lands in $\prod_{c\in\mathcal{C}} M^{\mathrm{h}}_{n,k,\nu}(\Gamma_c)$ will be called an *adelic hermitian modular form of weight (k,ν) and level $\mathcal{K}$*. The space of hermitian modular forms of weight (k,ν) will be denoted by $\mathcal{M}_{n,k,\nu}(\mathcal{K})$. Moreover, we set $\mathcal{M}_{n,k,\nu} := \mathcal{M}_{n,k,\nu}(U_n(\hat{\mathbf{Z}}))$. When $\nu = 0$ or $n = 2$ we drop them from notation.

We clearly have

$$\mathcal{M}_{n,k,\nu}(\mathcal{K}) \cong \prod_{c \in \mathcal{C}} M^{\mathrm{h}}_{n,k,\nu}(\Gamma_c). \tag{3.7}$$

Let $\chi : \mathrm{Cl}_K \to \mathbf{C}^\times$ be a character and choose a base $\mathcal{B}$ consisting of scalar matrices b such that $bb^* = I_n$. Such a base always exists when $(h_K, 2n) = 1$ by Corollary 3.9. Write Z_n for the center of G_n. Let $z = \gamma z_0 p_b \kappa$ with $\gamma \in Z_n(\mathbf{Q})$, $z_0 \in Z_n(\mathbf{R})$, $b \in \mathcal{B}$ and $\kappa \in (\mathcal{K} \cap Z_n(\mathbf{A}_{\mathrm{f}}))$ be an element of the center with $z_0 = \zeta I_{2n}$. If $f \in \mathcal{M}_{n,k,\nu}(\mathcal{K})$, then

$$f(zg) = f(z_0 p_b g) = \zeta^{-2n\nu - nk} f(p_b g).$$

Set

$$\mathcal{M}^{\chi}_{n,k,\nu}(\mathcal{K}) = \{f \in \mathcal{M}_{n,k,\nu}(\mathcal{K}) \mid f(p_b g) = \chi(b) f(g)\},$$

where we consider b as an element of Cl_K under the identification $\mathcal{B} = \mathrm{Cl}_K$ given by $b \mapsto c_K(\det b)$ with $c_K : \mathbf{A}_K^\times \twoheadrightarrow \mathrm{Cl}_K$. Then

$$\mathcal{M}_{n,k,\nu}(\mathcal{K}) = \bigoplus_{\chi \in \mathrm{Hom}(\mathrm{Cl}_K, \mathbf{C}^\times)} \mathcal{M}^{\chi}_{n,k,\nu}(\mathcal{K}). \tag{3.8}$$

By adapting the proof of Lemma A5.1 in [39] to the case of a trivial conductor one can show that for every integer s with $\mathcal{J}(K) \mid s$, there exists a Hecke character $\beta : \mathbf{A}_K^\times \to \mathbf{C}^\times$ unramified everywhere, trivial on $\mathbf{A}^\times$ and such that $\beta(z) = \frac{z^{2s}}{|z|^{2s}}$

for $z \in \mathbf{C}^\times = K_\infty^\times$. We will denote the set of such characters by $\mathrm{Char}(s)$. Note that every element β of $\mathrm{Char}(s)$ is unitary, i.e., $\overline{\beta(x)} = \beta(x)^{-1}$. If h_K is odd, one has $\#\mathrm{Char}(s) = h_K$ by [38], Lemma 8.14. Using the Iwasawa decomposition one sees that for $g \in U_n(\mathbf{A})$, one has $\det g = a\overline{a}^{-1} \det Y$ for some $a \in \mathbf{A}_K^\times$ and $Y \in U_n(\hat{\mathbf{Z}})$. By Lemma 5.11(4) in [38] we see that $\det Y = \kappa\overline{\kappa}^{-1}$ for some $\kappa \in \hat{\mathcal{O}}_K^\times$. So, absorbing κ into a we in fact we have $\det g = a\overline{a}^{-1}$ for some $a \in \mathbf{A}_K^\times$. Moreover, if $a\overline{a}^{-1} = b\overline{b}^{-1}$, then $\frac{a}{b} \in \mathbf{A}^\times$. Thus for every integer s, every $\beta \in \mathrm{Char}(s)$ and every $g \in U_n(\mathbf{A})$, the map $g \mapsto \beta(a)$ is a well-defined character on $U_n(\mathbf{A})$. Abusing notation we will write $\beta(g)$ instead of $\beta(a)$.

Proposition 3.13. *Assume $\mathcal{J}(K) \mid \nu$ and $(2n, h_K) = 1$. Let $\beta \in \mathrm{Char}(-\nu)$. Let $\mathcal{B}$ be a base as in Corollary 3.9 - note that then $\beta(p_b) = \beta(\det b)$ for $b \in \mathcal{B}$. Let $\mathcal{K} \subset G_n(\mathbf{A}_\mathrm{f})$ be an open compact subgroup. Then $\Gamma := U_n(\mathbf{Q}) \cap (G_n^+(\mathbf{R}) \times p_b\mathcal{K}p_b^{-1})$ is independent of $b \in \mathcal{B}$. Assume $\Gamma \subset U_n(\mathbf{Z})$. Then $M_{n,k,\nu}^\mathrm{h}(\Gamma) = M_{n,k}^\mathrm{h}(\Gamma)$ and one has the following commutative diagram in which all the maps are isomorphisms*

$$\begin{array}{ccc} \mathcal{M}_{n,k}^\mathrm{h}(\mathcal{K}) & \xrightarrow[\sim]{\Phi_0} & \prod_{b \in \mathcal{B}} M_{n,k}^\mathrm{h}(\Gamma) \\ \sim\downarrow \Psi_\beta & & \sim\downarrow \iota_\beta \\ \mathcal{M}_{n,k,\nu}^\mathrm{h}(\mathcal{K}) & \xrightarrow[\sim]{\Phi_\nu} & \prod_{b \in \mathcal{B}} M_{n,k}^\mathrm{h}(\Gamma) \end{array} \tag{3.9}$$

where $\iota_\beta(f_b) = \beta(\det b) f_b$, for $g = \gamma g_0 p_b \kappa \in U_n(\mathbf{Q})U_n(\mathbf{R})p_b\mathcal{K}$ one has $\Psi_\beta(f)(g) = \beta(g)f(g)$, and for $h \in U_n(\mathbf{R})$ $\Phi_0(f)_b(h\mathbf{i}_n) = j(h, \mathbf{i}_n)^k f(hp_b)$ and $\Phi_\nu(f)_b(h\mathbf{i}_n) = (\det h)^\nu j(h, \mathbf{i}_n)^k f(hp_b)$. The map Ψ_β is Hecke-equivariant; more precisely for $T = \mathcal{K}a\mathcal{K}$ with $a \in U_n(\mathbf{A})$ one has $\Psi_\beta(Tf)(x) = \beta(a)(T\Psi_\beta(f))(x)$ (for the definition of the Hecke action see section 4).

Proof. This is straightforward using the results of this section (cf. (3.7) and Remark 3.3). ☐

4. Hecke operators

4.1. **Hermitian Hecke operators.** We study Hecke operators acting on the space $\mathcal{M}_{k,\nu}$ of hermitian modular forms on $G(\mathbf{A}) = G_2(\mathbf{A})$. We also set $U = U_2$. Let p be a rational prime write $\mathcal{K}_p$ for $G(\mathbf{Z}_p)$. Let $\mathcal{H}_p$ be the $\mathbf{C}$-Hecke algebra generated by the double cosets $\mathcal{K}_p g \mathcal{K}_p$, $g \in G(\mathbf{Q}_p)$ with the usual law of multiplication (cf. [38], section 11), and $\mathcal{H}_p^+ \subset \mathcal{H}_p$ be the subalgebra generated by $\mathcal{K}_p g \mathcal{K}_p$ with $g \in U(\mathbf{Q}_p)$. If $\mathcal{K}_p g \mathcal{K}_p \in \mathcal{H}_p$, there exists a finite set $A_g \subset G(\mathbf{Q}_p)$ such that $\mathcal{K}_p g \mathcal{K}_p = \bigsqcup_{\alpha \in A_g} \mathcal{K}_p \alpha$. For $f \in \mathcal{M}_{k,\nu}$, $g \in G(\mathbf{Q}_p)$, $h \in G(\mathbf{A})$, set

$$([\mathcal{K}_p g \mathcal{K}_p] f)(h) = \sum_{\alpha \in A_g} f(h\alpha^{-1}).$$

It is clear that $[\mathcal{K}_p g \mathcal{K}_p] f \in \mathcal{M}_{k,\nu}$.

Remark 4.1. Let $\mathcal{K}_{0,p} := \mathcal{K}_p \cap U(\mathbf{Z}_p)$. Every element of $\mathcal{H}_p$ can be written as $\mathcal{K}_p g \mathcal{K}_p$ with g a diagonal matrix. For $\kappa \in \mathcal{K}_p$ write $m_\kappa = \mathrm{diag}(1, 1, \mu(\kappa), \mu(\kappa))$. Then $h_\kappa = \kappa m_\kappa^{-1} \in \mathcal{K}_{0,p}$. Since g is diagonal, m_κ commutes with g, hence we get

$$\mathcal{K}_p g \mathcal{K}_p = \mathcal{K}_p g \mathcal{K}_{0,p}.$$

From this it follows that

$$\mathcal{K}_{0,p} g \mathcal{K}_{0,p} = \bigsqcup_{\alpha \in A_g} \mathcal{K}_{0,p}\alpha \implies \mathcal{K}_p g \mathcal{K}_p = \bigsqcup_{\alpha \in A_g} \mathcal{K}_p g.$$

4.1.1. *The case of a split prime.* Let p be a prime which splits in K. Write $(p) = \mathfrak{p}\overline{\mathfrak{p}}$. Recall that $G(\mathbf{Q}_p) \cong \mathrm{GL}_4(\mathbf{Q}_p) \times \mathbf{G}_m(\mathbf{Q}_p)$. An element g of $G(\mathbf{Q}_p)$ can be written as $g = (g_1, g_2) \in \mathrm{GL}_4(\mathbf{Q}_p) \times \mathrm{GL}_4(\mathbf{Q}_p)$ with $g_2 = -\mu(g)J(g_1^t)^{-1}J$. Set

- $T_{\mathfrak{p}} := \mathcal{K}_p(\mathrm{diag}(1,p,p,p), \mathrm{diag}(1,1,p,1))\mathcal{K}_p$,
- $T_p := \mathcal{K}_p(\mathrm{diag}(1,1,p,p), \mathrm{diag}(1,1,p,p))\mathcal{K}_p$,
- $\Delta_{\mathfrak{p}} := \mathcal{K}_p(pI_4, I_4)\mathcal{K}_p$.

It is easy to see that the $\mathbf{C}$-algebra $\mathcal{H}_p$ is generated by the operators $T_{\mathfrak{p}}$, $T_{\overline{\mathfrak{p}}}$, T_p, $\Delta_{\mathfrak{p}}$, $\Delta_{\overline{\mathfrak{p}}}$ and their inverses.

Proposition 4.2. *We have the following decompositions*

$$\begin{aligned} T_{\mathfrak{p}} = & \bigsqcup_{a,b,c \in \mathbf{Z}/p\mathbf{Z}} \mathcal{K}_p \left(\begin{bmatrix} 1 & a & b & c \\ & p & & \\ & & p & \\ & & & p \end{bmatrix}, \begin{bmatrix} 1 & & b & \\ & 1 & c & \\ & & p & \\ & & -a & 1 \end{bmatrix} \right) \sqcup \\ & \bigsqcup_{d,e \in \mathbf{Z}/p\mathbf{Z}} \mathcal{K}_p \left(\begin{bmatrix} p & & & \\ & 1 & d & e \\ & & p & \\ & & & p \end{bmatrix}, \begin{bmatrix} 1 & & & d \\ & 1 & & e \\ & & 1 & \\ & & & p \end{bmatrix} \right) \sqcup \\ & \bigsqcup_{f \in \mathbf{Z}/p\mathbf{Z}} \mathcal{K}_p \left(\begin{bmatrix} p & & & \\ & p & & \\ & & 1 & f \\ & & & p \end{bmatrix}, \begin{bmatrix} p & & & \\ -f & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) \sqcup \\ & \mathcal{K}_p \left(\begin{bmatrix} p & & & \\ & p & & \\ & & p & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right). \end{aligned} \tag{4.1}$$

$$\begin{aligned} T_p = & \bigsqcup_{b,c,d,e \in \mathbf{Z}/p\mathbf{Z}} \mathcal{K}_p \left(\begin{bmatrix} 1 & & b & d \\ & 1 & c & e \\ & & p & \\ & & & p \end{bmatrix}, \begin{bmatrix} 1 & & b & c \\ & 1 & d & e \\ & & p & \\ & & & p \end{bmatrix} \right) \sqcup \\ & \bigsqcup_{a,c,f \in \mathbf{Z}/p\mathbf{Z}} \mathcal{K}_p \left(\begin{bmatrix} p & & & \\ -f & 1 & c & \\ & & p & \\ & & -a & p \end{bmatrix}, \begin{bmatrix} 1 & a & & c \\ & p & & \\ & & 1 & f \\ & & & p \end{bmatrix} \right) \sqcup \\ & \bigsqcup_{e,f \in \mathbf{Z}/p\mathbf{Z}} \mathcal{K}_p \left(\begin{bmatrix} p & & & \\ -f & 1 & & e \\ & & 1 & \\ & & & p \end{bmatrix}, \begin{bmatrix} p & & & \\ & 1 & & e \\ & & 1 & f \\ & & & p \end{bmatrix} \right) \sqcup \\ & \bigsqcup_{a,b \in \mathbf{Z}/p\mathbf{Z}} \mathcal{K}_p \left(\begin{bmatrix} 1 & & b & \\ & p & & \\ & & p & \\ & & -a & 1 \end{bmatrix}, \begin{bmatrix} 1 & a & b & \\ & p & & \\ & & p & \\ & & & 1 \end{bmatrix} \right) \sqcup \\ & \bigsqcup_{d \in \mathbf{Z}/p\mathbf{Z}} \mathcal{K}_p \left(\begin{bmatrix} 1 & & & d \\ & p & & \\ & & 1 & \\ & & & p \end{bmatrix}, \begin{bmatrix} p & & & \\ & 1 & d & \\ & & p & \\ & & & 1 \end{bmatrix} \right) \sqcup \\ & \mathcal{K}_p \left(\begin{bmatrix} p & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \begin{bmatrix} p & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right). \end{aligned} \tag{4.2}$$

Proof. This follows easily from the corresponding decompositions for the group $\mathrm{GL}_4(\mathbf{Q}_p)$. □

4.1.2. *The case of an inert prime.* Let p be a prime which is inert in K. Set

- $T_p := \mathcal{K}_p \, \mathrm{diag}(1,1,p,p)\mathcal{K}_p$,
- $T_{1,p} := \mathcal{K}_p \, \mathrm{diag}(1,p,p^2,p)\mathcal{K}_p$,
- $\Delta_p := \mathcal{K}_p pI_4 \mathcal{K}_p$.

The operators T_p, $T_{1,p}$, Δ_p and their inverses generate the $\mathbf{C}$-algebra $\mathcal{H}_p$.

Proposition 4.3. *We have the following decompositions*

$$
\begin{aligned}
T_p = & \bigsqcup_{\substack{b,d\in\mathbf{Z}/p\mathbf{Z}\\ c\in\mathcal{O}_K/p\mathcal{O}_K}} \mathcal{K}_p \begin{bmatrix} 1 & & b & c \\ & 1 & \bar{c} & d \\ & & p & \\ & & & p \end{bmatrix} \sqcup \bigsqcup_{e\in\mathbf{Z}/p\mathbf{Z}} \mathcal{K}_p \begin{bmatrix} p & & & \\ & 1 & & e \\ & & 1 & \\ & & & p \end{bmatrix} \sqcup \\
& \bigsqcup_{\substack{a\in\mathcal{O}_K/p\mathcal{O}_K\\ b\in\mathbf{Z}/p\mathbf{Z}}} \mathcal{K}_p \begin{bmatrix} 1 & a & b & \\ & p & & \\ & & p & \\ & & -\bar{a} & 1 \end{bmatrix} \sqcup \mathcal{K}_p \begin{bmatrix} p & & & \\ & p & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
T_{1,p} = & \bigsqcup_{\substack{a,c\in\mathcal{O}_K/p\mathcal{O}_K\\ b\in\mathbf{Z}/p^2\mathbf{Z}}} \mathcal{K}_p \begin{bmatrix} 1 & a & b+a\bar{c} & c \\ & p & p\bar{c} & \\ & & p^2 & \\ & & -\bar{a}p & p \end{bmatrix} \sqcup \bigsqcup_{\substack{c\in\mathcal{O}_K/p\mathcal{O}_K\\ d\in\mathbf{Z}/p^2\mathbf{Z}}} \mathcal{K}_p \begin{bmatrix} p & & & pc \\ & 1 & \bar{c} & d \\ & & p & \\ & & & p^2 \end{bmatrix} \\
& \bigsqcup_{a\in\mathcal{O}_K/p\mathcal{O}_K} \mathcal{K}_p \begin{bmatrix} p & pa & & \\ & p^2 & & \\ & & p & \\ & & -\bar{a} & 1 \end{bmatrix} \sqcup \mathcal{K}_p \begin{bmatrix} p^2 & & & \\ & p & & \\ & & 1 & \\ & & & p \end{bmatrix} \sqcup \\
& \bigsqcup_{\substack{b,d\in\mathbf{Z}/p\mathbf{Z}\\ bd\equiv 0 \pmod p\\ (b,d)\neq(0,0)}} \mathcal{K}_p \begin{bmatrix} p & & b & \\ & p & & d \\ & & p & \\ & & & p \end{bmatrix} \cup \bigsqcup_{\substack{b\in(\mathbf{Z}/p\mathbf{Z})^\times\\ c\in(\mathcal{O}_K/p\mathcal{O}_K)^\times}} \mathcal{K}_p \begin{bmatrix} p & & b & c \\ & p & \bar{c} & |c|^2b^{-1} \\ & & p & \\ & & & p \end{bmatrix}
\end{aligned} \tag{4.4}
$$

Proof. See the proof of Lemma 5.3 in [27] and references cited there. □

4.2. **Action of the Hecke operators on the Fourier coefficients.** Let $S = S_2$ be as in section 3. Write $\mathcal{S}_p := \mathcal{S}(\mathbf{Z}_p)$ for $\{h \in S(\mathbf{Q}_p) \mid \mathrm{tr}\,(S(\mathbf{Z}_p)h) \subset \mathbf{Z}_p\}$. For a matrix $h \in S(\mathbf{A})$ such that $h_p \in \mathcal{S}_p$ for every prime p, set

$$\epsilon_p(h) = \max\{n \in \mathbf{Z} \mid \frac{1}{p^n}h_p \in \mathcal{S}_p\}$$

and

$$\epsilon(h) = \prod_{p\nmid\infty} p^{\epsilon_p(h)}.$$

Note that $\epsilon_p(h) \geq 0$ for every p and $\epsilon(h) = \epsilon(h_{\mathrm{f}})$.

For $f \in \mathcal{M}_{k,\nu}$, $q \in \mathrm{GL}_2(\mathbf{A}_K)$ and $h \in S(\mathbf{Q})$ we write $c_f(h,q)$ for the (h,q)-Fourier coefficient of f as in (3.6).

4.2.1. *The case of a split prime.* Let p be a prime which splits in K. Let $\mathfrak{p}$ be a prime of K lying over p and denote by $\overline{\mathfrak{p}}$ its conjugate. As before we simultaneously identify $G(\mathbf{Q}_p)$ with a subgroup of $\mathrm{GL}_4(\mathbf{Q}_p)\times\mathrm{GL}_4(\mathbf{Q}_p)$ (the first factor corresponds to $\mathfrak{p}$ and the second one to $\overline{\mathfrak{p}}$) and with a subgroup of $G(\mathbf{A})$. Set

$$T_{\mathfrak{p},1} := \Delta_{\mathfrak{p}}^{-1}T_{\mathfrak{p}},$$

$$T_{\mathfrak{p},2} := \Delta_{\mathfrak{p}}^{-1}T_p.$$

Note that the operators $T_{\mathfrak{p},1}$, $T_{\mathfrak{p},2}$ and their inverses generate $\mathcal{H}_p^+$. Define the following elements of $\mathrm{GL}_2(\mathbf{Q}_p)\times\mathrm{GL}_2(\mathbf{Q}_p)$ which we regard as elements of $\mathrm{GL}_2(\mathbf{A}_K)$,

$$\begin{aligned}
\alpha_a &= \left(\begin{bmatrix} p & a \\ & 1 \end{bmatrix}, I_2\right), \quad a = 0,1,\dots,p-1,\\
\alpha_p &= \left(\begin{bmatrix} 1 & \\ & p \end{bmatrix}, I_2\right),\\
\beta_p &= \left(\begin{bmatrix} p & \\ & p \end{bmatrix}, I_2\right),\\
\gamma_a &= \left(\begin{bmatrix} 1 & a \\ & p \end{bmatrix}, I_2\right), \quad a = 0,1,\dots,p-1\\
\gamma_p &= \left(\begin{bmatrix} p & \\ & 1 \end{bmatrix}, I_2\right).
\end{aligned} \tag{4.5}$$

We will write $\pi_\mathfrak{p} \in \mathbf{A}_K^\times$ for the adele whose $\mathfrak{p}$th component is p and whose all other components are 1. Write $\pi_\mathfrak{p} = \gamma_\mathfrak{p}\pi_{\mathfrak{p},\infty}b_\mathfrak{p}\kappa_\mathfrak{p}$ with $\gamma_\mathfrak{p} \in K^\times$, $\pi_{\mathfrak{p},\infty} = \gamma_\mathfrak{p}^{-1} \in \mathbf{C}^\times$, $b_\mathfrak{p} \in \mathbf{A}_{K,\mathrm{f}}^\times$, $\kappa_\mathfrak{p} \in \hat{\mathcal{O}}_K^\times$, so that $\mathrm{val}_p(b_\mathfrak{p} b_\mathfrak{p}^*) = 0$ (this is always possible).

Proposition 4.4. *One has the following formulas*

$$c_{f'}(h,q) = \begin{cases} p^2\sum_{a=0}^p c_f(h,q\alpha_a) + \sum_{a=0}^p c_f(h,q\hat{\alpha}_a) & f' = T_{\mathfrak{p},1}f;\\ p^4 c_f(h,q\beta_p) + c_f(h,q\hat{\beta}_p) + p\sum_{a=0}^p\sum_{b=0}^p c_f(h,q\gamma_a\hat{\gamma}_b) & f' = T_{\mathfrak{p},2}f;\\ \gamma_\mathfrak{p}^{-2k-4\nu}c_f(h,qb_\mathfrak{p}^{-1}) & f' = \Delta_\mathfrak{p} f.\end{cases}$$

Proof. This is an easy calculation using Proposition 4.2. □

4.2.2. *The case of an inert prime.* Let p be a prime which is inert in K. Set

$$T_{p,0} := \Delta_p^{-1}T_{1,p}.$$

Define the following elements of $\mathrm{GL}_2(K_p) \subset \mathrm{GL}_2(\mathbf{A}_K)$:

$$\begin{aligned}
\alpha_a &= \begin{bmatrix} p & a \\ & 1 \end{bmatrix} \quad a \in \mathcal{O}_K/p\mathcal{O}_K,\\
\alpha_p &= \begin{bmatrix} 1 & \\ & p \end{bmatrix},\\
\beta_a &= \begin{bmatrix} 1 & ap^{-1} \\ & p^{-1} \end{bmatrix} \quad a \in \mathcal{O}_K/p\mathcal{O}_K,\\
\beta_p &= \begin{bmatrix} p^{-1} & \\ & 1 \end{bmatrix}.
\end{aligned} \tag{4.6}$$

Write $\mathbf{P}^1(\mathcal{O}_K/p\mathcal{O}_K)$ for the disjoint union of $\mathcal{O}_K/p\mathcal{O}_K$ and p. Let $h \in S(\mathbf{Q})$ and $q \in \mathrm{GL}_2(\mathbf{A}_{K,\mathrm{f}})$ and assume that $q^*hq \in \mathcal{S}_p$. Since $p \nmid D_K$, this implies that $q_p^*hq_p \in M_2(\mathcal{O}_{K,p})$, where q_p denotes the p-th component of q. Set

$$s(h,q) := \begin{cases} p & \mathrm{val}_p(\det(q^*hq)) = 0;\\ -p(p-1) & \mathrm{val}_p(\det(q^*hq)) > 0,\ \epsilon_p(q^*hq) = 0;\\ p^2(p-1) & \epsilon_p(q^*hq) > 0.\end{cases}$$

Proposition 4.5. *Assume that $q^*hq \in \mathcal{S}_p$. One has the following formulas:*

$$c_{f'}(h,q) = \begin{cases} s(h,q)c_f(h,q) + p^4\sum_{a\in\mathbf{P}^1(\mathcal{O}_K/p\mathcal{O}_K)} c_f(h,q\alpha_a) + \sum_{a\in\mathbf{P}^1(\mathcal{O}_K/p\mathcal{O}_K)} c_f(h,q\beta_a) & f' = T_{p,0}f;\\ p^{-2k+4}c_f(ph,q) + c_f(p^{-1}h,q) + p^{-k+1}\sum_{a\in\mathbf{P}^1(\mathcal{O}_K/p\mathcal{O}_K)} c_f(ph,\beta_a q) & f' = T_pf;\\ p^{-4\nu-2k}c_f(h,q) & f' = \Delta_p f.\end{cases}$$

If $q^*hq \notin \mathcal{S}_p$, $c_{f'}(h,q) = 0$ in all of the above cases.

Finally let us also note that in the inert case the algebra $\mathcal{H}_p^+$ is generated by the operators $T_{p,0}$ and $U_p := \Delta_p^{-1}T_p^2$ and their inverses.

5. Maass space

Let $S = S_2$, $\mathcal{S}$ be as in section 3 and $\mathcal{S}_p$, ϵ, ϵ_p as in section 4. In this section we assume that k is a positive integer divisible by $\#\mathcal{O}_K^\times$.

5.1. **Definition and basic properties.**

Definition 5.1. Let $\mathcal{B}$ be a base. We say that $f \in \mathcal{M}_{k,-k/2}$ is a $\mathcal{B}$-*Maass form* if there exist functions $c_{b,f} : \mathbf{Z}_{\geq 0} \to \mathbf{C}$, $b \in \mathcal{B}$, such that for every $q \in \mathrm{GL}_2(\mathbf{A}_K)$ and every $h \in S(\mathbf{Q})$ the Fourier coefficient $c_f(h,q)$ satisfies

$$(5.1)\quad c_f(h,q) = |\det q_\infty|^k e^{-2\pi\mathrm{tr}\,(q_\infty^* h q_\infty)}|\det\gamma_{b,q}|^{-k}\times \times \sum_{\substack{d\in\mathbf{Z}_+\\ d|\epsilon(q_{\mathrm{f}}^* h q_{\mathrm{f}})}} d^{k-1}c_{b,f}\left(D_K d^{-2}\det h\prod_p p^{\mathrm{val}_p(\det q_{\mathrm{f}}^* q_{\mathrm{f}})}\right),$$

where $q_{\mathrm{f}} = \gamma_{b,q}b\kappa_q \in \mathrm{GL}_2(K)b\mathcal{K}'$ for a unique $b \in \mathcal{B}$. Here $\mathcal{K}' = \mathrm{GL}_2(\hat{\mathcal{O}}_K)$ is a maximal compact subgroup of $\mathrm{GL}_2(\mathbf{A}_{K,\mathrm{f}})$. Also, here and in what follows we will often treat the K-points as embedded diagonally in the $\mathbf{A}_{K,\mathrm{f}}$-points (i.e., instead of writing $q_{\mathrm{f}} = \gamma_{b,q}q_\infty b\kappa_q$ with $q_\infty = \gamma_{b,q}^{-1} \in \mathrm{GL}_2(\mathbf{C})$ we will simply write $q_{\mathrm{f}} = \gamma_{b,q}b\kappa_q$ as above.)

Remark 5.2. Note that by [38], Proposition 18.3(2), $c_f(h,q) \neq 0$ only if $(q^*hq)_p \in \mathcal{S}_p$, so $\epsilon_p(q_{\mathrm{f}}^*hq_{\mathrm{f}}) \geq 0$. Also, note that Definition 5.1 is independent of the decomposition $q_{\mathrm{f}} = \gamma_{b,q}b\kappa_q \in \mathrm{GL}_2(K)b\mathcal{K}'$. Indeed, if $q_{\mathrm{f}} = \gamma'_{b,q}b\kappa'_q \in \mathrm{GL}_2(K)b\mathcal{K}'$ is another decomposition of q_{f}, then

$$\det\gamma'_{b,q}\det\gamma_{b,q}^{-1} = \det(\kappa_q(\kappa'_q)^{-1}) \in \hat{\mathcal{O}}_K^\times \cap K^\times = \mathcal{O}_K^\times,$$

so $\det(\gamma'_{b,q})^k = \det\gamma_{b,q}^k$.

Definition 5.3. The $\mathbf{C}$-subspace of $\mathcal{M}_{k,-k/2}$ consisting of $\mathcal{B}$-Maass forms will be called the $\mathcal{B}$-*Maass space*.

Definition 5.4. Let $\mathcal{B}$ be a base. We will say that $q \in \mathrm{GL}_2(\mathbf{A}_K)$ belongs to a class $b \in \mathcal{B}$ if there exist $\gamma \in \mathrm{GL}_2(K)$, $q_\infty \in \mathrm{GL}_2(\mathbf{C})$ and $\kappa \in \mathcal{K}'$ such that $q = \gamma b q_\infty \kappa$.

It is clear that the class of q depends only on q_{f}.

Lemma 5.5. *Suppose $r \in \mathrm{GL}_2(\mathbf{A}_K)$ and $q \in \mathrm{GL}_2(\mathbf{A}_K)$ belong to the same class and $r_{\mathrm{f}} = \gamma q_{\mathrm{f}}\kappa \in \mathrm{GL}_2(K)q_{\mathrm{f}}\,\mathrm{GL}_2(\hat{\mathcal{O}}_K)$. Then*

$$(5.2)\qquad c_f(h,r) = \left|\frac{\det r_\infty}{\det q_\infty}\right|^k e^{-2\pi\mathrm{tr}\,(r_\infty^* h r_\infty - q_\infty^*\gamma^* h\gamma q_\infty)}|\det\gamma|^{-k}c_f(\gamma^*h\gamma, q).$$

Proof. It follows from the proof of part (4) of Proposition 18.3 of [38], that

$$c_f(h,r) = |\det r_\infty|^k e^{-2\pi \mathrm{tr}\,(r_\infty^* h r_\infty)} c_{p_{r_{\mathrm{f}}}}(h), \tag{5.3}$$

where

$$f_{p_{r_{\mathrm{f}}}}(Z) = \sum_{h\in S} c_{p_{r_{\mathrm{f}}}}(h) e^{2\pi i \mathrm{tr}\, hZ}.$$

As is easy to see (cf. for example the Proof of Lemma 10.8 in [38]), $f_{p_{r_{\mathrm{f}}}} = f_{p_{q_{\mathrm{f}}}}|_k \begin{bmatrix} \gamma^{-1} & \\ & \gamma^* \end{bmatrix}$. Hence

$$c_{p_{r_{\mathrm{f}}}}(h) = |\det \gamma|^{-k} c_{p_{q_{\mathrm{f}}}}(\gamma^* h \gamma). \tag{5.4}$$

The Lemma follows from combining (5.3) with (5.4). □

Proposition 5.6. *Choose a base $\mathcal{B}$ and let $f \in \mathcal{M}_{k,-k/2}$. If there exist functions $c^*_{b,f} : \mathbf{Z}_{\geq 0} \to \mathbf{C}$, $b \in \mathcal{B}$, such that for every $b \in \mathcal{B}$ and every $h \in S(\mathbf{Q})$, the Fourier coefficient $c_f(h,b)$ satisfies condition (5.1) with $c^*_{b,f}$ in place of $c_{b,f}$, then f is a $\mathcal{B}$-Maass form and one has $c_{b,f} = c^*_{b,f}$ for every $b \in \mathcal{B}$.*

Proof. Fix $\mathcal{B}$ and $f \in \mathcal{M}_{k,-k/2}$. Suppose there exist $c^*_{b,f}$ such that (5.1) is satisfied for all pairs (h,b). Let $q = \gamma b x \kappa = (\gamma x, \gamma b \kappa) \in \mathrm{GL}_2(\mathbf{C}) \times \mathrm{GL}_2(\mathbf{A}_{K,\mathrm{f}})$, where $\gamma \in \mathrm{GL}_2(K)$, $x \in \mathrm{GL}_2(\mathbf{C})$ and $\kappa \in \mathcal{K}'$. Then by Lemma 5.5,

$$c_f(h,q) = |\det q_\infty|^k e^{-2\pi \mathrm{tr}\,(q_\infty^* h q_\infty - \gamma^* h \gamma)} |\det \gamma|^{-k} c_f(\gamma^* h \gamma, b).$$

Since condition (5.1) is satisfied for (h,b), we know that

$$c_f(h,b) = e^{-2\pi \mathrm{tr}\, h} \sum_{\substack{d \in \mathbf{Z}_+ \\ d \mid \epsilon(b^* h b)}} d^{k-1} c^*_{b,f} \left(D_K d^{-2} \det h \prod_p p^{\mathrm{val}_p(\det b^* b)} \right).$$

Thus

$$c_f(\gamma^* h \gamma, b) = e^{-2\pi \mathrm{tr}\,(\gamma^* h \gamma)} \sum_{\substack{d \in \mathbf{Z}_+ \\ d \mid \epsilon(b^* \gamma^* h \gamma b)}} d^{k-1} c^*_{b,f} \left(D_K d^{-2} \det(\gamma^* h \gamma) \prod_p p^{\mathrm{val}_p(\det b^* b)} \right).$$

So,

$$\begin{aligned} c_f(h,q) = {} & |\det q_\infty|^k e^{-2\pi \mathrm{tr}\,(q_\infty^* h q_\infty)} |\det \gamma|^{-k} \times \\ & \times \sum_{\substack{d \in \mathbf{Z}_+ \\ d \mid \epsilon(b^* \gamma^* h \gamma b)}} d^{k-1} c^*_{b,f} \left(D_K d^{-2} \det h \det(\gamma^* \gamma) \prod_p p^{\mathrm{val}_p(\det b^* b)} \right). \end{aligned} \tag{5.5}$$

The claim now follows since $\epsilon(b^* \gamma^* h \gamma b) = \epsilon(q_{\mathrm{f}}^* h q_{\mathrm{f}})$ and $\det(\gamma^* \gamma) \in \mathbf{Q}_+$, so $\det(\gamma^*\gamma) = \prod_p p^{\mathrm{val}_p(\det \gamma^* \gamma)}$. □

Proposition 5.7. *If $\mathcal{B}$ and $\mathcal{B}'$ are two bases, then the $\mathcal{B}$-Maass space and the $\mathcal{B}'$-Maass space coincide, i.e., the notion of a Maass form is independent of the choice of the base.*

Proof. Let $\mathcal{B}$ and $\mathcal{B}'$ be two bases. Write $q_{\mathrm{f}} = \gamma_{b,q} b \kappa_{\mathcal{B}} = \gamma_{b',q} b' \kappa_{\mathcal{B}'}$ with $b \in \mathcal{B}$, $b' \in \mathcal{B}'$, $\gamma_{b,q}, \gamma_{b',q} \in \mathrm{GL}_2(K)$ and $\kappa_{\mathcal{B}}, \kappa_{\mathcal{B}'} \in \mathcal{K}'$. Suppose f is a $\mathcal{B}$-Maass form, i.e., there exist functions $c_{b,f}$ for $b \in \mathcal{B}$, such that for every q and h,

$$c_f(h,q) = t\,|\det \gamma_{b,q}|^{-k} \sum_{\substack{d \in \mathbf{Z}_+ \\ d \mid \epsilon(q_{\mathrm{f}}^* h q_{\mathrm{f}})}} d^{k-1} c_{b,f}(s),$$

where $t = |\det q_\infty|^k e^{-2\pi \mathrm{tr}\,(q_\infty^* h q_\infty)}$ and $s = D_K d^{-2} \det h \prod_p p^{\mathrm{val}_p(\det q_{\mathrm{f}}^* q_{\mathrm{f}})}$. Our goal is to show that there exist functions $c_{b',f}$ for $b' \in \mathcal{B}'$, such that for every q and h,

$$c_f(h,q) = t\,|\det \gamma_{b',q}|^{-k} \sum_{\substack{d \in \mathbf{Z}_+ \\ d \mid \epsilon(q_{\mathrm{f}}^* h q_{\mathrm{f}})}} d^{k-1} c_{b',f}(s). \tag{5.6}$$

We have

$$\det \gamma_{b,q} = \det \gamma_{b',q} \det(b' b^{-1}) \det(\kappa_{\mathcal{B}}^{-1} \kappa_{\mathcal{B}'}). \tag{5.7}$$

Since $\det(b'b^{-1})$ corresponds to a principal fractional ideal, say $(\alpha_{b,b'})$, under the map $((\alpha_{b,b'})_{\mathfrak{p}}) \mapsto \prod_{\mathfrak{p}} \mathfrak{p}^{\mathrm{val}_{\mathfrak{p}}((\alpha_{b,b'})_{\mathfrak{p}})}$, using [10], Theorem 3.3.1, we can write $\det(b'b^{-1}) = \alpha_{b,b'} \kappa_{b,b'} \in \mathbf{A}_{K,\mathrm{f}}^\times$ with $\kappa_{b,b'} \in \hat{\mathcal{O}}_K^\times$. Then it follows from (5.7) that

$$\beta := \kappa_{b,b'} \det(\kappa_{\mathcal{B}}^{-1} \kappa_{\mathcal{B}'}) \in \hat{\mathcal{O}}_K^\times \cap K^\times = \mathcal{O}_K^\times.$$

Hence $\beta^k = 1$. Thus $|\det \gamma_{b,q}|^{-k} = |\det \gamma_{b',q}|^{-k} |\alpha_{b,b'}|^{-k}$. Note that $|\alpha_{b,b'}|^{-k}$ is well defined and only depends on b and b' (i.e., it is independent of q and h). Set $c_{b',f}(n) = |\alpha_{b,b'}|^{-k} c_{b,f}(n)$. Then it is clear that $c_{b',f}$ satisfies (5.6). □

Definition 5.8. From now on we will refer to $\mathcal{B}$-Maass forms simply as *Maass forms*. Similarly we will talk about the *Maass space* instead of $\mathcal{B}$-Maass spaces. This is justified by Proposition 5.7. The Maass space will be denoted by $\mathcal{M}_{k,-k/2}^{\mathrm{M}}$.

We now recall the definition of Maass space introduced in [29]. We will refer to it as the $U(\mathbf{Z})$-Maass space. Assume $\mathcal{J}(K) \mid \frac{k}{2}$, so that by Proposition 3.13 the space $M_k^{\mathrm{h}} = M_{k,-k/2}^{\mathrm{h}} = M_{k,-k/2}^{\mathrm{h}}(U(\mathbf{Z}))$. We say that $F(Z) = \sum_{h \in \mathcal{S}} c_F(h) e^{2\pi i \mathrm{tr}\,(hZ)} \in M_k^{\mathrm{h}}$ is a $U(\mathbf{Z})$*-Maass form* if there exists a function $\alpha_F : \mathbf{Z}_{\geq 0} \to \mathbf{C}$ such that for every $h \in \mathcal{S}$, one has

$$c_F(h) = \sum_{\substack{d \in \mathbf{Z}_+ \\ d \mid \epsilon(h)}} d^{k-1} \alpha_F(D_K d^{-2} \det h). \tag{5.8}$$

The subspace of M_k^{h} consisting of $U(\mathbf{Z})$-Maass forms will be denoted by $M_k^{\mathrm{h,M}}$.

Proposition 5.9. *If $2 \nmid h_K$, then the Maass space $\mathcal{M}_{k,-k/2}^{\mathrm{M}}$ is isomorphic (as a $\mathbf{C}$-linear space) to $\#\mathcal{B}$ copies of the $U(\mathbf{Z})$-Maass space $M_k^{\mathrm{h,M}}$.*

Proof. Since the Maass space is independent of the choice of a base $\mathcal{B}$ by Proposition 5.7, we may choose $\mathcal{B}$ as in Corollary 3.9 and $\#\mathcal{B} = \#\mathcal{C} = h_K$, with $\mathcal{C}$ as in Proposition 3.8. The map $\Phi_{\mathcal{B}} : \mathcal{M}_{k,-k/2} \to \prod_{b \in \mathcal{B}} M_k^{\mathrm{h}}$ is an isomorphism. Let $f \in \mathcal{M}_{k,-k/2}^{\mathrm{M}}$ and set $(f_b)_{b \in \mathcal{B}} = \Phi_{\mathcal{B}}(f)$. Set $\alpha_{f_b} := c_{b,f}$. Then using (5.3), and the fact that the matrices b commute with h and $b^* b = I_2$, we see that condition (5.1) for $c_f(h,b)$ translates into condition (5.8) for $c_{f_b}(h)$. Hence $\Phi_{\mathcal{B}}(\mathcal{M}_{k,-k/2}^{\mathrm{M}}) \subset \prod_{b \in \mathcal{B}} M_k^{\mathrm{h,M}}$. On the other hand if $(f_b)_{b \in \mathcal{B}} \in \prod_{b \in \mathcal{B}} M_k^{\mathrm{h,M}}$, set $c_{b,f} := \alpha_{f_b}$. Then conditions (5.8) for $c_{f_b}(h)$ translate into conditions (5.1) for $c_f(h,b)$. By Proposition 5.6 this implies that f is a Maass form. □

5.2. **Invariance under Hecke action.** It was proved in [29] that the $U(\mathbf{Z})$-Maass space is invariant under the action of a certain Hecke operator T_p associated with a prime p which is inert in F. On the other hand Gritsenko in [17] proved the invariance of the $U(\mathbf{Z})$-Maass space under all the Hecke operators when $K = \mathbf{Q}(i)$. In this section we show that the Maass space $\mathcal{M}^{\mathrm{M}}_{k,-k/2}$ is in fact invariant under all the local Hecke algebras (for primes $p \nmid D_K$) without imposing restrictions on the class number.

Theorem 5.10. *Let $p \nmid D_K$ be a rational prime. The Maass space is invariant under the action of $\mathcal{H}_p^+$, i.e., if $f \in \mathcal{M}^{\mathrm{M}}_{k,-k/2}$, and $g \in U(\mathbf{Q}_p)$, then $[\mathcal{K}_p g \mathcal{K}_p] f \in \mathcal{M}^{\mathrm{M}}_{k,-k/2}$.*

Remark 5.11. In fact (at least when D_K is a prime) the Maass space is invariant under $\mathcal{H}_p$, which is an easy consequence of the invariance under $\mathcal{H}_p^+$. In this case it is also invariant under $\mathcal{H}_{D_K}$, hence under *all* local Hecke algebras. For both of these facts see Remark 5.20.

Proof of Theorem 5.10. We will only present the proof in the case when p splits in K. For such a prime p the invariance of $\mathcal{M}^{\mathrm{M}}_{k,-k/2}$ under the action of $\mathcal{H}_p^+$ follows from Propositions 5.14, 5.15 below. If p is inert one can proceed along the same lines, however, it is the case when p splits that is essentially new. Indeed, if p is inert, the elements of $\mathcal{H}_p^+$ respect the decomposition (3.7), hence the statement of the theorem reduces to an assertion about the action of $\mathcal{H}_p^+$ on $M_k^{\mathrm{h,M}}$. Then the method used in [17] can be adapted to prove the theorem. See also Theorem 7 in [29] which proves the invariance of the $U(\mathbf{Z})$-Maass space for a certain family of Hecke operators in $\mathcal{H}_p^+$. □

Let p be a prime which splits in K. Write $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$. It suffices to prove the invariance of the Maass space under the operators $T_{\mathfrak{p},1}$ and $T_{\mathfrak{p},2}$. In the rest of section 5.2, l will denote a rational prime.

5.3. **Diagonalizing hermitian matrices mod l^n.** We begin by generalizing Proposition 7 of [29] to split primes.

Proposition 5.12. *Let n be a positive integer, and assume $l \nmid D_K$. Let $h \in \mathcal{S}_l$, $h \neq 0$. Then there exist a, $d \in \mathbf{Z}_l$ with $l \nmid a$ and $u \in \mathrm{SL}_2(\mathcal{O}_{K,l})$ such that*

$$u^* h u \equiv l^{\epsilon_l(h)} \begin{bmatrix} a & \\ & d \end{bmatrix} \pmod{l^n \mathcal{S}_l}.$$

In fact it is enough to prove the following lemma.

Lemma 5.13. *Proposition 5.12 holds if $\mathcal{S}_l$ is replaced by the subgroup of hermitian matrices inside $M_2(\mathcal{O}_{K,l})$.*

Proof. For inert l, this is Proposition 7 of [29]. So, assume that $l\mathcal{O}_K = \lambda\overline{\lambda}$ with $\lambda \neq \overline{\lambda}$. Without loss of generality we may assume that $\epsilon_l(h) = 0$. Let (M, M^t) be the image of h under the composite

$$M_2(\mathcal{O}_{K,l}) \twoheadrightarrow M_2(\mathcal{O}_K/l^n\mathcal{O}_K) \xrightarrow{\sim} M_2(\mathcal{O}_K/\lambda^n) \oplus M_2(\mathcal{O}_K/\overline{\lambda}^n) \cong \cong M_2(\mathbf{Z}/l^n\mathbf{Z}) \oplus M_2(\mathbf{Z}/l^n\mathbf{Z}). \tag{5.9}$$

Since the canonical map

$$\mathrm{SL}_2(\mathcal{O}_{K,l}) \to \mathrm{SL}_2(\mathcal{O}_K/l^n\mathcal{O}_K) \cong \mathrm{SL}_2(\mathbf{Z}/l^n\mathbf{Z}) \oplus \mathrm{SL}_2(\mathbf{Z}/l^n\mathbf{Z})$$

is surjective ([36], p. 490), it is enough to find $A_1, A_2 \in \mathrm{SL}_2(\mathbf{Z}/l^n\mathbf{Z})$ such that

$$A_2^t M A_1 = \left[\begin{smallmatrix} \alpha & \\ & \delta \end{smallmatrix}\right] \tag{5.10}$$

with $\alpha \not\equiv 0 \bmod p$. The existence of such A_1 and A_2 is clear. □

5.4. **Invariance under $T_{\mathfrak{p},1}$.** Let $g = T_{\mathfrak{p},1}f$. Then for $q \in \mathrm{GL}_2(\mathbf{A}_K)$ and $\sigma \in S(\mathbf{A})$, we can write

$$g\left(\begin{bmatrix} q & \sigma\hat{q} \\ & \hat{q} \end{bmatrix}\right) = \sum_{h\in S} c_g(h,q) e_{\mathbf{A}}(\mathrm{tr}\,(h\sigma)).$$

Define the following matrices

$$\alpha'_{\mathfrak{p},a} = \left(\begin{bmatrix} p & -a \\ & 1 \end{bmatrix}, I_2\right) \in \mathrm{GL}_2(\mathbf{Q}_p)\times \mathrm{GL}_2(\mathbf{Q}_p) \quad a = 0,1,\dots,p-1$$

and

$$\alpha'_{\mathfrak{p},p} = \left(\begin{bmatrix} 1 & \\ & p \end{bmatrix}, I_2\right), \in \mathrm{GL}_2(\mathbf{Q}_p)\times \mathrm{GL}_2(\mathbf{Q}_p).$$

For $a = 0,1,\dots,p$, set $\alpha_{\mathfrak{p},a}$ to be the images of $\alpha'_{\mathfrak{p},a}$ in $\mathrm{GL}_2(F_p)$. To simplify notation in this section we drop the subscript $\mathfrak{p}$ and simply write α_a for $\alpha_{\mathfrak{p},a}$.

Proposition 5.14. *The Maass space is invariant under the action of $T_{\mathfrak{p},1}$, i.e., if $f \in \mathcal{M}^{\mathrm{M}}_{k,-k/2}$, then $g \in \mathcal{M}^{\mathrm{M}}_{k,-k/2}$.*

Proof. Choose a base $\mathcal{B}$ in such a way that for all $b \in \mathcal{B}$ we have $b_l = I_2$ if $l \mid D_K$. For $b \in \mathcal{B}$ write $b_{\mathbf{Q}}$ for $\prod_l l^{\mathrm{val}_l(\det b^*b)}$. By Propositions 5.6 and 5.7 it is enough to show that there exist functions $c_{b,g} : \mathbf{Z}_+ \to \mathbf{C}$ ($b \in \mathcal{B}$) such that

$$c_g(h,b) = e^{-2\pi \mathrm{tr}\, h} \sum_{\substack{d\in\mathbf{Z}_+ \\ d\mid\epsilon(b^*hb)}} d^{k-1} c_{b,g}\left(D_K d^{-2} b_{\mathbf{Q}} \det h\right). \tag{5.11}$$

For $b \in \mathcal{B}$, set $b' = b\alpha_p$. Note that all of the matrices: $b\alpha_a$, $b\hat{\alpha}_a$, ($a = 0,1,\dots,p$) belong to the same class b'. Denote any of these matrices by q. Then $q = \gamma_{b',q} b' \kappa_q \in \mathrm{GL}_2(K) b' \mathcal{K}'$ and it is easy to see that

$$\det \gamma_{b',q}^k = \begin{cases} 1 & q = b\alpha_a, \;\; a = 0,1,\dots,p \\ p^{-k} & q = b\hat{\alpha}_a \;\; a = 0,1,\dots,p, \end{cases}$$

and

$$\prod_l \mathrm{val}_l(\det q^*q) = b_{\mathbf{Q}} \times \begin{cases} p & q = b\alpha_a \;\; a = 0,1,\dots,p \\ p^{-1} & q = b\hat{\alpha}_a \;\; a = 0,1,\dots,p. \end{cases}$$

Set $h_0 := b^*hb$ and write $h_0 = \epsilon(h_0)h'$ with $\epsilon(h') = 1$. Set $D = D_K \det h$ and $D' = D_K \prod_l l^{\mathrm{val}_l(\det h')}$. Using Proposition 4.4 and the fact that f is a Maass form, we obtain

$$c_g(h,b) = e^{-2\pi \mathrm{tr}\, h} \times \left(p^2 \sum_{a=0}^{p} \sum_{\substack{d \in \mathbf{Z}_+ \\ d \mid \epsilon(\alpha_a^* b^* h b \alpha_a)}} d^{k-1} c_{b',f}(Dd^{-2} b_{\mathbf{Q}} p) + \right. \\ \left. + p^k \sum_{a=0}^{p} \sum_{\substack{d \in \mathbf{Z}_+ \\ d \mid \epsilon(\hat{\alpha}_a^* b^* h b \hat{\alpha}_a)}} d^{k-1} c_{b',f}(Dd^{-2} b_{\mathbf{Q}} p^{-1}) \right). \tag{5.12}$$

Using Proposition 5.12, one can relate $\epsilon(\alpha_a^* h_0 \alpha_a)$ and $\epsilon(\hat{\alpha}_a^* h_0 \hat{\alpha}_a)$ to $\epsilon(h_0)$ for $a = 0, 1, \dots, p$, and then (5.12) becomes

$$c_g(h,b) = e^{-2\pi \mathrm{tr}\, h} p^2 \sum_0 A_d^{(1)} + e^{-2\pi \mathrm{tr}\, h} \times \\ \times \begin{cases} p^3 \sum_0 A_d^{(1)} + p^k (p+1) \sum_{-1} A_d^{(-1)} & p \nmid D', \\ p^2(p-1) \sum_0 A_d^{(1)} + p^2 \sum_1 A_d^{(1)} + p^k \sum_0 A_d^{(-1)} + p^{k+1} \sum_{-1} A_d^{(-1)} & p \mid D', \end{cases} \tag{5.13}$$

where $\sum_n A_d^{(m)} := \sum_{d \mid p^n \epsilon(h_0)} A_d^{(m)}$, $A_d^{(m)} = d^{k-1} c_{b',f}(Dd^{-2} b_{\mathbf{Q}} p^m)$ and if there is no d dividing $p^n \epsilon(h_0)$, we set $\sum_n = 0$.

For D in the image of the map $h \mapsto D_K \epsilon(h)^{-2} b_{\mathbf{Q}} \det h$ and $b \in \mathcal{B}$ we make the following definition

$$c_{b,g}(D) = p^2(p+1) c_{b',f}(Dp) + p^k (p+1) c_{b',f}(Dp^{-1}), \tag{5.14}$$

where we assume that $c_{b',f}(n) = 0$ when $n \notin \mathbf{Z}_+$. If D is not in the image of that map, we set $c_{b,g}(D) = 0$. Note that we clearly have

$$c_g(h,b) = e^{-2\pi \mathrm{tr}\, h} c_{b,g}(D_K b_{\mathbf{Q}} \det h)$$

for every h with $\epsilon(b^* h b) = 1$. Thus to check if g lies in the Maass space we just need to check that (5.11) holds with $c_{b,g}$ defined by (5.14). This is an easy calculation using (5.13). □

5.5. **Invariance under $T_{\mathfrak{p},2}$.** This is completely analogous to the proof for $T_{\mathfrak{p},1}$, hence we only include the relevant formulas for the reader's convenience.

Let $g = T_{\mathfrak{p},2} f$. Define matrices:

$$\beta_p' = \left(\begin{bmatrix} p & \\ & p \end{bmatrix}, I_2 \right) \in \mathrm{GL}_2(\mathbf{Q}_p) \times \mathrm{GL}_2(\mathbf{Q}_p),$$

$$\gamma_a' = \left(\begin{bmatrix} 1 & \\ a & p \end{bmatrix}, I_2 \right) \in \mathrm{GL}_2(\mathbf{Q}_p) \times \mathrm{GL}_2(\mathbf{Q}_p), \quad a = 0, 1, \dots, p-1,$$

$$\gamma_p' = \left(\begin{bmatrix} p & \\ & 1 \end{bmatrix}, I_2 \right) \in \mathrm{GL}_2(\mathbf{Q}_p) \times \mathrm{GL}_2(\mathbf{Q}_p),$$

and set β_p to be the image of β_p' in $\mathrm{GL}_2(K_p)$, and γ_a to be the image of γ_a' in $\mathrm{GL}_2(K_p)$ $(a = 0, 1, \dots, p)$.

Proposition 5.15. *The Maass space is invariant under the action of $T_{\mathfrak{p},2}$.*

Proof. This is similar to the proof of Proposition 5.14. Let b, D, D', h, h' and h_0 be as in that proof. Then for all a, c, we see that $b\beta_p$, $b\hat{\beta}_p$, $b\gamma_a\hat{\gamma}_c$ all lie in the same class $b' = b\beta_p$. One has

$$\det \gamma^k_{b',q} = \begin{cases} 1 & q = b\beta_p \\ p^{-k} & q = b\gamma_a\hat{\gamma}_c, \quad a, c \in \{0, 1, \dots, p\} \\ p^{-2k} & q = b\hat{\beta}_p, \end{cases}$$

and

$$\prod_l \mathrm{val}_l(\det q^* q) = b_{\mathbf{Q}} \times \begin{cases} p^2 & q = b\beta_p \\ 1 & q = b\gamma_a\hat{\gamma}_c, \quad a, c \in \{0, 1, \dots, p\} \\ p^{-2} & q = b\hat{\beta}_p. \end{cases}$$

Using Proposition 5.12 as in the proof of Proposition 5.14 we obtain

$$\begin{aligned} c_g(h, b) = e^{-2\pi \mathrm{tr}\, h} \Bigg(& p^4 \sum_1 A_d^{(2)} + p^{2k} \sum_{-1} A_d^{(-2)} + \\ & + p^{k+1}(p+1) \sum_0 A_d^{(0)} + p^{k+3} \sum_{-1} A_d^{(0)} \Bigg) + \\ & + e^{-2\pi \mathrm{tr}\, h} p^{k+1} \begin{cases} p \sum_{-1} A_d^{(0)} & p \nmid D' \\ p \sum_0 A_d^{(0)} & p \mid D',\ p^2 \nmid D' \\ \sum_1 A^{(0)} + (p-1)\sum_0 A^{(0)} & p^2 \mid D', \end{cases} \end{aligned} \tag{5.15}$$

where if there is no d dividing $p^n \epsilon(h_0)$, we set $\sum_n = 0$. For D in the image of the map $h \mapsto D_K \epsilon(h)^{-2} b_{\mathbf{Q}} \det h$, we make the following definition:

$$\begin{aligned} c_{b,g}(D) = {} & p^4 c_{b',f}(Dp^2) + (p^{k+3} + p^{k+2} + p^{k+1}) c_{b',f}(D) + \\ & + \begin{cases} 0 & p \nmid D \\ p^{k+2} c_{b'f}(D) & p \mid D,\ p^2 \nmid D \\ p^{k+2} c_{b',f}(D) + p^{2k} c_{b',f}(Dp^{-2}) & p^2 \mid D. \end{cases} \end{aligned} \tag{5.16}$$

We now check as in the proof of Proposition 5.14 that g is a Maass form. □

5.6. **Descent.** In this section we assume that h_K is odd and choose a base $\mathcal{B}$ as in Corollary 3.9. One has $\#\mathcal{B} = \#\mathcal{C} = h_K$ with $\mathcal{C} = \{p_b \mid b \in \mathcal{B}\}$. We also assume that $I_2 \in \mathcal{B}$. Let χ_K be the quadratic Dirichlet character associated with the extension $K/\mathbf{Q}$. For a positive integer n, set

$$a_K(n) = \#\{\alpha \in (iD_K^{-1/2}\mathcal{O}_K)/\mathcal{O}_K \mid D_K N_{K/\mathbf{Q}}(\alpha) \equiv -n \pmod{D_K}\}.$$

Theorem 5.16. *There exists a* $\mathbf{C}$*-linear injection of vector spaces*

$$\mathrm{Desc}_{\mathcal{B}} : \mathcal{M}^{\mathrm{M}}_{k,-k/2} \hookrightarrow \prod_{b \in \mathcal{B}} M_{k-1}(D_K, \chi_K),$$

such that $\mathrm{Desc}_{\mathcal{B}}(f) = (\phi_b)_{b \in \mathcal{B}}$ *with*

$$a_{\phi_b}(n) = i \frac{a_K(n)}{\sqrt{D_K}} c_{b,f}(n),$$

where $a_{\phi_b}(n)$ *is the* n*-th Fourier coefficient of* ϕ_b*. The map* $\mathrm{Desc}_{\mathcal{B}}$ *depends on the choice of* $\mathcal{B}$*.*

Proof. This follows immediately from [29], Theorem 6 and formula (4) using our assumption on $\mathcal{B}$ and (5.3). □

Remark 5.17. Krieg in [29] explicitly describes the image of the descent map he defines and denotes it by $G_{k-1}(D_K,\chi_K)^*$. The image of $\mathrm{Desc}_{\mathcal{B}}$ is exactly

$$\prod_{b\in\mathcal{B}} G_{k-1}(D_K,\chi_K)^* \subset \prod_{b\in\mathcal{B}} M_{k-1}(D_K,\chi_K).$$

Identify $\mathcal{B}$ with Cl_K via the map that sends b to $c_K(\det b)$, where $c_K : \mathbf{A}_K^\times \twoheadrightarrow \mathrm{Cl}_K$ is the canonical projection. Then Cl_K acts faithfully on $\mathcal{B}$ by multiplication and defines an embedding $s : \mathrm{Cl}_K \hookrightarrow S'_{\mathcal{B}}$, where $S'_{\mathcal{B}}$ is the group of permutations of the elements of $\mathcal{B}$. Write $S_{\mathcal{B}}$ for the image of s. For a split p with $p\mathcal{O}_K = \mathfrak{p}\overline{\mathfrak{p}}$, let $\alpha_{\mathfrak{p},p}$ be as in section 5.4. Note that $\alpha_{\overline{\mathfrak{p}},p} = \alpha^*_{\mathfrak{p},p}$. Write $\sigma_{\mathfrak{p},n} := s\circ c_K(\det \alpha^n_{\mathfrak{p},p})$. If $A=(a_b)_{b\in\mathcal{B}}$ is an ordered tuple indexed by elements of $\mathcal{B}$, and $\sigma\in S_{\mathcal{B}}$, define $\sigma(A)$ to be the $\mathcal{B}$-tuple whose b-component is $a_{\sigma(b)}$. Write $\alpha^n_{\mathfrak{p},p} = \gamma_{\mathfrak{p},n}\sigma_{\mathfrak{p},n}(I_2)\kappa_{\mathfrak{p},n}$ for $\gamma_{\mathfrak{p},n}\in \mathrm{GL}_2(K)$ and $\kappa_{\mathfrak{p},n}\in\mathcal{K}'$.

Write T_p for the classical Hecke operator acting on elliptic modular forms, i.e., for $\phi(z) = \sum_{n=1}^\infty a(n)e^{2\pi i n z} \in M_m(N,\psi)$ define $\phi' := T_p\phi$ by $\phi'(z) = \sum_{n=1}^\infty a'(n)e^{2\pi i n z}$ with $a'(n) = a(np) + \psi(p)p^{m-1}a(n/p)$. Here $a(n)=0$ if $n\not\in\mathbf{Z}_{\geq 0}$. The action of T_p on $M_{k-1}(D_K,\chi_K)$ extends component-wise to an action on $\prod_{b\in\mathcal{B}} M_{k-1}(D_K,\chi_K)$ which we will again denote by T_p. Write $\mathbf{T}_p$ for the $\mathbf{C}$-subalgebra of endomorphisms of $\prod_{b\in\mathcal{B}} M_{k-1}(D_K,\chi_K)$ generated by T_p if p is split (resp. by T_p^2 if p is inert) and the group $S_{\mathcal{B}}$.

Theorem 5.18. *Let p be a rational prime which splits in K. There exists a $\mathbf{C}$-algebra map*

$$\mathrm{Desc}_{\mathcal{B},p} : \mathcal{H}_p^+ \to \mathbf{T}_p,$$

such that for every $T\in\mathcal{H}_p^+$ the following diagram

$$\begin{array}{ccc} \mathcal{M}^{\mathrm{M}}_{k,-k/2} & \xrightarrow{\quad T \quad} & \mathcal{M}^{\mathrm{M}}_{k,-k/2} \\ \downarrow{\scriptstyle \mathrm{Desc}_{\mathcal{B}}} & & \downarrow{\scriptstyle \mathrm{Desc}_{\mathcal{B}}} \\ \prod_{b\in\mathcal{B}} M_{k-1}(D_K,\chi_K) & \xrightarrow{\mathrm{Desc}_{\mathcal{B},p}(T)} & \prod_{b\in\mathcal{B}} M_{k-1}(D_K,\chi_K) \end{array}$$

commutes. Moreover, one has

$$\begin{aligned} \mathrm{Desc}_{\mathcal{B},p}(T_{\mathfrak{p},1}) &= p^{2-k/2}(p+1)T_p\circ\sigma_{\mathfrak{p},1}, \\ \mathrm{Desc}_{\mathcal{B},p}(T_{\mathfrak{p},2}) &= p^{4-k}(T_p^2 + p^{k-1} + p^{k-3})\circ\sigma_{\mathfrak{p},2}. \end{aligned} \tag{5.17}$$

Proof. This follows from Theorem 5.16, and formulas (5.14) and (5.16). Just for illustration, we include the argument in the case of $T_{\mathfrak{p},1}$. Let $f\in\mathcal{M}^{\mathrm{M}}_{k,-k/2}$ and set $g = T_{\mathfrak{p},1}f$. Fix $b\in\mathcal{B}$ and write b_1 for $\sigma_{\mathfrak{p},1}(b)$. One has $\alpha_{\mathfrak{p},p} = \gamma b'\kappa$, where $\gamma=\gamma_{\mathfrak{p},1}\in\mathrm{GL}_2(K)$, $\kappa=\kappa_{\mathfrak{p},1}\in\mathcal{K}'$, $b'=\sigma_{\mathfrak{p},1}(I_2)\in\mathcal{B}$ with γ,κ diagonal. In fact one can choose γ to be of the form $\left[\begin{smallmatrix}1 & \\ & *\end{smallmatrix}\right]$. Write $bb' = \gamma_{b,b'}b_1\kappa_{b,b'}\in\mathrm{GL}_2(K)b_1\mathcal{K}'$, where $\gamma_{b,b'}$, $\kappa_{b,b'}$ are scalars. Then

$$b\alpha_{\mathfrak{p},p} = \gamma bb'\kappa = \gamma\gamma_{b,b'}b_1\kappa\kappa_{b,b'}.$$

Identify $\mathcal{M}^{\mathrm{M}}_{k,-k/2}$ with $\prod_{c\in\mathcal{B}} M_k^{\mathrm{h,M}}$ via $f'\mapsto (f'_c)_{c\in\mathcal{B}}$. For $f'_c\in M_k^{\mathrm{h,M}}$, $h\in\mathcal{S}$, denote by $c_{f'_c}(h)$ the h-Fourier coefficient of f'_c. We will study the action of $T_{\mathfrak{p},1}$ on $c_{f_{b_1}}(h)$.

Since f is a Maass form it is enough to consider h of the form $\left[\begin{smallmatrix}1&*\\ *&*\end{smallmatrix}\right]$. Fix such an h. Set $D = D_K \det h$. Then by (5.3) and (5.14),

$$c_{g_b}(h) = e^{2\pi \operatorname{tr} h} c_g(h, b) = p^2(p+1)(c_{b\alpha_{\mathfrak{p},p},f}(Dp) + p^{k-2}c_{b\alpha_{\mathfrak{p},p},f}(Dp^{-1})).$$

By (5.3) and (5.1), we have

(5.18)

$$c_{f_{b_1}}\left(h\left[\begin{smallmatrix}1&\\&p\end{smallmatrix}\right]\right) = e^{2\pi \operatorname{tr} h\left[\begin{smallmatrix}1&\\&p\end{smallmatrix}\right]} c_f\left(h\left[\begin{smallmatrix}1&\\&p\end{smallmatrix}\right], b_1\right) = c_{b_1,f}(Dp) = |\det \gamma\gamma_{b,b'}|^k c_{b\alpha_{\mathfrak{p},p},f}(Dp)$$

$$c_{f_{b_1}}\left(h\left[\begin{smallmatrix}1&\\&1/p\end{smallmatrix}\right]\right) = e^{2\pi \operatorname{tr} h\left[\begin{smallmatrix}1&\\&1/p\end{smallmatrix}\right]} c_f\left(h\left[\begin{smallmatrix}1&\\&1/p\end{smallmatrix}\right], b_1\right) = c_{b_1,f}(Dp^{-1}) = = |\det \gamma\gamma_{b,b'}|^k c_{b\alpha_{\mathfrak{p},p},f}(Dp^{-1}),$$

where we have used the fact that $\epsilon(h) = \epsilon\left(h\left[\begin{smallmatrix}1&\\&p\end{smallmatrix}\right]\right) = 1 = \epsilon\left(h\left[\begin{smallmatrix}1&\\&1/p\end{smallmatrix}\right]\right)$ for h as above (the last equality holding for h such that $h\left[\begin{smallmatrix}1&\\&1/p\end{smallmatrix}\right] \in \mathcal{S}$). This gives us

$$c_{g_b}(h) = p^2(p+1)|\det \gamma\gamma_{b,b'}|^{-k}(c_{b_1,f}(Dp) + p^{k-2}c_{b_1,f}(Dp^{-1})).$$

Note that $|\det \gamma|^k = p^{k/2}$ since if we write x for the adele whose $\mathfrak{p}$th and $\overline{\mathfrak{p}}$th component is p and all the other components are 1, then we have

$$x^{k/2} = |\det \alpha_{\mathfrak{p},p}|^k = |\det \gamma|^k |\det b'|^k |\det \kappa|^k.$$

The claim now follows from Theorem 5.16 and the fact that $|\det \gamma_{b,b'}|^k = 1$, which follows from $bb^* = b'(b')^* = I_2$. □

For completeness we also include the analogue of Theorem 5.18 for an inert p. It can be proved in the same way or can be deduced from the results of section 3 of [17].

Theorem 5.19. *Let p be a rational prime which is inert in K. There exists a* $\mathbf{C}$*-algebra map*

$$\mathrm{Desc}_{\mathcal{B},p} : \mathcal{H}_p^+ \to \mathbf{T}_p,$$

such that for every $T \in \mathcal{H}_p$ the following diagram

$$\begin{array}{ccc} \mathcal{M}^{\mathrm{M}}_{k,-k/2} & \xrightarrow{\quad T \quad} & \mathcal{M}^{\mathrm{M}}_{k,-k/2} \\ \downarrow{\scriptstyle \mathrm{Desc}_{\mathcal{B}}} & & \downarrow{\scriptstyle \mathrm{Desc}_{\mathcal{B}}} \\ \prod_{b\in\mathcal{B}} M_{k-1}(D_K, \chi_K) & \xrightarrow{\mathrm{Desc}_{\mathcal{B},p}(T)} & \prod_{b\in\mathcal{B}} M_{k-1}(D_K, \chi_K) \end{array}$$

commutes. Moreover, one has

$$\begin{aligned} \mathrm{Desc}_{\mathcal{B},p}(T_{p,0}) &= p^{-k+4}(p^2+1)T_p^2 + p^4 + p^3 + p - 1, \\ \mathrm{Desc}_{\mathcal{B},p}(U_p) &= p^8(T_p^4 + (p+3)p^{k-2}T_p^2 + p^{2k-4}(p^2+p+1)). \end{aligned} \tag{5.19}$$

5.7. ***L*-functions.** In this section we study eigenforms in $\mathcal{M}^{\mathrm{M}}_{k,-k/2}$ and give a formula for the standard L-function of such an eigenform.

From now on assume that D_K is prime. It is well-known that this implies that h_K is odd, hence we can (and will) choose $\mathcal{B}$ be as in Corollary 3.9. One has $\#\mathcal{B} = \#\mathcal{C} = h_K$ with $\mathcal{C} = \{p_b \mid b \in \mathcal{B}\}$. On the other hand, for such a D_K the space $S_{k-1}(D_K, \chi_K)$ of cusp forms inside $M_{k-1}(D_K, \chi_K)$ has a basis $\mathcal{N}$ consisting of newforms. In particular, if $\phi \in S_{k-1}(D_K, \chi_K)$ is an eigenform for almost all T_p, it is so for all T_p. For $\phi = \sum_{n=1}^\infty a_\phi(n)e^{2\pi i n z} \in S_{k-1}(D_K, \chi_K)$, set $\phi^\rho(z) := \sum_{n=1}^\infty \overline{a_\phi(n)}e^{2\pi i n z}$. Let $\mathcal{N}' \subset \mathcal{N}$ denote the set formed by choosing one element from each pair (ϕ, ϕ^ρ) such that $\phi \in \mathcal{N}$ and $\phi \neq \phi^\rho$. Then the set $\{\phi - \phi^\rho \mid \phi \in \mathcal{N}'\}$ is a basis of $S^*_{k-1}(D_K, \chi_K) := G^*_{k-1}(D_K, \chi_K) \cap S_{k-1}(D_K, \chi_K)$ (cf. [29], Remark (b) on p. 670). Let $\chi \in \mathrm{Hom}(\mathrm{Cl}_K, \mathbf{C}^\times)$. Recall (cf. (3.8)) that we have the decomposition

$$\mathcal{M}_{k,-k/2} = \bigoplus_{\chi \in \mathrm{Hom}(\mathrm{Cl}_K, \mathbf{C}^\times)} \mathcal{M}^\chi_{k,-k/2}. \tag{5.20}$$

Let $\mathcal{S}_{k,-k/2}$ denote the subspace of cusp forms in $\mathcal{M}_{k,-k/2}$ and write $\mathcal{S}^{\mathrm{M}}_{k,-k/2}$ for $\mathcal{S}_{k,-k/2} \cap \mathcal{M}^{\mathrm{M}}_{k,-k/2}$. It is clear that a decomposition analogous to (5.20) holds for $\mathcal{S}_{k,-k/2}$, with $\mathcal{S}^\chi_{k,-k/2}$ having the obvious meaning. Write $\mathcal{S}^{\mathrm{M},\chi}_{k,-k/2}$ for $\mathcal{S}^{\mathrm{M}}_{k,-k/2} \cap \mathcal{M}^\chi_{k,-k/2}$.

Let $\mathbf{T}_p$ be as in section 5.6 and write $\mathbf{T}'_{\mathbf{C}}$ for $\mathbf{C}$-subalgebra of the ring of endomorphisms of $S_{k-1}(D_K, \chi_K)$ generated by T_p (for p split) and T_p^2 (for p inert). The algebra $\mathbf{T}'_{\mathbf{C}}$ acts on $S^*_{k-1}(D_K, \chi_K)$. For $\phi \in \mathcal{N}'$, the element $\phi - \phi^\rho \in S^*_{k-1}(D_K, \chi_K)$ is a non-zero eigenform for $\mathbf{T}'_{\mathbf{C}}$ and for every $\chi \in \mathrm{Hom}(\mathrm{Cl}_K, \mathbf{C}^\times)$, the $\mathcal{B}$-tuple

$$\phi_\chi := (\chi(c_K(\det b))(\phi - \phi^\rho))_{b \in \mathcal{B}}$$

is an eigenform for $\mathbf{T}_p$ for every $p \neq D_K$. Below we will write $\chi(b)$ instead of $\chi(c_K(\det b))$. Hence $f_{\phi,\chi} := \mathrm{Desc}^{-1}_{\mathcal{B}}(\phi_\chi)$ lies in $\mathcal{S}^{\mathrm{M},\chi}_{k,-k/2}$ and is an eigenform for $\mathcal{H}_p$ for every $p \neq D_K$.

Remark 5.20. Since $2 \nmid h_K$, the prime $\mathfrak{p}$ such that $D_K \mathcal{O}_K = \mathfrak{p}^2$ is principal. Hence one can use the calculations in [17] to conclude that the Maass space is also invariant under the action of $\mathcal{H}_{D_K}$ and hence that $f_{\phi,\chi}$ is an eigenform for $\mathcal{H}_p$ for all p. Also note that for a split prime p and a prime $\mathfrak{p}$ of K lying over p the operator $\Delta_{\mathfrak{p}}$ acts on $f_{\phi,\chi}$ via multiplication by a scalar, so we conclude that the Maass space is in fact invariant under $\mathcal{H}_p$.

We have proved the following proposition.

Proposition 5.21. *Let* $\mathrm{Desc}_{\mathcal{B}}$ *be the map from Theorem 5.16. The composite*

$$\mathcal{M}^{\mathrm{M}}_{k,-k/2} \xrightarrow{\mathrm{Desc}_{\mathcal{B}}} \prod_{b \in \mathcal{B}} M_{k-1}(D_K, \chi_K) \xrightarrow{\mathrm{pr}_{I_2}} M_{k-1}(D_K, \chi_K)$$

induces $\mathbf{C}$*-linear isomorphisms* $\mathcal{S}^{\mathrm{M},\chi}_{k,-k/2} \cong S^*_{k-1}(D_K, \chi_K)$ *for every* $\chi \in \mathrm{Hom}(\mathrm{Cl}_K, \mathbf{C}^\times)$. *The inverse of such an isomorphism (for a fixed* χ*) is induced by*

$$\phi - \phi^\rho \mapsto f_{\phi,\chi} := \mathrm{Desc}^{-1}_{\mathcal{B}}(\phi_\chi)$$

for any $\phi \in \mathcal{N}'$. *Moreover, these isomorphisms are Hecke-equivariant with respect to the Hecke algebra maps* (5.17) *and* (5.19) *except that in* (5.17) *we replace composition with* $\sigma_{\mathfrak{p},n}$ *by multiplication by* $\chi(\alpha^n_{\mathfrak{p}})$.

It follows that if ϕ runs over $\mathcal{N}'$ and F_ϕ denotes the Maass lift of ϕ in the sense of Krieg [29], i.e., $F_\phi = \mathrm{Desc}^{-1}_K(\phi - \phi^\rho) \in M^{\mathrm{h,M}}_k$, where $\mathrm{Desc}_K : M^{\mathrm{h,M}}_k \xrightarrow{\sim} S^*_{k-1}(D_K, \chi_K)$ is the (non-Hecke-equivariant) isomorphism constructed in Theorem on p.676 of [29], then the set of $\mathcal{B}$-tuples $\{(\chi(b)F_\phi)_{b \in \mathcal{B}}\}$ is basis of eigenforms of $\mathcal{S}^{\mathrm{M},\chi}_{k,-k/2}$ after we identify $\mathcal{S}^{\mathrm{M},\chi}_{k,-k/2}$ with its image inside $\prod_{b \in \mathcal{B}} M^{\mathrm{h,M}}_k$.

Let $\phi \in S_{k-1}(D_K, \chi_K)$ be a newform and write $f_{\phi,\chi} \in \mathcal{S}^{\mathrm{M},\chi}_{k,-k/2}$ for its Maass lift as above. For a unitary Hecke character $\psi : K^\times \setminus \mathbf{A}^\times_K \to \mathbf{C}^\times$ and a Hecke eigenform $f \in \mathcal{S}^{\chi}_{k,-k/2}$ denote by $L_{\rm st}(f,s,\psi) = Z(s,f,\psi)$ the standard L-function of f twisted by ψ as defined in [39], section 20.6 with the Euler factor at D_K removed. Moreover, for the newform $\phi \in S_{k-1}(D_K,\chi_K)$, and a prime $\mathfrak{p}$ of $\mathcal{O}_K$ of characteristic $p \neq D_K$, set $\alpha_{\mathfrak{p},j} := \alpha_{p,j}^d$, where $\alpha_{p,j}$, $j = 1, 2$ are the p-Satake parameters of ϕ and $d := [\mathcal{O}_K/\mathfrak{p} : \mathbf{F}_p]$. For $s \in \mathbf{C}$ with $\mathrm{Re}(s)$ sufficiently large, define

$$L(\mathrm{BC}(\phi), s, \psi) := \prod_{\mathfrak{p}\nmid D_K} \prod_{j=1}^{2} (1 - \psi^*(\mathfrak{p})\alpha_{\mathfrak{p},j}(N\mathfrak{p})^{-s})^{-1},$$

where ψ^* denotes the ideal character associated to ψ and $N\mathfrak{p}$ denotes the norm of $\mathfrak{p}$. It is well-known that $L(\mathrm{BC}(\phi), s, \psi)$ can be continued to the entire $\mathbf{C}$-plane.

Proposition 5.22. *Let ψ be a unitary Hecke character of K. The following identity holds:*

$$L_{\rm st}(f_{\phi,\chi}, s, \psi) = L(\mathrm{BC}(\phi), s - 2 + k/2, \chi\psi)L(\mathrm{BC}(\phi), s - 3 + k/2, \chi\psi).$$

Proof. This is an easy calculation involving Satake parameters. □

Remark 5.23. In [25] Ikeda has studied liftings from the space of elliptic cusp forms into the space of hermitian cusp forms defined on the group U_n with no assumptions on the class number of K. In particular he constructs a lifting $S_{k-1}(D_K,\chi_K) \to \mathcal{S}_{k,-k/2}$, which agrees with the map $\phi \mapsto f_{\phi,\mathbf{1}}$, where $\mathbf{1}$ denotes the trivial character. The method used in [25] is different from ours.

5.8. **The Petersson norm of a Maass lift.** Let $\phi \in S_{k-1}(D_K,\chi_K)$ be a newform such that $\phi \neq \phi^\rho$. Let $\chi : \mathrm{Cl}_K \to \mathbf{C}^\times$ be a character. Write $f_{\phi,\chi} \in \mathcal{S}^{\mathrm{M},\chi}_{k,-k/2}$ for the Maass lift of ϕ.

Theorem 5.24. *Let $\ell > 3$ be an odd prime and assume $\ell \nmid h_K D_K$. Then one has*

$$\langle f_{\phi,\chi}, f_{\phi,\chi}\rangle = C\pi^{-k-2} \cdot \langle \phi, \phi\rangle\, L(\mathrm{Symm}^2 \phi, k),$$

where $C_\chi \in \mathbf{Q}^\times$ is a ℓ-adic unit and

$$L(\mathrm{Symm}^2 \phi, s)^{-1} := \prod_{p\nmid D_K} (1-\alpha_p^2 p^{-s})(1-\alpha_p\beta_p p^{-s})(1-\beta_p^2 p^{-s}) \times \tag{5.21}$$
$$\times \prod_{p\mid D_K} (1-a(p)^2 p^{-s})(1-\overline{a(p)}^2 p^{-s}).$$

Here α_p, β_p are the classical p-Satake parameters of ϕ and $a(p)$ is the p-th Fourier coefficient of ϕ.

Proof. This is Proposition 17.4 in [25], which is essentially due to Sugano - see the references cited in [loc.cit.]. □

6. Completed Hecke algebras

Let ℓ, as before, be a fixed prime such that $\ell \nmid 2D_K$. Suppose D_K is prime. Then the space $S_{k-1}(D_K,\chi_K)$ has a canonical basis $\mathcal{N}$ consisting of newforms. The goal of this section is to construct a Hecke operator $T^{\rm h}$ acting on the space $\mathcal{S}^{\chi}_{k,-k/2}$ such that $T^{\rm h}$ preserves the ℓ-integrality of the Fourier coefficients of the hermitian modular forms in $\mathcal{S}^{\chi}_{k,-k/2}$ and such that $T^{\rm h} f_{\phi,\chi} = \eta f_{\phi,\chi}$ for a Maass lift $f_{\phi,\chi}$ of an elliptic modular form ϕ and $T^{\rm h} f = 0$ for all the $f \in \mathcal{S}^{\mathrm{M},\chi}_{k,-k/2}$ with $\langle f, f_{\phi,\chi}\rangle = 0$. Here η is a generator of the Hida's congruence ideal.

6.1. **Elliptic Hecke algebras.** Let $\mathbf{T}_{\mathbf{Z}}$ be the $\mathbf{Z}$-subalgebra of $\mathrm{End}_{\mathbf{C}}(S_{k-1}(D_K, \chi_K))$ generated by the (standard) Hecke operators T_n, $n = 1, 2, 3, \dots$ (for the action of T_p on the Fourier coefficients see section 5.6).

Definition 6.1. For every $\mathbf{Z}$-algebra A we set

(i) $\mathbf{T}_A := \mathbf{T}_{\mathbf{Z}} \otimes_{\mathbf{Z}} A$;

(ii) $\mathbf{T}'_A$ to be the A-subalgebra of $\mathbf{T}_A$ generated by the set

$$\Sigma' := \{T_p\}_{p \text{ split in } K} \cup \{T_{p^2}\}_{p \text{ inert in } K}.;$$

(iii) $\tilde{\mathbf{T}}'_A$ to be the A-subalgebra of $\mathbf{T}'_A$ generated by $\tilde{\Sigma}'$, where $\tilde{\Sigma}' = \Sigma' \setminus \{T_\ell\}$.

Suppose $\phi = \sum_{n=1}^{\infty} a_\phi(n) q^n \in \mathcal{N}$. For $T \in \mathbf{T}_{\mathbf{C}}$, set $\lambda_{\phi,\mathbf{C}}(T)$ to denote the eigenvalue of T corresponding to ϕ. It is a well-known fact that $\lambda_{\phi,\mathbf{C}}(T_n) = a_\phi(n)$ for all $\phi \in \mathcal{N}$ and that the set $\{a_\phi(n)\}_{n \in \mathbf{Z}_{>0}}$ is contained in the ring of integers of a finite extension L_ϕ of $\mathbf{Q}$. Let E be a finite extension of $\mathbf{Q}_\ell$ containing the fields L_ϕ for all $\phi \in \mathcal{N}$. Denote by $\mathcal{O}$ the valuation ring of E and by ϖ a uniformizer of $\mathcal{O}$. Then $\{a_\phi(n)\}_{\phi \in \mathcal{N}, n \in \mathbf{Z}_{>0}} \subset \mathcal{O}$. Moreover, one has

$$\mathbf{T}_E = \prod_{\phi \in \mathcal{N}} E \tag{6.1}$$

and

$$\mathbf{T}_{\mathcal{O}} = \prod_{\mathfrak{m}} \mathbf{T}_{\mathcal{O},\mathfrak{m}}, \tag{6.2}$$

where $\mathbf{T}_{\mathcal{O},\mathfrak{m}}$ denotes the localization of $\mathbf{T}_{\mathcal{O}}$ at $\mathfrak{m}$ and the product runs over all maximal ideals of $\mathbf{T}_{\mathcal{O}}$. Analogous decompositions hold for $\mathbf{T}'_{\mathcal{O}}$ and $\tilde{\mathbf{T}}'_{\mathcal{O}}$. Every $\phi \in \mathcal{N}$ gives rise to an $\mathcal{O}$-algebra homomorphism $\mathbf{T}_{\mathcal{O}} \to \mathcal{O}$ assigning to T the eigenvalue of T corresponding to ϕ. We denote this homomorphism by λ_ϕ and its reduction mod ϖ by $\overline{\lambda}_\phi$. If $\mathfrak{m} = \ker \overline{\lambda}_\phi$, we write $\mathfrak{m}_\phi$ for $\mathfrak{m}$. For simplicity in this section we will drop the subscript $\mathcal{O}$ from notation, so for example, we will simply write $\mathbf{T}$ instead of $\mathbf{T}_{\mathcal{O}}$.

Fix a maximal ideal $\mathfrak{m}$ of $\mathbf{T}$. Write $\mathfrak{m}'$ (resp. $\tilde{\mathfrak{m}}'$) for the maximal ideal of $\mathbf{T}'$ (resp. $\tilde{\mathbf{T}}'$) corresponding to $\mathfrak{m}$. We have the following commutative diagram

$$\begin{array}{ccccc} \tilde{\mathbf{T}}' & \longrightarrow & \mathbf{T}' & \longrightarrow & \mathbf{T} \\ \big\downarrow{\scriptstyle \tilde{\pi}'} & & \big\downarrow{\scriptstyle \pi'} & & \big\downarrow{\scriptstyle \pi} \\ \tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'} & \xrightarrow{i'} & \mathbf{T}'_{\mathfrak{m}'} & \xrightarrow{i} & \mathbf{T}_{\mathfrak{m}} \end{array} \tag{6.3}$$

where the top arrows are the natural inclusions and the vertical arrows are the canonical surjections coming from the decomposition (6.2) and its analogues. Note that the localizations of the Hecke algebras in the bottom row of (6.3) are Noetherian, local, complete $\mathcal{O}$-algebras. In [27], we proved the following properties of the maps i and i' (the proofs in [loc.cit.] are for $D_K = 4$, but they generalize verbatim to the general case).

Theorem 6.2. *The map i in (6.3) is an injection. Moreover if $\phi \in \mathcal{N}$ is ordinary at ℓ and $\overline{\rho}_\phi|_{G_K}$ is absolutely irreducible, then both i and i' are surjective.*

Proof. The first statement is Proposition 8.5 in [27]. The surjectivity statement for i is the main result of section 8.2 in [loc.cit.] - see Corollary 8.12. The surjectivity statement for i' follows from Corollary 8.10 in [loc.cit.]. $\square$

We also record the following result from [27], which again works for any D_K.

Proposition 6.3 ([27], Proposition 8.13)**.** *If $\overline{\rho}_\phi|_{G_K}$ is absolutely irreducible, then $\phi \not\equiv \phi^\rho \pmod{\varpi}$.*

Fix $\phi \in \mathcal{N}$ and set $\mathcal{N}_\phi := \{\psi \in \mathcal{N} \mid \mathfrak{m}_\psi = \mathfrak{m}_\phi\}$, where $\mathfrak{m}_\phi$ (resp. $\mathfrak{m}_\psi$) is the maximal ideal of $\mathbf{T}$ corresponding to ϕ (resp. to ψ). Similarly, we define $\mathcal{N}'_\phi$ and $\tilde{\mathcal{N}}'_\phi$. It is easy to see that we can identify $\mathbf{T}_{\mathfrak{m}_\phi}$ with the image of $\mathbf{T}$ inside $\mathrm{End}_{\mathbf{C}}(S_{k-1,\phi})$, where $S_{k-1,\phi} \subset S_{k-1}(D_K, \chi_K)$ is the subspace spanned by $\mathcal{N}_\phi$. Similarly we define $S'_{\phi,k-1}$ and $\tilde{S}'_{\phi,k-1}$.

Lemma 6.4. *Suppose that ϕ is ordinary at ℓ and that $\overline{\rho}_\phi|_{G_K}$ is absolutely irreducible. Then $\tilde{\mathcal{N}}'_\phi = \mathcal{N}'_\phi$ and the set $\mathcal{N}_\phi$ is formed from the set $\mathcal{N}'_\phi$ by choosing one element from each pair (ψ, ψ^ρ) such that $\psi \in \mathcal{N}_\phi$.*

Proof. The last statement can be directly deduced from Corollary 8.4 in [27], so we just need to show that if $\psi \in \mathcal{N}'_\phi$, then $\psi \in \tilde{\mathcal{N}}'_\phi$. This, as we will demonstrate, follows from ordinarity of ϕ. Indeed, since the forms ϕ and ψ have congruent Hecke eigenvalues for all the operators in $\tilde{\mathbf{T}}'$, one can easily show (using Tchebotarev Density Theorem and the Brauer-Nesbitt Theorem) that $\overline{\rho}_\phi|_{G_K} \cong \overline{\rho}_\psi|_{G_K}$. We will explain the case when ℓ is inert in K (the split case being very similar). Using ordinarity at ℓ we conclude that both of the representations ρ_ϕ and ρ_ψ when restricted to the decomposition groups at the prime $\mathfrak{l}$ of K lying over ℓ have a one-dimensional unramified quotient on which G_K operates by the character which sends $\mathrm{Frob}_\mathfrak{l} = \mathrm{Frob}_\ell^2$ to the square of the unique unit root α_h of the polynomial $X^2 - a_h(\ell) + \chi_K(\ell)\ell^{k-2}$, where $h \in \{\phi, \psi\}$ and $a_h(\ell)$ is the eigenvalue of the operator T_ℓ corresponding to h. So, $a_h(\ell)^2 = \alpha_h^2 + 2\chi_K(\ell)\ell^{k-2} + \ell^{2k-4}\alpha_h^{-2}$. Since $\overline{\rho}_\phi|_{G_K} \cong \overline{\rho}_\psi|_{G_K}$, we conclude that $\alpha_\phi^2 \equiv \alpha_\psi^2 \pmod{\varpi}$ and hence $a_\phi(\ell)^2 \equiv a_\psi^2(\ell) \pmod{\varpi}$. □

Write $\mathbf{T}_{\mathfrak{m}_\phi} \otimes E = E \times B_E$, where $B_E = \prod_{\psi \in \mathcal{N}_\phi \setminus \{\phi\}} E$ and let B denote the image of $\mathbf{T}_{\mathfrak{m}_\phi}$ under the composite $\mathbf{T}_{\mathfrak{m}_\phi} \hookrightarrow \mathbf{T}_{\mathfrak{m}_\phi} \otimes E \xrightarrow{\pi_\phi} B_E$, where π_ϕ is projection. Denote by $\delta : \mathbf{T}_{\mathfrak{m}_\phi} \hookrightarrow \mathcal{O} \times B$ the map $T \mapsto (\lambda_\phi(T), \pi_\phi(T))$. If E is sufficiently large, there exists $\eta \in \mathcal{O}$ such that $\mathrm{coker}\,\delta \cong \mathcal{O}/\eta\mathcal{O}$. This cokernel is usually called the *congruence module of* ϕ.

Proposition 6.5. *Assume $\phi \in \mathcal{N}$ is ordinary at ℓ and the associated Galois representation ρ_ϕ is such that $\overline{\rho}_\phi|_{G_K}$ is absolutely irreducible. Then there exists $T \in \tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_\phi}$ such that $T\phi = \eta\phi$, $T\phi^\rho = \eta\phi^\rho$ and $T\psi = 0$ for all $\psi \in \tilde{\mathcal{N}}'_\phi \setminus \{\phi, \phi^\rho\}$.*

Proof. By Theorem 6.2, the natural $\mathcal{O}$-algebra map $\tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_\phi} \to \mathbf{T}_{\mathfrak{m}_\phi}$ is surjective. So by Lemma 6.4, it is enough to find $T \in \mathbf{T}_{\mathfrak{m}_\phi}$ such that $T\phi = \eta\phi$ and $T\psi = 0$ for every $\psi \in \mathcal{N}_\phi \setminus \{\phi\}$. (Note that by Proposition 6.3, $\phi^\rho \notin \mathcal{N}_\phi$.) It follows from the exactness of the sequence $0 \to \mathbf{T}_{\mathfrak{m}_\phi} \xrightarrow{\delta} \mathcal{O} \times B \to \mathcal{O}/\eta\mathcal{O} \to 0$, that $(\eta, 0) \in \mathcal{O} \times B$ is in the image of $\mathbf{T}_{\mathfrak{m}_\phi} \hookrightarrow \mathcal{O} \times B$. Let T be the preimage of $(\eta, 0)$ under this injection. Then T has the desired property. □

Proposition 6.6 ([20], Theorem 2.5)**.** *Suppose $\ell > k$. If $\phi \in \mathcal{N}$ is ordinary at ℓ, then*

$$\eta = (*)\frac{\langle \phi, \phi \rangle}{\Omega_\phi^+ \Omega_\phi^-},$$

where $\Omega_\phi^+, \Omega_\phi^-$ *denote the "integral" periods defined in* [46] *and* $(*)$ *is a* ϖ*-adic unit.*

6.2. **Galois representations attached to hermitian modular forms.** Let $f \in \mathcal{S}^{\chi}_{k,-k/2}$ be an eigenform for the local Hecke algebra $\mathcal{H}_p$ for every $p \nmid D_K$. For every rational prime p, let $\lambda_{p,j}(f)$, $j = 1, \dots, 4$, denote the p-Satake parameters of f. (For the definition of p-Satake parameters when p inerts or ramifies in K, see [24], and for the case when p splits in K, see [18].) Let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$ lying over p.

Theorem 6.7. *There exists a finite extension* E_f *of* $\mathbf{Q}_\ell$ *and a* 4*-dimensional semisimple Galois representation* $\rho_f : G_K \to \mathrm{GL}_{E_F}(V)$ *unramified away from the primes of* K *dividing* $D_K\ell$ *and such that*

(i) *for any prime* $\mathfrak{p}$ *of* K *such that* $\mathfrak{p} \nmid D_K\ell$*, the set of eigenvalues of* $\rho_f(\mathrm{Frob}_\mathfrak{p})$ *coincides with the set of the Satake parameters of* f *at* $\mathfrak{p}$*, i.e., one has*

$$L(\rho_f, s)_\mathfrak{p} = L_{\mathrm{st}}(f, s)_\mathfrak{p},$$

where $L_{\mathrm{st}}(f,s)_\mathfrak{p}$ *is the* $\mathfrak{p}$*-component of the function* $L_{\mathrm{st}}(f, s, \mathbf{1})$ *introduced in section 5.7 and* $L(\rho_f, s)_\mathfrak{p} = \det(I_4 - \rho_f(\mathrm{Frob}_\mathfrak{p})(N\mathfrak{p})^{-s})^{-1}$*;*

(ii) *if* $\mathfrak{p}$ *is a place of* K *over* ℓ*, the representation* $\rho_f|_{D_\mathfrak{p}}$ *is crystalline (cf. section 9).*

(iii) *if* $\ell > k$*, and* $\mathfrak{p}$ *is a place of* K *over* ℓ*, the representation* $\rho_f|_{D_\mathfrak{p}}$ *is short. (For a definition of short we refer the reader to* [12]*, section 1.1.2.)*

(iv) *one has* $\rho^\vee(3) \cong \rho^c\chi^{-2}$*, where* c *denotes the lift to* $G_\mathbf{Q}$ *of the generator of* $\mathrm{Gal}(K/\mathbf{Q})$ *and* $\rho^c(g) = \rho(cgc^{-1})$*.*

Remark 6.8. Theorem 6.7 is stated as Theorem 7.1.1 in [41], where it is attributed to Skinner as a consequence of the work of Morel and Shin. We refer the reader to [41], section 7 for further discussion. Galois representations attached to hermitian modular forms are also discussed in [5] or [3]. We will assume Theorem 6.7 without a proof in what follows.

Remark 6.9. It is not known if the representation ρ_f is also unramified at the prime D_K. See [2] for a discussion of this issue.

6.3. **Integral lifts of Hecke operators.** Fix a character $\chi : \mathrm{Cl}_K \to \mathbf{C}^\times$ and for every prime $p \nmid D_K$ write $\mathcal{H}_p^\chi$ for the quotient of $\mathcal{H}_p^+$ acting on $\mathcal{S}^\chi_{k,-k/2}$.

Definition 6.10. For a prime p which splits in K as $\mathfrak{p}\overline{\mathfrak{p}}$ set $\Sigma_p = \{T_{\mathfrak{p},1}, T_{\mathfrak{p},2}, T_{\overline{\mathfrak{p}},1}, T_{\overline{\mathfrak{p}},2}\}$ and for a prime p which is inert in K set $\Sigma_p = \{T_{p,0}, U_p\}$. Set $\mathcal{H}^\chi_\mathbf{Z}$ to be the $\mathbf{Z}$-subalgebra of $\mathrm{End}_\mathbf{C}(\mathcal{S}^\chi_{k,-k/2})$ generated by the set $\bigcup_{p\nmid D_K\ell} \Sigma_p$. For any $\mathbf{Z}$-algebra A set $\mathcal{H}^\chi_A := \mathcal{H}^\chi_\mathbf{Z} \otimes_\mathbf{Z} A$.

Note that $\mathcal{H}^\chi_\mathbf{Z}$ is a finite free $\mathbf{Z}$-algebra. As before let E be a sufficiently large finite extension of $\mathbf{Q}_\ell$ with valuation ring $\mathcal{O}$. We fix a uniformizer $\varpi \in \mathcal{O}$. To ease notation put $\mathcal{H}^\chi = \mathcal{H}^\chi_\mathcal{O}$.

Lemma 6.11. *Let* $\ell \nmid 2D_K$ *be a rational prime,* E *a finite extension of* $\mathbf{Q}_\ell$ *and* $\mathcal{O}$ *the valuation ring of* E*. Suppose that* $f \in \mathcal{S}^\chi_{k,-k/2}$*,* $T \in \mathcal{H}^\chi$ *and* $c_f(h,q) \in \mathbf{C}$ *is the* (h,q)*-Fourier coefficient of* f*. Write* $c_{Tf}(h,q)$ *for the corresponding Fourier coefficient of* Tf*. Assume that there exists* $\alpha \in \mathbf{C}$ *such that* $\alpha c_f(h,q) \in \mathcal{O}$*. Then* $\alpha c_{Tf}(h,q) \in \mathcal{O}$*.*

Proof. This follows directly from Propositions 4.4 and 4.5 (note that the powers of p in Proposition 4.5 are ℓ-adic units). □

Proposition 6.12. *The space $\mathcal{S}^{\chi}_{k,-k/2}$ has a basis consisting of eigenforms.*

Proof. This is a standard argument, which uses the fact that $\mathcal{H}^{\chi}_{\mathbf{C}}$ is commutative and all $T \in \mathcal{H}^{\chi}_{\mathbf{C}}$ are self-adjoint. □

From now on $\mathcal{N}^{\mathrm{h}}$ will denote a fixed basis of eigenforms of $\mathcal{S}^{\chi}_{k,-k/2}$.

Theorem 6.13. *Let $f \in \mathcal{N}^{\mathrm{h}}$. There exists a number field L_f with ring of integers $\mathcal{O}_{L_f}$ such that the f-eigenvalue of every Hecke operator $T \in \mathcal{H}^{\chi}_{\mathcal{O}_{L_f}}$ lies in $\mathcal{O}_{L_f}$.*

Proof. This can be seen as a consequence of Theorem 6.7. □

Let ℓ be a rational prime and E a finite extension of $\mathbf{Q}_\ell$ containing the fields L_f from Theorem 6.13 for all $f \in \mathcal{N}^{\mathrm{h}}$. Denote by $\mathcal{O}$ the valuation ring of E and by ϖ a uniformizer of $\mathcal{O}$. As in the case of elliptic modular forms, $f \in \mathcal{N}^{\mathrm{h}}$ gives rise to an $\mathcal{O}$-algebra homomorphism $\mathcal{H}^{\chi} \to \mathcal{O}$ assigning to T the eigenvalue of T corresponding to the eigenform f. We denote this homomorphism by λ_f. Proposition 6.12 and Theorem 6.13 imply that we have

$$\mathcal{H}^{\chi}_E \cong \prod_{f \in \mathcal{N}^{\mathrm{h}}} E.$$

Moreover, as in the elliptic modular case, we have

$$\mathcal{H}^{\chi} \cong \prod_{\mathfrak{m}} \mathcal{H}^{\chi}_{\mathfrak{m}}, \tag{6.4}$$

where the product runs over the maximal ideals of $\mathcal{H}^{\chi}$ and $\mathcal{H}^{\chi}_{\mathfrak{m}}$ denotes the localization of $\mathcal{H}^{\chi}$ at $\mathfrak{m}$.

The descent map Desc defined in section 5.6 induces the following map (for which we use the same name):

$$\mathrm{Desc} : \mathcal{H}^{\chi} \to \tilde{\mathbf{T}}', \tag{6.5}$$

given by the following formulas (cf. Theorems 5.18, 5.19 and Proposition 5.21)

$$\begin{aligned} \mathrm{Desc}(T_{\mathfrak{p},1}) &= u_1(p+1)T_p, \\ \mathrm{Desc}(T_{\mathfrak{p},2}) &= u_2(T_p^2 + p^{k-1} + p^{k-3}) \\ \mathrm{Desc}(T_{p,0}) &= u_3(p^2+1)T_p^2 + p^4 + p^3 + p - 1, \\ \mathrm{Desc}(U_p) &= u_4 T_p^4 + u_5(p+3)T_p^2 + p^{2k+4}(p^2+p+1) \end{aligned} \tag{6.6}$$

where $u_1, u_2, u_3, u_4, u_5 \in \mathcal{O}^{\times}$.

The first two formulas in (6.6) are for a prime $\mathfrak{p}$ of K lying over a split prime $p \neq \ell$ and the last two for an inert prime $p \neq \ell$. The map (6.5) factors through

$$\mathrm{Desc} : \mathcal{H}^{\chi} \twoheadrightarrow \mathcal{H}^{\mathrm{M},\chi} \to \tilde{\mathbf{T}}', \tag{6.7}$$

where $\mathcal{H}^{\mathrm{M},\chi}$ is the quotient of $\mathcal{H}^{\chi}$ acting on the space $\mathcal{S}^{\mathrm{M},\chi}_{k,-k/2}$. Now fix a newform $\phi \in S_{k-1}(D_K, \chi_K)$ such that $\overline{\rho}_\phi|_{G_K}$ is absolutely irreducible. Then by Proposition 6.3 we in particular have that $\phi \neq \phi^\rho$. Write $f_{\phi,\chi}$ for the Maass lift of ϕ lying in the space $\mathcal{S}^{\mathrm{M},\chi}_{k,-k/2}$. Write $\tilde{\mathfrak{m}}'_\phi$ for the maximal ideal of $\tilde{\mathbf{T}}'$ corresponding to ϕ and $\mathfrak{m}_\phi$ (resp. $\mathfrak{m}^{\mathrm{M}}_\phi$) for the corresponding maximal ideals of $\mathcal{H}^{\chi}$ (resp. $\mathcal{H}^{\mathrm{M},\chi}$). The map (6.7) induces the corresponding map on localizations:

$$\mathrm{Desc} : \mathcal{H}^{\chi}_{\mathfrak{m}_\phi} \twoheadrightarrow \mathcal{H}^{\mathrm{M},\chi}_{\mathfrak{m}^{\mathrm{M}}_\phi} \to \tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_\phi}. \tag{6.8}$$

Proposition 6.14. *Let $\phi \in \mathcal{N}$ be such that $\overline{\rho}_\phi|_{G_K}$ is absolutely irreducible. Assume $\ell \nmid (k-1)(k-2)(k-3)$ and that ϕ is ordinary at ℓ. Then the map (6.8) is surjective.*

Proof. Let $p \nmid \ell D_K$ be a prime. We need to show that T_p is in the image of Desc. Note that by the first formula in (6.6) this is clear if $\ell \nmid (p+1)$ (for a split p) and (by the third formula in (6.6)) if $\ell \nmid (p^2+1)$ (for an inert p). To prove this result we will work with Galois representations. As discussed in section 6.2 to every eigenform $f \in \mathcal{S}^{\chi}_{k,-k/2}$ one can attach an ℓ-adic Galois representation $\rho_f : G_K \to \mathrm{GL}_4(E)$ and it follows from Theorem 6.7(i) and Proposition 5.22 together with the Tchebotarev Density Theorem and the Brauer-Nesbitt Theorem that

$$\rho_{f_{\phi,\chi}} = \begin{bmatrix} \rho_\phi|_{G_K} & \\ & (\rho_\phi \otimes \epsilon)|_{G_K} \end{bmatrix} \otimes \chi\epsilon^{2-k/2}, \tag{6.9}$$

where we treat χ as an $\mathfrak{l}$-adic Galois character via class field theory (here $\mathfrak{l}$ denotes a prime of K lying over ℓ). Note that the reason for the $(p+1)$- and (p^2+1)-factors in (6.6) is exactly the presence of the cyclotomic character in the lower-right corner of (6.9), because for a prime $\mathfrak{p}$ of K lying over p we get $\mathrm{tr}\,\rho_{f_{\phi,\chi}}(\mathrm{Frob}_\mathfrak{p}) = u(\epsilon(\mathrm{Frob}_\mathfrak{p})+1)\mathrm{tr}\,\rho_\phi(\mathrm{Frob}_\mathfrak{p})$ for $u = \epsilon^{2-k/2}(\mathrm{Frob}_\mathfrak{p}) \in \mathcal{O}^\times$. We will now construct an idempotent in the group algebra $\mathcal{H}^{\mathrm{M},\chi}_{\mathfrak{m}^{\mathrm{M}}_\phi}[G_K]$ that will kill off the ϵ-part from the expression $\epsilon(\mathrm{Frob}_\mathfrak{p})+1$.

From now on assume that p is split (the inert case being analogous) and fix a prime $\mathfrak{p}$ of $\mathcal{O}_K$ over it. Let $\tilde{\mathcal{N}}'_\phi \subset \mathcal{N}$, as before, be the subset consisting of those forms $\psi \in S_{k-1}(D_K, \chi_K)$ whose corresponding maximal ideal of $\tilde{\mathbf{T}}'$ is $\tilde{\mathfrak{m}}'_\phi$. The set $\tilde{\mathcal{N}}'_\phi$ is in one-to-one correspondence with the set consisting of mutually orthogonal Hecke eigenforms in $\mathcal{S}^{\mathrm{M},\chi}_{k,-k/2}$ whose corresponding maximal ideal in $\mathcal{H}^{\mathrm{M},\chi}$ is $\mathfrak{m}^{\mathrm{M}}_\phi$. Set $R' := \prod_{\psi \in \tilde{\mathcal{N}}'_\phi} \mathcal{O}$ and let R be the $\mathcal{O}$-subalgebra of R' generated by the tuples $(\lambda_{f_{\psi,\chi}}(T))_{\psi \in \tilde{\mathcal{N}}'_\phi}$ for all $T \in \mathcal{H}^{\mathrm{M},\chi}$. Then R is a complete Noetherian local $\mathcal{O}$-algebra with residue field $\mathbf{F} = \mathcal{O}/\varpi$. It is a standard argument to show that $R \cong \mathcal{H}^{\mathrm{M},\chi}_{\mathfrak{m}^{\mathrm{M}}_\phi}$. Let I_ℓ denote the inertia group at ℓ. For every $\psi \in \mathcal{N}$, ordinary at ℓ, we have by (6.9) and Theorem 3.26 (2) in [22] that (note that χ is unramified)

$$\rho_{f_{\psi,\chi}}|_{I_\ell} \cong \begin{bmatrix} \epsilon^{k/2} & * & & \\ & \epsilon^{2-k/2} & & \\ & & \epsilon^{1+k/2} & * \\ & & & \epsilon^{3-k/2} \end{bmatrix}.$$

If $\ell \nmid (k-1)(k-2)(k-3)$ it is easy to see that there exists $\sigma \in I_\ell$ such that the elements $\beta_1 := \epsilon^{k/2}(\sigma)$, $\beta_2 := \epsilon^{2-k/2}(\sigma)$, $\beta_3 := \epsilon^{1+k/2}(\sigma)$, $\beta_4 := \epsilon^{3-k/2}(\sigma)$ are all distinct mod ϖ and non-zero mod ϖ. For every ψ as above, we choose a basis of the space of $\rho_{f_{\psi,\chi}}$ so that $\rho_{f_{\psi,\chi}}$ is $\mathcal{O}$-valued and $\rho_{f_{\psi,\chi}}(\sigma) = \mathrm{diag}(\beta_1, \beta_2, \beta_3, \beta_4)$. Let S be the set consisting of the places of K lying over ℓ and the primes dividing D_K. Note that we can treat $\rho_{f_{\psi,\chi}}$ as a representation of $G_{K,S}$, the Galois group of the maximal Galois extension of K unramified away from S. Moreover, $\mathrm{tr}\,\rho_{f_{\psi,\chi}}(G_{K,S}) \subset R$, since $G_{K,S}$ is generated by conjugates of $\mathrm{Frob}_\mathfrak{p}$, $\mathfrak{p} \notin S$ and for such a $\mathfrak{p}$, $\mathrm{tr}\,\rho_{f_{\psi,\chi}}(\mathrm{Frob}_\mathfrak{p}) \in R$ by Theorem 6.7 (i) and the fact that the coefficients of the characteristic polynomial of $\rho_{f_{\psi,\chi}}(\mathrm{Frob}_{\mathfrak{p}})$ belong to $\mathcal{H}^{\chi}$. Set

$$e_j = \prod_{l\neq j} \frac{\sigma-\beta_l}{\beta_j-\beta_l} \in \mathcal{O}[G_{K,S}] \hookrightarrow R[G_{K,S}]$$

and $e := e_1+e_2$. Let

$$\rho := \prod_{\psi\in\tilde{\mathcal{N}}'_{\phi}} \rho_{f_{\psi,\chi}} : G_{K,S} \to \prod_{\psi\in\tilde{\mathcal{N}}'_{\phi}} \mathrm{GL}_4(\mathcal{O}).$$

We extend ρ to an R-algebra map $\rho' : R[G_{K,S}] \to M_4(R')$.

Set

$$r_e(\mathfrak{p}) := \epsilon^{k/2-2}(\mathrm{Frob}_{\mathfrak{p}})\mathrm{tr}\,\rho'(e\,\mathrm{Frob}_{\mathfrak{p}}) \in R'.$$

We claim that $r_e(\mathfrak{p}) \in R$. Note that $\rho'(e\,\mathrm{Frob}_{\mathfrak{p}})$ is a polynomial in $\rho'(\sigma^i\,\mathrm{Frob}_{\mathfrak{p}})$, $i=0,1,2,3$, with coefficients in $\mathcal{O}$, so it is enough to show that $\mathrm{tr}\,\rho'(\sigma^i\,\mathrm{Frob}_{\mathfrak{p}}) \in R$. Fix i, set $\tau = \sigma^i\,\mathrm{Frob}_{\mathfrak{p}} \in G_{K,S}$. Then by the Tchebotarev Density Theorem, $G_{K,S}$ is generated by conjugacy classes of Frobenii away from S, so $\mathrm{tr}\,\rho'(\tau)$ is the limit of $\mathrm{tr}\,\rho'(\mathrm{Frob}_{\mathfrak{l}}) \in R$ for some sequence of primes $\mathfrak{l} \not\in S$. So, we get $\mathrm{tr}\,\rho'(\tau) \in R$ by completeness of R.

Note that

$$\rho'(\mathrm{Frob}_{\mathfrak{p}}\,e) = \prod_{\psi\in\tilde{\mathcal{N}}'_{\phi}} \rho_{f_{\psi,\chi}}(\mathrm{Frob}_{\mathfrak{p}})\rho'_{f_{\psi,\chi}}(e) = \prod_{\psi\in\tilde{\mathcal{N}}'_{\phi}} \rho_{\psi}(\mathrm{Frob}_{\mathfrak{p}})\epsilon^{2-k/2}(\mathrm{Frob}_{\mathfrak{p}})$$

and thus

$$r_e(\mathfrak{p}) = (a_{\psi}(p))_{\psi\in\tilde{\mathcal{N}}'_{\phi}} \in R,$$

where $\psi = \sum_{n=1}^{\infty} a_{\psi}(n)q^n$. Define $T^{\mathrm{M}}(p)$ to be the image of $r_e(\mathfrak{p})$ under the $\mathcal{O}$-algebra isomorphism $R \xrightarrow{\sim} \mathcal{H}^{\mathrm{M},\chi}_{\mathfrak{m}^{\mathrm{M}}_{\phi}}$. Note that $\mathrm{Desc}(T^{\mathrm{M}}(p)) = T_p$. □

The above arguments yield the following result.

Theorem 6.15. *Let $\phi \in \mathcal{N}$ be such that $\overline{\rho}_{\phi}|_{G_K}$ is absolutely irreducible. Assume $\ell \nmid 2D_K(k-1)(k-2)(k-3)$ and that ϕ is ordinary at ℓ. Let $f_{\phi,\chi} \in \mathcal{S}^{\mathrm{M},\chi}_{k,-k/2}$ be the Maass lift of ϕ. Then there exists $T^{\mathrm{h}} \in \mathcal{H}^{\chi}_{\mathcal{O}}$ such that $T^{\mathrm{h}} f_{\phi,\chi} = \eta f_{\phi,\chi}$ and $T^{\mathrm{h}} f = 0$ for any eigenform $f \in \mathcal{S}^{\mathrm{M},\chi}_{k,-k/2}$ orthogonal to $f_{\phi,\chi}$.*

Proof. Let $T \in \tilde{\mathbf{T}}'_{\tilde{\mathfrak{m}}'_{\phi}}$ be as in Proposition 6.5 and let $T^{\mathrm{h}}_0 \in \mathcal{H}^{\chi}_{\mathfrak{m}_{\phi}}$ be an element of the inverse image of T under the map (6.8), which by Proposition 6.14 is surjective. Pulling back T^{h}_0 to $\mathcal{H}$ via the canonical projection induced by decomposition (6.4) we obtain an operator T^{h} with the desired property. □

7. Eisenstein series and theta series

The goal of this section is to express the inner product of a hermitian Siegel Eisenstein series of level N multiplied by a certain hermitian theta series against an eigenform $f \in \mathcal{M}_{n,k}(N)$ in terms of the standard L-function of f. In this section we also prove that the Fourier coefficients of the Eisenstein series and the theta series that we will use are ℓ-adically integral. We derive the desired formulas and properties from certain calculations carried out by Shimura in [38] and [39]. We will often refer the reader to [loc.cit.] for some definitions, facts and formulas, but whenever we do so, we will explain how the statements in [loc.cit.] referenced here imply what we need. We will set $h_K = \# \operatorname{Cl}_K$. The results of this section are valid for U_n for a general $n > 1$.

7.1. **Some coset decompositions.** Let Q be any finite subset of $\mathrm{GL}_n(\mathbf{A}_{K,\mathrm{f}})$ of cardinality h_K such that $\det Q = \operatorname{Cl}_K$ under the canonical map $c_K : \mathbf{A}_K^\times \twoheadrightarrow \operatorname{Cl}_K$. Then we have by (3.2)

$$\mathrm{GL}_n(\mathbf{A}_K) = \bigsqcup_{q\in Q} \mathrm{GL}_n(K)\,\mathrm{GL}_n(\mathbf{C})q\,\mathrm{GL}_n(\hat{\mathcal{O}}_K). \tag{7.1}$$

For $r \in \mathrm{GL}_n(\mathbf{A}_{K,\mathrm{f}})$, the group $r\,\mathrm{GL}_n(\hat{\mathcal{O}}_K)r^{-1}$ is also an open compact subgroup of $\mathrm{GL}_n(\mathbf{A}_{K,\mathrm{f}})$ with $\det r\,\mathrm{GL}_n(\hat{\mathcal{O}}_K)r^{-1} = \hat{\mathcal{O}}_K^\times$. Hence by the Strong Approximation Theorem for GL_n ([10], Theorem 3.3.1) we also have

$$\mathrm{GL}_n(\mathbf{A}_K) = \bigsqcup_{q\in Q} \mathrm{GL}_n(K)\,\mathrm{GL}_n(\mathbf{C})qr\,\mathrm{GL}_n(\hat{\mathcal{O}}_K)r^{-1}. \tag{7.2}$$

As before, for any $q \in \mathrm{GL}_n$ we put $p_q := \left[\begin{smallmatrix} q & \\ & \hat{q}\end{smallmatrix}\right] \in U_n$. Write P for the Siegel parabolic of U_n.

Lemma 7.1. *For any $r \in \mathrm{GL}_n(\mathbf{A}_{K,\mathrm{f}})$ the following decomposition holds:*

$$P(\mathbf{A}) = \bigsqcup_{q\in Q} P(\mathbf{Q})P(\mathbf{R})p_q p_r \mathcal{K}_P p_r^{-1},$$

where $\mathcal{K}_P := U_n(\hat{\mathbf{Z}}) \cap P(\mathbf{A})$.

Proof. Write $P = MN$ for the Levi decomposition. As $M \cong \operatorname{Res}_{K/\mathbf{Q}} \mathrm{GL}_{n/K}$, and $M \cap N = \{I_{2n}\}$, we get by (7.1):

$$P(\mathbf{A}) = M(\mathbf{A})N(\mathbf{A}) = \bigsqcup_{q\in Q} M(\mathbf{Q})M(\mathbf{R})p_q p_r \mathcal{K}_M p_r^{-1} N(\mathbf{A}),$$

where $\mathcal{K}_M := \{p_x \mid x \in \mathrm{GL}_n(\hat{\mathcal{O}}_K)\} \subset \mathcal{K}_P$. Set $\mathcal{K}_{P,r} := p_r \mathcal{K}_P p_r^{-1}$. This is a compact open subgroup of $P(\mathbf{A}_\mathrm{f})$. Let

$$X := \bigcap_{q\in Q} p_q \mathcal{K}_{P,r} p_q^{-1} \cap N(\mathbf{A}).$$

(Note if $(2n, h_K) = 1$, Corollary 3.9 implies that we can find Q so that p_q are scalars, and then $X = \mathcal{K}_{P,r} \cap N(\mathbf{A})$.) By [38], Lemma 9.6(1), we know that $N(\mathbf{A}) = N(\mathbf{Q})XN(\mathbf{R})$, since $XN(\mathbf{R}) = N(\mathbf{R})X$ is open in $N(\mathbf{A})$. Thus we have

$$\begin{aligned} P(\mathbf{A}) &= \bigsqcup_{q\in Q} M(\mathbf{Q})N(\mathbf{A})M(\mathbf{R})p_q\mathcal{K}_{P,r} \\ &= \bigsqcup_{q\in Q} M(\mathbf{Q})N(\mathbf{Q})XN(\mathbf{R})M(\mathbf{R})p_q\mathcal{K}_{P,r} = \bigsqcup_{q\in Q} P(\mathbf{Q})P(\mathbf{R})Xp_q\mathcal{K}_{P,r}, \end{aligned} \tag{7.3}$$

where the first equality follows from normality of N in P and the fact that $\mathcal{K}_M \subset \mathcal{K}_P \subset P(\mathbf{A})$ while the third one follows from the fact that X has trivial infinite components. Note that every $x \in X$ can be written as $x = p_q k p_q^{-1}$ for some $k \in \mathcal{K}_{P,r}$. Hence $Xp_q\mathcal{K}_{P,r} \subset p_q\mathcal{K}_{P,r}$. The other containment is obvious, so we have $Xp_q\mathcal{K}_{P,r} = p_q\mathcal{K}_{P,r}$. Thus finally $P(\mathbf{A}) = \bigsqcup_{q\in Q} P(\mathbf{Q})P(\mathbf{R})p_q\mathcal{K}_{P,r}$, as desired. □

Fix $r \in \mathrm{GL}_n(\mathbf{A}_{K,\mathrm{f}})$ and an integer $N > 1$. Set

$$\Gamma^{\mathrm{h}}_{j,r}(N) = U_n(\mathbf{Q}) \cap U_n(\mathbf{R}) p_r \mathcal{K}_j(N) p_r^{-1}, \quad \text{for } j = 0, 1.$$

and for any subgroup Γ of $U_n(\mathbf{Q})$ we put $\Gamma^P := \Gamma \cap P(\mathbf{Q})$. Note that $\Gamma^{\mathrm{h}}_{j,I_n}(N) = \Gamma^{\mathrm{h}}_{j,n}(N)$ for $j = 0, 1$ with $\Gamma^{\mathrm{h}}_{j,n}(N)$ defined as in section 2.2. In the discussion below we keep N fixed and to shorten notation we write $\Gamma_r = \Gamma^{\mathrm{h}}_{0,r}(N)$.

Lemma 7.2. *The canonical injection*

$$\Gamma^P_r \setminus \Gamma_r \hookrightarrow P(\mathbf{Q}) \setminus (U_n(\mathbf{Q}) \cap P(\mathbf{A}) U_n(\mathbf{R}) p_r \mathcal{K}_{0,n}(N) p_r^{-1})$$

is a bijection.

Proof. We need to prove surjectivity. Set $\mathcal{K}_{P,r} = p_r \mathcal{K}_P p_r^{-1}$ and $\mathcal{K}_{0,r}(N) := p_r \mathcal{K}_{0,n}(N) p_r^{-1}$. By Lemma 7.1 we have

$$P(\mathbf{A}) = \bigsqcup_{q \in Q} P(\mathbf{Q}) P(\mathbf{R}) p_q \mathcal{K}_{P,r},$$

so

$$P(\mathbf{A}) U_n(\mathbf{R}) p_r \mathcal{K}_0(N) p_r^{-1} = \bigcup_{q \in Q} P(\mathbf{Q}) P(\mathbf{R}) p_q \mathcal{K}_{P,r} U_n(\mathbf{R}) \mathcal{K}_{0,r}(N).$$

Note that $\det U_n(\mathbf{Q}) \subset H(\mathbf{Q})$, where

$$H = \{x \in \mathrm{Res}_{K/\mathbf{Q}} \mathbf{G}_{m/K} \mid x\overline{x} = 1\},$$

and

$$\det(P(\mathbf{Q}) P(\mathbf{R}) p_q \mathcal{K}_{P,r} U_n(\mathbf{R}) \mathcal{K}_{0,r}(N)) \subset \det p_q H(\mathbf{Q}) \det D,$$

with $D = U_n(\mathbf{R}) \mathcal{K}_{0,r}(N)$. Thus

$$U_n(\mathbf{Q}) \cap P(\mathbf{Q}) P(\mathbf{R}) p_q \mathcal{K}_{P,r} U_n(\mathbf{R}) \mathcal{K}_{0,r}(N) = \emptyset$$

unless $\det p_q \in H(\mathbf{Q}) \det D$. However, Q is chosen so that $\det p_q$ runs over all the ideal classes of K. It follows from Lemma 8.14 in [38] that there is a bijection between Cl_K and $H(\mathbf{A})/H(\mathbf{Q}) \det D$, thus $\det p_q \in H(\mathbf{Q}) \det D$ only for one q (which without loss of generality we can take to equal I_{2n}). Thus

$$\begin{aligned}(7.4) \quad & P(\mathbf{Q}) \setminus (U_n(\mathbf{Q}) \cap P(\mathbf{A}) U_n(\mathbf{R}) p_r \mathcal{K}_{0,n}(N) p_r^{-1}) \\ &= P(\mathbf{Q}) \setminus (U_n(\mathbf{Q}) \cap P(\mathbf{Q}) P(\mathbf{R}) \mathcal{K}_{P,r} U_n(\mathbf{R}) \mathcal{K}_{0,r}(N) \\ &= P(\mathbf{Q}) \setminus (U_n(\mathbf{Q}) \cap P(\mathbf{Q}) U_n(\mathbf{R}) \mathcal{K}_{0,r}(N)).\end{aligned}$$

Thus if $g \in U_n(\mathbf{Q})$ can be written as $g = pk$ with $p \in P(\mathbf{Q})$ and $k \in U_n(\mathbf{R}) \mathcal{K}_{0,r}(N)$, then clearly $p^{-1} g \in \Gamma_r$. □

We will need more congruence subgroups. Let $\Gamma(N) := \Gamma^{\mathrm{h}}_n(N)$ be the subgroup introduced in section 2 and set

$$\Gamma_u(N) := \left\{ \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \Gamma^{\mathrm{h}}_{0,n}(N) \mid 1 - \det D \in N\mathcal{O}_K \right\}.$$

Lemma 7.3. *The canonical injection*

$$\Gamma(N)^P \setminus \Gamma(N) \hookrightarrow \Gamma_u(N)^P \setminus \Gamma_u(N)$$

is a bijection.

Proof. We need to prove surjectivity. Let $g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \Gamma_u(N)$. We have $D \in M_n(\mathcal{O}_K)$ with $1 - \det D \in N\mathcal{O}_K$. Note that the reduction of D mod $N\mathcal{O}_K$ lies in $\mathrm{SL}_n(\mathcal{O}_K/N\mathcal{O}_K)$. By the strong approximation for SL_n, the reduction map $\mathrm{SL}_n(\mathcal{O}_K) \to \mathrm{SL}_n(\mathcal{O}_K/N\mathcal{O}_K)$ is surjective. Thus there exists $q \in \mathrm{SL}_n(\mathcal{O}_K)$ such that $D - q \in M_n(N\mathcal{O}_K)$. Put

$$h = \begin{bmatrix} q^* & -D^*Bq^{-1} \\ & q^{-1} \end{bmatrix}.$$

Then $h \in \Gamma_u(N)^P$ and $hg \in \Gamma(N)$. □

7.2. **Eisenstein series.** As before, let $N > 1$ be an integer and set $\mathcal{O}_{K,p} = \mathbf{Z}_p \otimes \mathcal{O}_K$. Let ψ be a Hecke character of K satisfying

$$\psi_\infty(x) = x^m |x|^{-m} \tag{7.5}$$

for a positive integer m and

$$\psi_p(x) = 1 \text{ if } p \neq \infty,\ x \in \mathcal{O}_{K,p}^\times \textit{ and } x - 1 \in N\mathcal{O}_{K,p}. \tag{7.6}$$

Set $\psi_N = \prod_{p|N} \psi_p$. Let δ_P denote the modulus character of P. We define

$$\mu_P : M(\mathbf{Q})N(\mathbf{A}) \setminus U_n(\mathbf{A}) \to \mathbf{C}$$

by setting

$$\mu_P(g) = \begin{cases} 0 & g \notin P(\mathbf{A})\mathcal{K}_{0,n,\infty}^+ \mathcal{K}_{0,n}(N) \\ \psi(\det d_q)^{-1} \psi_N(\det d_k)^{-1} j(k_\infty, \mathbf{i}_n)^{-m} & g = qk \in P(\mathbf{A})(\mathcal{K}_{0,n,\infty}^+ \mathcal{K}_{0,n}(N)). \end{cases}$$

Note that μ_P has a local decomposition $\mu_P = \prod_p \mu_{P,p}$, where

$$\mu_{P,p}(q_p k_p) = \begin{cases} \psi_p(\det d_{q_p})^{-1} & \text{if } p \nmid N\infty, \\ \psi_p(\det d_{q_p})^{-1} \psi_p(\det d_{k_p}) & \text{if } p \mid N, p \neq \infty, \\ \psi_\infty(\det d_{q_\infty})^{-1} j(k_\infty, \mathbf{i})^{-m} & \text{if } p = \infty \end{cases} \tag{7.7}$$

and δ_P has a local decomposition $\delta_P = \prod_p \delta_{P,p}$, where

$$\delta_{P,p}\left(\begin{bmatrix} A & \\ & \hat{A} \end{bmatrix} uk \right) = |\det A \det \overline{A}|_{\mathbf{Q}_p}. \tag{7.8}$$

Definition 7.4. The series

$$E(g, s, N, m, \psi) := \sum_{\gamma \in P(\mathbf{Q}) \setminus U_n(\mathbf{Q})} \mu_P(\gamma g) \delta_P(\gamma g)^{s/2}$$

is called the (*hermitian*) *Siegel Eisenstein series of weight* m*, level* N *and character* ψ.

For $x \in U_n(\mathbf{A}_\mathrm{f})$, $g \in U_n(\mathbf{R})$ and $Z = g\mathbf{i}_n$ we define ([39], (17.23a))

$$E_x(Z, s, m, \psi, N) = j(g, \mathbf{i}_n)^m E(xg, s, N, m, \psi).$$

Fix $r \in \mathrm{GL}_n(\mathbf{A}_{K,\mathrm{f}})$, write $A_r = P(\mathbf{Q}) \setminus (U_n(\mathbf{Q}) \cap P(\mathbf{A}) p_r U_n(\mathbf{R}) K_0(N) p_r^{-1}$. Then

$$E_{p_r}(Z, s, m, \psi, N) = \psi_\mathrm{f}(\det r^*) |\det(rr^*)|_{\mathbf{Q}}^s \sum_{a \in A_r} N(\mathfrak{a}_{p_r}(a))^s \psi[a]_{p_r} (\det(\mathrm{Im} Z))^{s-m/2} |_m a, \tag{7.9}$$

where $\mathfrak{a}_{p_r}(a)$, $\psi[a]_{p_r}$ are defined in section 18 of [38]. (Our notation differs slightly from that in [38], which we quote here. In particular our r corresponds to $\hat{g}$ in [loc.cit.] and one has $\delta(Z) = \det \eta(Z)$ by (6.3.11) in [loc.cit.] and $\eta(Z) = 2\mathrm{Im}(Z)$

by (6.1.8) in [loc.cit.].) As stated in the proof of Lemma 17.13 of [39], there exists a finite set $B \subset U_n(\mathbf{Q})$ such that $A_r = \bigsqcup_{b\in B} S_b b$, where $S_b = (P(\mathbf{Q}) \cap U_n(\mathbf{R}) b\Gamma_r b^{-1}) \setminus b\Gamma_r b^{-1}$. By Lemma 7.2, we can take $B = \{I_{2n}\}$. It follows then from the proof of Lemma 17.13 of [39], that $\mathfrak{a}_{p_r}(\gamma) = \mathfrak{a}_{p_r}(I_{2n})$ for $\gamma \in S_{I_{2n}} = (P(\mathbf{Q}) \cap U_n(\mathbf{R})\Gamma_r) \setminus \Gamma_r$. By the definition of $\mathfrak{a}_{p_r}(a)$ in [38], Lemma 18.7(3), we get $\mathfrak{a}_{p_r}(I_{2n}) = \mathcal{O}_K$ ([38], (18.4.4)), so for $\gamma \in S_{I_{2n}}$, we have

$$N(\mathfrak{a}_{p_r}(\gamma)) = N(\mathfrak{a}_{p_r}(I_{2n})) = 1.$$

Moreover, for $\gamma \in S_{I_{2n}}$, we have by [38], Lemma 18.7(3) and (12.8.2)

(7.10)
$$\psi[\gamma]_{p_r} = \psi_\infty(\det d_\gamma)\psi^*(\det d_\gamma \mathfrak{a}_{p_r}(\gamma)^{-1}) = \psi_\infty(\det d_\gamma)\psi^*(\det d_\gamma \mathfrak{a}_{p_r}(I_{2n})^{-1}) = \\ = \psi_\infty(\det d_\gamma)\psi^*(\det d_\gamma \mathcal{O}_K) = \psi_N^{-1}(\det d_\gamma).$$

Hence we get
(7.11)
$$E_{p_r}(Z,s,m,\psi,N) = \psi_{\mathrm{f}}(\det r^*)|\det(rr^*)|_{\mathbf{Q}}^s \sum_{\gamma\in\Gamma_r^P\setminus\Gamma_r} \psi_N(\det d_\gamma)^{-1}\det(\mathrm{Im}(Z))^{s-m/2}|_m\gamma.$$

In what follows we will write E_r instead of E_{p_r} for $r \in \mathrm{GL}_n(\mathbf{A}_{K,\mathrm{f}})$. For any congruence subgroup Γ of $U_n(\mathbf{Q})$ we define an Eisenstein series (cf. [39] (17.3), (17.3a), where a similar definition is made in the case when Γ is a congruence subgroup of $SU_n(\mathbf{Q})$):

$$E(Z,s,m,\Gamma) = \sum_{\gamma\in\Gamma^P\setminus\Gamma} \det(\mathrm{Im})(Z)^{s-m/2}|_\gamma. \tag{7.12}$$

Let $X = X_{m,N}$ be the set of all Hecke characters ψ of K satisfying (7.5) and (7.6).

Lemma 7.5. *Assume $r \in \mathrm{GL}_n(\mathbf{A}_{K,\mathrm{f}})$ is such that p_r is a scalar. Then*

$$\sum_{\psi\in X} \psi_{\mathrm{f}}(\det r^*)^{-1}|\det(rr^*)|_{\mathbf{Q}}^{-s} E_r(Z,s,m,\psi,N) = \#XE(Z,s,m,\Gamma_{1,n}(N)).$$

Proof. Note that $\#X \neq \emptyset$ because of our assumption that $N > 1$ by Lemma 11.14(1) in [38]. Let $x \in \mathcal{O}_K$, $(x,N) = 1$, be such that there exists $\psi' \in X$ with $\psi'_N(x) \neq 1$. Then $\sum_{\psi\in X}\psi_N(x) = 0$. Thus,

$$A := \sum_{\psi\in X} \psi_{\mathrm{f}}(\det r^*)^{-1}|\det(rr^*)|_{\mathbf{Q}}^{-s} E_r(Z,s,m,\psi,N) = \\ = \sum_{\psi\in X}\sum_{\gamma\in\Gamma_r^P\setminus\Gamma_r} \psi_N(\det d_\gamma)^{-1}\det(\mathrm{Im}(Z))^{s-m/2}|_m\gamma = \\ = \sum_{\gamma\in\Gamma_r^P\setminus\Gamma_r}\left(\sum_{\psi\in X}\psi_N(\det d_\gamma)^{-1}\right)\det(\mathrm{Im}(Z))^{s-m/2}|_m\gamma. \tag{7.13}$$

By our assumption on r we have $\Gamma_r = \Gamma_{0,n}^{\mathrm{h}}(N)$. Thus the inner sum equals 0 unless $\gamma \in \Gamma_u(N)$, in which case it equals $\#X$. Hence we get

$$A = \#X \sum_{\gamma\in\Gamma_u(N)^P\setminus\Gamma_u(N)} \det(\mathrm{Im}(Z))^{s-m/2}|_m\gamma.$$

Using Lemma 7.3 we further get

$$A = \#X \sum_{\gamma \in \Gamma(N)^P \backslash \Gamma(N)} \det(\mathrm{Im}(Z))^{s-m/2}|_m \gamma = \#X E(Z, s, m, \Gamma(N)). \tag{7.14}$$

Now apply $\sum_{\gamma \in \Gamma(N) \backslash \Gamma^{\mathrm{h}}_{1,n}(N)} |_m \gamma$ to both sides of (7.14). We have $E_r(Z, s, m, \psi, N)|_m \gamma = E_r(Z, s, m, \psi, N)$ for every $\gamma \in \Gamma^{\mathrm{h}}_{1,n}(N)$, and

$$\sum_{\gamma \in \Gamma(N) \backslash \Gamma^{\mathrm{h}}_{1,n}(N)} E(Z, s, m, \Gamma(N))|_m \gamma = [\Gamma^{\mathrm{h}}_{1,n}(N)^P : \Gamma(N)^P] E(Z, s, m, \Gamma^{\mathrm{h}}_{1,n}(N))$$

by [39] (17.5) together with Remark 17.12(2). Note also that $[\Gamma^{\mathrm{h}}_{1,n}(N)^P : \Gamma(N)^P] = [\Gamma^{\mathrm{h}}_{1,n}(N) : \Gamma(N)]$, hence we finally get $A = \#X E(Z, s, m, \Gamma^{\mathrm{h}}_{1,n}(N))$. □

Remark 7.6. Lemma 7.5 is true even without assuming that p_r is a scalar. Its conclusion can then be stated as

$$\sum_{\psi \in X} \psi_{\mathrm{f}}(\det r^*)^{-1} |\det(rr^*)|^{-s}_{\mathbf{Q}} E_r(Z, s, m, \psi, N) = E(Z, s, m, \Gamma_{1,r}(N)).$$

We omit the proof however, as we will have no need for this result.

7.3. **Theta series and inner products.** Let Γ be a congruence subgroup of $U_n(\mathbf{Q})$. Let $F, G \in M^{\mathrm{sh}}_k(\Gamma)$ (with at least one of the forms cuspidal), where the superscript 'sh', as before, indicates that the forms are not necessarily holomorphic. We set

$$\langle F, G \rangle_\Gamma := \int_{\Gamma \backslash \mathbf{H}_n} F(Z) \overline{G(Z)} \det(\mathrm{Im}(Z))^k dZ,$$

and

$$\langle F, G \rangle = \left(\int_{\Gamma \backslash \mathbf{H}_n} dZ \right)^{-1} \int_{\Gamma \backslash \mathbf{H}_n} F(Z) \overline{G(Z)} \det(\mathrm{Im}(Z))^k dZ,$$

(compare with [38], (10.9.2)). Let ξ be an (algebraic) Hecke character of K. Then ξ^c defined by $\xi^c(x) = \xi(\overline{x})$ is also an algebraic Hecke character of K whose infinity type is the conjugate of the infinity type of ξ. Let Q be as in (7.1). Let $f, g \in \mathcal{M}'_{n,k,\nu}(\mathcal{K})$ for some open compact subgroup $\mathcal{K}$ of $U_n(\mathbf{A}_{\mathrm{f}})$. Then by Proposition 3.11 the forms f and g correspond to $\#Q$-tuples of functions on $\mathbf{H}_n$ which we denote by (f_{p_q}) and (g_{p_q}) respectively or simply by (f_q) and (g_q). If either f or g is cuspidal, set

$$\langle f, g \rangle = (\#Q)^{-1} \sum_{q \in Q} \langle f_q, g_q \rangle,$$

(compare with [38], (10.9.6)), and

$$\langle f, g \rangle_\Gamma = (\#Q)^{-1} \sum_{q \in Q} \langle f_q, g_q \rangle_\Gamma$$

if for all $q \in Q$ the integrand $f_q(Z)\overline{g_q(Z)} \det(\mathrm{Im}(Z))^k dZ$ is Γ-invariant (for example if $f, g \in \mathcal{M}'(\mathcal{K}_{0,n}(N))$ and all $q \in Q$ are scalars, then one can take $\Gamma = \Gamma^{\mathrm{h}}_{0,n}(N)$).

Let k be a positive integer. Fix a Hecke character ξ of K with conductor $\mathfrak{f}_\xi$ and infinity type $|x|^t x^{-t}$ for an integer t about which we for now only assume that

$t \geq -k$. We will now define a theta series associated with the character ξ. Let $\lambda : M_n(\mathbf{A}_{K,\mathrm{f}}) \to \mathbf{C}$ be a function given by

$$\lambda(g) = \begin{cases} \xi_{\mathfrak{f}_\xi}(\det g) & \text{if } g \in M_n(\hat{\mathcal{O}}_K) \text{ and } g_v \in \mathrm{GL}_n(\mathcal{O}_v) \text{ for all } v \mid \mathfrak{f}_\xi \\ 0 & \text{otherwise.} \end{cases}$$

The map λ is a Schwarz function (cf. [39], section A5 for a more precise statement). Fix $\tau \in \mathcal{S}$. For $Z \in \mathbf{H}_n$ set

$$\theta_\xi(Z, \lambda) = \sum_{\alpha \in M_n(K)} \lambda(\alpha)\overline{\det \alpha} \cdot e(\mathrm{tr}\,(\alpha^* \tau \alpha Z)).$$

For $g \in U_n(\mathbf{A})$ set

$$\theta_\xi(g) = j(g, i)^{-l} \theta(gi, \lambda^g),$$

where $l = t + k + n$ and the automorphism of the space of Schwarz functions on $M_n(\mathbf{A}_{K,\mathrm{f}})$ given by $\lambda \mapsto \lambda^g$ is defined in Theorem A5.4 of [39].

Remark 7.7. Note that θ_ξ depends on the choice of the matrix τ. If $\{g^*\tau g\}_{g \in \mathcal{O}_K^2} = \mathbf{Z}$ and c is a positive integer such that $\{g^*\tau^{-1}g\}_{g \in \mathcal{O}_K^2} \subset \frac{1}{c}\mathbf{Z}$, then $\theta_\xi \in \mathcal{M}_l(N, \psi')$, where $N = D_K c N_{K/\mathbf{Q}}(\mathfrak{f}_\xi)$ by [39], section A5.5 and [38], Proposition A7.16. Note that such a c always exists (for example one can take $c = \det \tau$). In what follows we fix τ and c so that $\{g^*\tau g\}_{g \in \mathcal{O}_K^2} = \mathbf{Z}$ and we fix N as above. In particular, we have $\mathfrak{f}_\xi \mid N$.

By [39], (22.14b), $\psi' = \xi^{-1}\varphi^{-n}$, where φ is a Hecke character of K with infinity type $\frac{|a_\infty|}{a_\infty}$ and such that $\varphi|_{\mathbf{A}}^\times = \chi_K$ (such a character always exists, but is not unique - cf. [39], Lemma A.5.1). Thus

$$\psi'_\infty(x) = x^{t+n}|x|^{-t-n}.$$

Let $Q = \mathcal{B}$ be as in Corollary 3.9. (In fact, we believe our result holds for a more general Q, but for simplicity we proceed with Q as in that corollary.) Then θ_ξ corresponds to a $\#Q$-tuple of functions, which we denote following [39] by (θ_{χ,p_q}) or simply by $(\theta_{\chi,q})$.

Set $m = -t - n$. Let $\gamma \in \Gamma^{\mathrm{h}}_{0,n}(N)$. Note that if ψ is a Hecke character of K, then

$$\psi_N(\det a_\gamma) = \psi_N(\overline{\det d_\gamma}^{-1}) = \psi_N^{\mathrm{c}}(\det d_\gamma)^{-1}. \tag{7.15}$$

Note that this makes sense because $N \in \mathbf{Z}$. Let $f \in \mathcal{M}_{n,k}(N)$. Then f corresponds to a $\#Q$-tuple of functions (f_q). We have

$$\langle E(\cdot, s, m, \Gamma^{\mathrm{h}}_{1,n}(N))\theta_{\chi,q}, f_q\rangle_{\Gamma^{\mathrm{h}}_{1,n}(N)} = \tag{7.16}$$

$$= \int_{\Gamma^{\mathrm{h}}_{0,n}(N)\backslash\mathcal{H}} \theta_{\chi,q}(Z) \left(\sum_{\gamma \in \Gamma^{\mathrm{h}}_{1,n}(N)\backslash\Gamma^{\mathrm{h}}_{0,n}(N)} \psi'_N(\det a_\gamma) E(Z, s, m, \Gamma^{\mathrm{h}}_{1,n}(N))|_m \gamma \right) \overline{f_q(Z)}\delta(Z)^k dZ$$

By Lemma 7.5 one has

$$\sum_{\gamma \in \Gamma^{\mathrm{h}}_{1,n}(N)\backslash\Gamma^{\mathrm{h}}_{0,n}(N)} \psi'_N(\det a_\gamma) E(Z, s, m, \Gamma^{\mathrm{h}}_{1,n}(N))|_m \gamma = \tag{7.17}$$

$$(\#X)^{-1} \sum_{\psi \in X} \sum_{\gamma \in \Gamma^{\mathrm{h}}_{1,n}(N)\backslash\Gamma^{\mathrm{h}}_{0,n}(N)} \psi_{\mathrm{f}}(\det q^*)^{-1} |\det(qq^*)|_{\mathbf{Q}}^{-s} E_q(Z, s, m, \psi, N)|_\gamma.$$

Note that for $Z = g_\infty \mathbf{i}_n$ with $g = (g_\infty, 1)$, we have

$$E_q(Z, s, m, \psi, N) = j(g_\infty, \mathbf{i}_n)^m E(p_q g, s, N, m, \psi),$$

and

$$\begin{aligned} E_q(Z, s, m, \psi, N)|_m \gamma &= j(\gamma, Z)^{-m} E_q(\gamma Z, s, m, \psi, N) \\ &= j(g_\infty, \mathbf{i}_n)^m j(\gamma, Z)^{-m} E(p_q(\gamma g)_\infty, s, N, m, \psi) \\ &= j(g_\infty, \mathbf{i}_n)^m j(\gamma, Z)^{-m} E((\gamma g_\infty, p_q), s, N, m, \psi) \\ &= j(g_\infty, \mathbf{i}_n)^m j(\gamma, Z)^{-m} E((g_\infty, p_q \gamma^{-1}), s, N, m, \psi) \end{aligned} \tag{7.18}$$

where we have used the assumption that p_q is a scalar. Then by [38], (18.6.2), we have

$$E((g_\infty, p_q\gamma^{-1}), s, N, m, \psi) =$$
$$= \psi_N(\det d_{\gamma^{-1}})^{-1} E((g_\infty, p_q), s, N, m, \psi) = \psi_N(\det d_{\gamma^{-1}})^{-1} E(p_q g, s, N, m, \psi).$$

Note that

$$\det d_{\gamma^{-1}} = \det d_\gamma^{-1} \mod N.$$

Hence finally we get

$$E_q(Z, s, m, \psi, N)|_m \gamma = \psi_N(\det d_\gamma) E_q(Z, s, m, \psi, N). \tag{7.19}$$

Then (7.17) equals

$$\begin{aligned} &(\#X)^{-1} \sum_{\psi \in X} \psi_{\mathrm{f}}(\det q^*)^{-1} |\det(qq^*)|_{\mathbf{Q}}^{-s} E_q(Z, s, m, \psi, N) \times \\ &\qquad \times \sum_{\gamma \in \Gamma_{1,n}^{\mathrm{h}}(N) \backslash \Gamma_{0,n}^{\mathrm{h}}(N)} \psi'_N(\det a_\gamma) \psi_N(\det d_\gamma) = \\ &= (\#X)^{-1} \sum_{\psi \in X} \psi_{\mathrm{f}}(\det q^*)^{-1} |\det(qq^*)|_{\mathbf{Q}}^{-s} E_q(Z, s, m, \psi, N) \times \\ &\qquad \times \sum_{\gamma \in \Gamma_{1,n}^{\mathrm{h}}(N) \backslash \Gamma_{0,n}^{\mathrm{h}}(N)} (\psi')^{\mathrm{c}}_N(\det d_\gamma)^{-1} \psi_N(\det d_\gamma), \end{aligned} \tag{7.20}$$

where the last equality follows from (7.15), according to which $\psi'_N(\det a_\gamma) = (\psi')^{\mathrm{c}}_N(\det d_\gamma)^{-1}$. Using the fact that

$$\sum_{\gamma \in \Gamma_{1,n}^{\mathrm{h}}(N) \backslash \Gamma_{0,n}^{\mathrm{h}}(N)} (\psi')^{\mathrm{c}}_N(\det d_\gamma)^{-1} \psi_N(\det d_\gamma) = 0$$

unless $\psi = (\psi')^{\mathrm{c}}$, we obtain

$$\begin{aligned} &\sum_{\gamma \in \Gamma_{1,n}^{\mathrm{h}}(N) \backslash \Gamma_{0,n}^{\mathrm{h}}(N)} \psi'_N(\det a_\gamma) E(Z, s, m, \Gamma_{1,n}^{\mathrm{h}}(N))|_m \gamma = \\ &= (\#X)^{-1} [\Gamma_{0,n}^{\mathrm{h}}(N) : \Gamma_{1,n}^{\mathrm{h}}(N)] (\psi')^c (\det q^*)^{-1} |\det(qq^*)|_{\mathbf{Q}}^{-s} E_q(Z, s, m, \psi, N). \end{aligned} \tag{7.21}$$

Hence finally

$$\begin{aligned} &\left\langle E(\cdot, s, m, \Gamma_{1,n}^{\mathrm{h}}(N)) \theta_{\chi,q}, f_q \right\rangle_{\Gamma_{1,n}^{\mathrm{h}}(N)} \\ &= (\#X)^{-1} [\Gamma_{0,n}^{\mathrm{h}}(N) : \Gamma_{1,n}^{\mathrm{h}}(N)] (\psi')^{-1} (\det q) |\det(qq^*)|_{\mathbf{Q}}^{-s} \left\langle E_q(\cdot, s, m, ((\psi')^c), N) \theta_{\chi,q}, f_q \right\rangle_{\Gamma_{0,n}^{\mathrm{h}}(N)}. \end{aligned} \tag{7.22}$$

Note that the inner product (7.22) makes sense, because first of all $(\psi')^c_\infty(x) = x^{-t-n}|x|^{t+n} = x^m|x|^m$, so the definition of $E(Z,s,m,(\psi')^c,\Gamma^{\rm h}_{1,n}(N))$ makes sense, and secondly,

$$\begin{aligned} E(Z,s,m,(\psi')^c,N)|_\gamma &= (\psi')^c_N(\det d_{\gamma^{-1}})^{-1}E_q(Z,s,m,(\psi')^c,N) \\ &= (\psi')^c_N(\det d_\gamma)E_q(Z,s,m,(\psi')^c,N) \\ &= (\psi')^{-1}_N(\det a_\gamma)E_q(Z,s,m,(\psi')^c,N), \end{aligned} \tag{7.23}$$

where the first equality follows from (7.19) and the last one from (7.15). Hence $(E_q(Z,s,m,(\psi')^c,N)\theta_{\chi,q})|_k\gamma = E_q(Z,s,m,(\psi')^c,N)\theta_{\chi,q}$ for every $\gamma \in \Gamma^{\rm h}_{0,n}(N)$.

Set $\Gamma := \Gamma^{\rm h}_{1,n}(N) \cap SU_n(\mathbf{Q})$. We now relate $\left\langle E(\cdot,s,m,\Gamma^{\rm h}_{1,n}(N))\theta_{\chi,q},f_q\right\rangle_{\Gamma^{\rm h}_{1,n}(N)}$ to $\langle E(\cdot,s,m,\Gamma)\theta_{\chi,q},f_q\rangle_\Gamma$. By [39], formula (17.5) and Remark 17.12(2), we have

$$E(Z,s,m,\Gamma^{\rm h}_{1,n}(N)) = \frac{1}{[\Gamma^{\rm h}_{1,n}(N):\Gamma]}\sum_{\alpha\in\Gamma\backslash\Gamma^{\rm h}_{1,n}(N)} E(Z,s,m,\Gamma)|_m\alpha.$$

Hence

$$\begin{aligned} &\left\langle E(\cdot,s,m,\Gamma^{\rm h}_{1,n}(N))\theta_{\chi,q},f_q\right\rangle_{\Gamma^{\rm h}_{1,n}(N)} = [\Gamma^{\rm h}_{1,n}(N):\Gamma]^{-1}\left\langle E(\cdot,s,m,\Gamma^{\rm h}_{1,n}(N))\theta_{\chi,q},f_q\right\rangle_\Gamma = \\ &= [\Gamma^{\rm h}_{1,n}(N):\Gamma]^{-2}\left\langle\left(\sum_{\alpha\in\Gamma\backslash\Gamma^{\rm h}_{1,n}(N)} E(Z,s,m,\Gamma)|_m\alpha\right)\theta_{\chi,q},f_q\right\rangle_\Gamma. \end{aligned} \tag{7.24}$$

Since $\theta_{\chi,q}|_l\alpha = \theta_{\chi,q}$ and $f_q|_k\alpha = f_q$ for $\alpha\in\Gamma^{\rm h}_{1,n}(N)$ we finally have

$$\begin{aligned} &\left\langle E(\cdot,s,m,\Gamma^{\rm h}_{1,n}(N))\theta_{\chi,q},f_q\right\rangle_{\Gamma^{\rm h}_{1,n}(N)} = \\ &= \frac{1}{[\Gamma^{\rm h}_{1,n}(N):\Gamma]^2}\sum_{\alpha\in\Gamma\backslash\Gamma^{\rm h}_{1,n}(N)}\langle(E(\cdot,s,m,\Gamma)|_m\alpha)\,(\theta_{\chi,q}|_l\alpha)\,,f_q|_k\alpha\rangle_\Gamma = \\ &= \frac{1}{[\Gamma^{\rm h}_{1,n}(N):\Gamma]}\,\langle E(\cdot,s,m,\Gamma)\theta_{\chi,q},f_q\rangle_\Gamma\,. \end{aligned} \tag{7.25}$$

7.4. **The standard L-function.** Let Q and f be as before and let $q\in Q$. From now on we assume that f is a Hecke eigenform. Let $D(s,f,\xi)$ and $D_q(s,f,\theta_\xi)$ denote the Dirichlet series defined in [39] by formulas (22.11) and (22.4) respectively. Let $r\in \mathrm{GL}_n(\mathbf{A}_{K,\mathrm{f}})$ and $\tau\in\mathcal{S}^+ := \{h\in\mathcal{S}\mid h>0\}$. Then (22.18b) in [loc. cit.] gives

$$D(s+3n/2,f,\xi) = (\det\tau)^{s+(k+l)/2}|\det r|_K^{-s-n/2}\sum_{q\in Q}(\psi')^{-1}(\det q)|\det qq^*|^s_{\mathbf{Q}}D_q(s,f,\theta_\chi), \tag{7.26}$$

while [39], (22.9) gives

$$D_q(s,f,\theta_\xi) = A_N\Gamma((s))^{-1}\,\langle f_q,\theta_{\chi,q}E(\cdot,\overline{s}+n,m,\Gamma)\rangle_\Gamma\,, \tag{7.27}$$

where A_N and $\Gamma((s))$ are defined as follows. Let $X_{\rm re} = \{h\in M_n(\mathbf{C})\mid h=h^*\}/\{h\in M_n(\mathcal{O}_K)\mid h=h^*\}$ and $X_{\rm im} = \{h\in M_n(\mathbf{C})\mid h=h^*, h>0\}/\sim$, where $h\sim h'$ if there exists $g\in\mathrm{GL}_n(\mathcal{O}_K)$ such that $h'=ghg^*$. Then $X_{\rm re}\times X_{\rm im}$ is commensurable with $\Gamma^{\rm h}_{0,n}(N)\cap P(\mathbf{Q})\backslash\mathbf{H}_n$ i.e, the ratio of their volumes is a positive rational number

([39], p. 179). We set A_N to be this rational number times the $\mathrm{vol}(X_{\mathrm{re}})^{-1}$. Note that $A_N \in \mathbf{Q}$. We also set (cf. [39], p.179 and formulas (22.4a), (16.47))

$$\Gamma((s)) = (4\pi)^{-\frac{n}{2}(2s+k+l)}\pi^{2n(n-1)/4}\prod_{i=0}^{n-1}\Gamma(s-i).$$

Combining (7.25) with (7.22), we obtain

$$\langle f_q, \theta_{\chi,q}E(\cdot, s, m, \Gamma)\rangle_{\Gamma} =$$
$$(\#X)^{-1}[\Gamma^{\mathrm{h}}_{0,n}(N):\Gamma]\overline{(\psi')^{-1}(\det q)}|\det(qq^*)|_{\mathbf{Q}}^{-\overline{s}}\langle f_q, E_q(\cdot, s, m, (\psi')^c, N)\theta_{\chi,q}\rangle_{\Gamma^{\mathrm{h}}_{0,n}(N)} =$$
$$(\#X)^{-1}[\Gamma^{\mathrm{h}}_{0,n}(N):\Gamma]\psi'(\det q)|\det(qq^*)|_{\mathbf{Q}}^{-\overline{s}}\langle f_q, E_q(\cdot, s, m, (\psi')^c, N)\theta_{\chi,q}\rangle_{\Gamma^{\mathrm{h}}_{0,n}(N)}.$$

Hence

$$(7.28)\quad \langle f_q, \theta_{\chi,q}E(\cdot, \overline{s}+n, m, \Gamma)\rangle_{\Gamma} =$$
$$(\#X)^{-1}[\Gamma^{\mathrm{h}}_{0,n}(N):\Gamma]\psi'(\det q)|\det(qq^*)|_{\mathbf{Q}}^{-s-n}\langle f_q, E_q(\cdot, \overline{s}+n, m, (\psi')^c, N)\theta_{\chi,q}\rangle_{\Gamma^{\mathrm{h}}_{0,n}(N)}.$$

Combining (7.26), (7.27) and (7.28) we finally obtain

(7.29)
$$D(s+3n/2, f, \xi) = A_N(\#X)^{-1}[\Gamma^{\mathrm{h}}_{0,n}(N):\Gamma]\Gamma((s))^{-1}(\det\tau)^{s+(k+l)/2}|\det r|_K^{-s-n/2}\times$$
$$\times\sum_{q\in Q}|\det qq^*|_{\mathbf{Q}}^{-n}\langle f_q, E_q(\cdot, \overline{s}+n, m, (\psi')^c, N)\theta_{\chi,q}\rangle_{\Gamma^{\mathrm{h}}_{0,n}(N)}.$$

By our assumption $\det qq^* = 1$, so we get

(7.30)
$$\begin{aligned} D(s+3n/2, f, \xi) =& A_N(\#X)^{-1}[\Gamma^{\mathrm{h}}_{0,n}(N):\Gamma]\Gamma((s))^{-1}(\det\tau)^{s+(k+l)/2}|\det r|_K^{-s-n/2} \\ &\times\sum_{q\in Q}\langle f_q, E_q(\cdot, \overline{s}+n, m, (\psi')^c, N)\theta_{\chi,q}\rangle_{\Gamma^{\mathrm{h}}_{0,n}(N)} = \\ =& A_N(\#X)^{-1}[\Gamma^{\mathrm{h}}_{0,n}(N):\Gamma]\Gamma((s))^{-1}(\det\tau)^{s+(k+l)/2}|\det r|_K^{-s-n/2} \\ &\times\langle f, E(\cdot, \overline{s}+n, m, (\psi')^c, N)\theta_{\xi}\rangle_{\Gamma^{\mathrm{h}}_{0,n}(N)}. \end{aligned}$$

We now relate $D(s, f, \xi)$ to the standard L-function $L_{\mathrm{st}}(f, s, \psi) = Z(s, f, \psi)$ of f as defined in [39], section 20.6. In this section we assume that $(D_K, \mathfrak{f}_{\xi}) = 1$. (This assumption is not strictly necessary, but our formulas complicate if we do not make it.) Let ξ, φ, ψ' be Hecke characters of K as we defined them above. Let $N = \mathbf{Z}\cap(\mathrm{cond}\,\psi')$.

One has (by [39], formula (22.19))

$$(7.31)\quad D(s, f, \xi) = \frac{c_{f_r}(\tau)Z(s, f, \xi)}{\prod_{p\in\mathbf{b}}g_p(\xi^*(p\mathcal{O}_K)p^{-2s})\prod_{j=1}^{n}L(2s-n-j+1, \chi_K^{n+j-1}\xi^*_{\mathbf{Q}})}.$$

Here $c_{f_r}(\tau)$ is the τ-Fourier coefficient of f_r, and $L(s, \psi)$ is the usual Dirichlet L-function of ψ while $\prod_{p\in\mathbf{b}}g_p(\xi^*(p\mathcal{O}_K)p^{-2s})$ is defined in [39], Lemma 20.5. Note that the definition of Fourier coefficients in [39] (cf. Proposition 20.2 in [loc.cit.]) differs from ours and from the one given in (18.6.6) in [38] (which in turn agrees with ours) by the factor $e^{-2\pi\mathrm{tr}\,\tau}$ (so, $c_f(\tau, r)$ in (22.19) of [39] agrees with our $c_{f_r}(\tau)$).

Combining (7.31) with (7.30) we obtain

$$\langle f, E(\cdot, s, m, (\psi')^c, N)\theta_\xi\rangle_{\Gamma^{\rm h}_{0,n}(N)} = C(s)\frac{Z(\overline{s}+n/2, f, \xi)}{\prod_{j=1}^n L(2\overline{s}-j+1, \chi_K^{n+j-1}\xi^*_{\mathbf{Q}})}, \tag{7.32}$$

where

$$C(s) = \frac{(\#X)\Gamma((\overline{s}-n))(\det\tau)^{-\overline{s}+n-(k+l)/2}|\det r|_K^{\overline{s}-n/2}c_{f_r}(\tau)}{A_N[\Gamma^{\rm h}_{0,n}(N):\Gamma]\prod_{p\in\mathbf{b}} g_p(\xi^*(p\mathcal{O}_K)p^{-2\overline{s}-n})}. \tag{7.33}$$

Following [39] (17.24), we define

$$D(g,s,N,m,\psi) := E(g,s,N,m,\psi)\prod_{j=1}^n L(2s-j+1, \psi_{\mathbf{Q}}\chi_K^{j-1}).$$

Using (7.32) we finally get

$$\langle D(\cdot, s, m, (\psi')^c, N)\theta_\xi, f\rangle_{\Gamma^{\rm h}_{0,n}(N)} = \overline{C(s)Z(\overline{s}+n/2, f, \xi)}. \tag{7.34}$$

We record this as a theorem.

Theorem 7.8. *Assume $(h_K, 2n) = 1$. Let $f \in \mathcal{M}_{n,k}(N)$ be a Hecke eigenform. Let ξ, ψ' be as above. Then*

$$\langle D(\cdot, s, m, (\psi')^c, N)\theta_\xi, f\rangle_{\Gamma^{\rm h}_{0,n}(N)} = \overline{C(s)}\cdot\overline{L_{\rm st}(f, \overline{s}+n/2, \xi)}$$

with $C(s)$ defined by (7.33).

7.5. **ℓ-integrality of Fourier coefficients of Eisenstein series and theta series.** Let $D(g,s,N,m,\psi)$ be as above. Fix $r \in {\rm GL}_n(\mathbf{A}_{K,{\rm f}})$. As for the Eisenstein series E, we define

$$D_r(Z,s,m,\psi,N) = D_{p_r}(Z,s,m,\psi,N) = j(g,\mathbf{i}_n)^m D(p_r g, s, N, m, \psi),$$

where $g \in U_n(\mathbf{R})$ and $Z = g\mathbf{i}_n$

Theorem 7.9. *Let $r \in {\rm GL}_n(\mathbf{A}_{K,{\rm f}})$. Then $D_r(Z, n-m/2, N, m, \psi)$ is a holomorphic function of Z.*

Proof. This follows from Theorem 17.12(iii) in [39]. □

For a function $A : U_n(\mathbf{A}) \to \mathbf{C}$ set $A^*(g) = A(g\eta_{\rm f}^{-1})$, where $\eta_{\rm f} \in U_n(\mathbf{A})$ is a matrix with trivial infinity component and all finite components equal to the matrix $J = \left[\begin{smallmatrix} & -I_n \\ I_n & \end{smallmatrix}\right]$.

We write

$$D^*\left(\begin{bmatrix} q & \sigma q \\ & \hat{q}\end{bmatrix}, n-m/2, N, m, \psi\right) = \sum_{h\in S} c(h,q)e_{\mathbf{A}}(h\sigma).$$

Theorem 7.10. *Suppose we take $q = (y^{1/2}, q_1)$, where $y^{1/2}$ (resp. q_1) denotes the infinite (resp. finite) component of q. One has*

$$\begin{aligned} c(h,q) = (*)e^{-2\pi{\rm tr}\,((qq^*)_\infty h)}\det(qq^*)_\infty^{m/2}\psi(\det q_1)|\det q_1|_K^{m/2}\times \\ N^{-n^2}\Phi\pi^{n(n+1)/2}\cdot\frac{\prod_{i=0}^{n-1-{\rm rk}(h)} L_N(n-m-i, \psi, \chi_K^{n+i-1})}{\prod_{i=0}^{n-1}\Gamma(n-i)}, \end{aligned} \tag{7.35}$$

where Φ is an algebraic integer and $(*)$ is an ℓ-adic unit. If $r < 1$ we set $\prod_{j=0}^r = 1$.

Proof. The theorem follows from Propositions 18.14 and 19.2 in [38], combined with Lemma 18.7 of [38] and formulas (4.34K) and (4.35K) in [37]. It is a long but straightforward calculation. □

Definition 7.11. Let $\mathcal{B}$ be a base. We will say that $\mathcal{B}$ is *admissible* if

- All $b \in \mathcal{B}$ are scalar matrices with $bb^* = I_n$, which implies p_b are scalar matrices with $p_b p_b^* = I_{2n}$;
- For every $b \in \mathcal{B}$ there exists a rational prime $p \nmid 2D_K\ell$ such that $b_\mathfrak{q} = I_n$ for all $\mathfrak{q} \nmid p$ and $b_\infty = I_n$.

The set of primes p for which $b_\mathfrak{q} \neq I_n$ with $\mathfrak{q} \mid p$ (or by a slight abuse of terminology the product of such primes) will be called the *support* of $\mathcal{B}$.

Remark 7.12. If $(h_K, 2n) = 1$ it follows from Corollary 3.9 together with the Tchebotarev Density Theorem that an admissible base exists.

For $r \in \mathrm{GL}_2(\mathbf{A}_{K,\mathrm{f}})$ write $D_r^*(Z)$ for $D_r^*(Z, n - m/2, , N, m, \psi)$.

Corollary 7.13. *Assume $\ell \nmid N(n-2)!$. If $\mathcal{B}$ is an admissible base whose support is relatively prime to* $\operatorname{cond}\psi$, *then for every $h \in S$ and for every $b \in \mathcal{B}$, the product*

$$\pi^{-n(n+1)/2} c_{h,b}$$

lies in the ring of integers of a finite extension of $\mathbf{Q}_\ell$. Here $c_{h,b}$ stands for the h-Fourier coefficient of $D_b(Z)$.

Proof. Let $\mathcal{O}$ be the ring of integers in some sufficiently large finite extension of $\mathbf{Q}_\ell$. For any admissible base $\mathcal{B}$, one has by Theorem 7.10

$$c_h = \psi(\det b)\pi^{n(n+1)/2} \cdot \prod_{i=0}^{n-1-\mathrm{rk}(h)} L_N(n - m - i, \psi, \chi_K^{n+i-1}) \cdot x,$$

where $x \in \mathcal{O}$. So, the corollary follows from the fact that $\psi(\det b) \in \mathcal{O}^\times$ upon noting that for every Dirichlet character ψ' of conductor dividing N and every $n \in \mathbf{Z}_{<0}$, one has $L(n, \psi') \in \mathbf{Z}_\ell[\psi']$ (by a simple argument using [47], Corollary 5.13) and $(1 - \psi'(p)p^{-n}) \in \mathbf{Z}_\ell[\psi']$ for every $p \mid N$. □

We now turn to the theta series. First note that $\xi_{\mathfrak{f}_\xi}(\det g)^r = 1$ for a sufficiently large integer r (because $\xi_{\mathfrak{f}_\xi}$ is a character of finite order). So $\lambda(\alpha) \neq 0$ only if $\alpha \in M_n(K) \cap M_n(\hat{\mathcal{O}}_K) = M_n(\mathcal{O}_K)$ and for such α one has $\lambda(\alpha)\overline{\det\alpha} \in \mathcal{O}$ (more precisely $\lambda(\alpha)$ is a root of unity in $\mathcal{O}$ and $\det\alpha \in \mathcal{O}_K$).

Fix r, τ as in section 7.4. Let $q \in \mathrm{GL}_n(\mathbf{A}_{K,\mathrm{f}})$.

Proposition 7.14. *Assume $q_v = r_v = I_n$ for every $v \mid \mathfrak{f}_\xi$. Write*

$$A(\sigma, q, r) = \{\alpha \in M_n(K) \cap r M_n(\hat{\mathcal{O}}_K) q^{-1} \mid \alpha^*\tau\alpha = \sigma,\ \alpha_v \in \mathrm{GL}_n(\mathcal{O}_{K,v}) \text{ for all } v \mid \mathfrak{f}_\xi\}.$$

Then the Fourier coefficient $c_{\theta_\xi}(\sigma, q) = 0$ if $\det\sigma = 0$. If $\det\sigma \neq 0$, one gets

$$c_{\theta_\xi}(\sigma, q) = e^{-2\pi \mathrm{tr}\,\sigma} |\det q|_K^{n/2} \varphi^n(\det q)\xi_{\mathfrak{f}_\xi}(\det q)\xi(\det r) \sum_{\alpha \in A(\sigma,q,r)} \xi_{\mathfrak{f}_\xi}(\det\alpha)\overline{\det\alpha},$$

where $|\cdot|_K$ denotes the idele norm on $\mathbf{A}_K^\times$ (cf. [39], *p.180).*

Proof. The Fourier coefficient of $\theta_\xi(g)$ is computed in section A5 of [39] (formula (A.5.11)), where we have $\omega' = \xi^{-1}\varphi^n$. Our formula is slightly reordered and simplified due to the assumptions we imposed, but this is an easy calculation. Again note the discrepancy in the definitions of Fourier coefficients between us and [39] pointed out after formula (7.31). □

Corollary 7.15. *Assume $r_v = I_n$ for all $v \mid \mathfrak{f}_\xi D_K \ell$. For an admissible base $\mathcal{B}$ we have that $c_{(\theta_\xi)_b}(\sigma) = e^{2\pi \mathrm{tr}\, \sigma} c_{\theta_\xi}(\sigma, b) \in \mathcal{O}$ for all $b \in \mathcal{B}$ and all $\sigma \in \mathcal{S}$.*

Proof. Let us fix an admissible $\mathcal{B}$. Then we automatically have $|\det b|_K = 1$. Since $\mathrm{cond}\,\varphi \mid D_K$, we see that $\varphi(\det b) \in \mathcal{O}^\times$. Similarly we get $\xi_{\mathfrak{f}_\xi}(\det b) = 1$ and $\xi(\det b) \in \mathcal{O}^\times$. Since $\xi_{\mathfrak{f}_\xi}$ is of finite order, we see that $\xi_{\mathfrak{f}_\xi}(\det \alpha) \in \mathcal{O}^\times$ for any matrix α as remarked above. Finally, since $r_v = b_v = I_n$ for all $v \mid \ell$, we get that for such v and $\alpha \in A(\sigma, b, r)$ one has $\alpha_v \in M_n(\mathcal{O}_{K,v})$. Since $\alpha \in M_n(K)$, we get that $\alpha \in M_n(\mathcal{O}_{K,v})$, so $\det \alpha \in \mathcal{O}$. Since $c_{(\theta_\xi)_b}(\sigma) = e^{2\pi \mathrm{tr}\, \sigma} c_{\theta_\xi}(\sigma, b)$ by (5.3), we are done. □

8. Congruence

In this section we set $n = 2$ and write $U = U_2$. Let K be an imaginary quadratic field of discriminant $-D_K$, which we assume to be prime. It is a well-known fact that this implies that the class number of K is odd. Then the space $S_{k-1}(D_K, \chi_K)$ has a (unique) basis of newforms, which we, as before, denote by $\mathcal{N}$. We fix the following set of data:

- a positive even integer k divisible by $\#\mathcal{O}_K^\times$;
- a rational prime $\ell > k$ such that $\ell \nmid h_K D_K$;
- $\phi \in \mathcal{N}$, which is ordinary at ℓ such that $\overline{\rho}_\phi|_{G_K}$ is absolutely irreducible;
- $\chi \in \mathrm{Hom}(\mathrm{Cl}_K, \mathbf{C}^\times)$ and write $f_{\phi,\chi}$ for the Maass lift of ϕ lying in the space $\mathcal{S}^\chi_{k,-k/2}$;
- ξ a Hecke character of K of ∞-type $z^t|z|^{-t}$ for an integer $-6 > t \geq -k$; we write $\mathfrak{f}_\xi$ for the conductor of ξ and set $N = D_K h_K N_{K/\mathbf{Q}}(\mathfrak{f}_\xi)$;
- $\beta \in \mathrm{Char}(k/2)$;
- an admissible base $\mathcal{B}$ whose support is prime to N.

Let E be a finite extension of $\mathbf{Q}_\ell$, which we will always assume to be sufficiently large to contain any (finite number of) number fields that we encounter. Write $\mathcal{O}$ for its valuation ring. We also fix a uniformizer $\varpi \in \mathcal{O}$.

Lemma 8.1. *Let $b \in \mathcal{B}$ and $\tau \in \mathcal{S}$. Write $c_{f_{\phi,\chi}}(\tau, b)$ for the (τ, b)-Fourier coefficient of $f_{\phi,\chi}$ Then $e^{2\pi \mathrm{tr}\, \tau} c_{f_{\phi,\chi}}(\tau, b) \in \mathcal{O}$ for arbitrary $\tau \in S$.*

Proof. It is a standard fact that the Fourier coefficients of ϕ are algebraic integers (the field which they all generate is a finite extension of $\mathbf{Q}$), so by our assumption that $\mathcal{O}$ be sufficiently large we may assume that they lie in $\mathcal{O}$. Then the Lemma follows from (5.1) and the formula in Theorem 5.16. □

Definition 8.2. Let $\tau \in \mathcal{S}$ and $b \in \mathcal{B}$. We will call (τ, b) an *ordinary pair* if the following two conditions are simultaneously satisfied:

(1) $\mathrm{val}_\ell(\overline{e^{2\pi \mathrm{tr}\,(\tau)} c_{f_{\phi,\chi}}(\tau, b)}) = 0$;
(2) $(\det \tau, N) = 1$.

Lemma 8.3. *An ordinary pair exists.*

Proof. This is proved like Lemma 7.10 in [27]. Note that the prime p_0 chosen in the proof of that lemma can be taken to be arbitrarily large because its existence is guaranteed by the Tchebotarev Density Theorem. Hence condition (2) can also be satisfied. □

Definition 8.4. Let f and g be two hermitian modular forms. We say that f is congruent to g modulo ϖ^n, a property which we denote by $f \equiv g \pmod{\varpi^n}$ if there exists a base $\mathcal{B}$ such that f_b and g_b both have Fourier coefficients lying in $\mathcal{O}$ for all $b \in \mathcal{B}$ and $f_b \equiv g_b \pmod{\varpi^n}$ for all $b \in \mathcal{B}$. The latter congruence means that $\varpi^n \mid (c_{f_b}(h) - c_{g_b}(h))$ for all $h \in \mathcal{S}$.

Remark 8.5. Note that if f and g are congruent $(\mathrm{mod}\ \varpi^n)$ with respect to one admissible base, say $\mathcal{B}$, and $\mathcal{B}'$ is another admissible base such that f_b and g_b both have Fourier coefficients lying in $\mathcal{O}$ for all $b \in \mathcal{B}'$, then $f_b \equiv g_b \pmod{\varpi^n}$ for all $b \in \mathcal{B}'$. Indeed this follows from admissibility of $\mathcal{B}$ and $\mathcal{B}'$ and formula (5.4).

8.1. **The inner product ratio.** Let $\Psi_\beta : \mathcal{M}_k \xrightarrow{\sim} \mathcal{M}_{k,-k/2}$ be the isomorphism defined in Proposition 3.13. Denote the restriction of $f_{\phi,\chi}$ to $U(\mathbf{A})$ again by $f_{\phi,\chi}$ (cf. Proposition 3.10). Fix $\tau \in \mathcal{S}$ and assume that the theta series θ_ξ was defined using that τ (cf. section 7.3). For the moment we will not assume that we are working with an ordinary pair to obtain a more general formula for the ratio of the inner products. Set $c = \det \tau$. Set $\psi' = \xi^{-1}\varphi^{-2}$ with φ as in section 7.3. Then $\theta_\xi \in \mathcal{M}_l(Nc, \psi')$, where $l = t + k + 2$ (see Remark 7.7). Set $m = k - l$. To shorten notation write $D(g) := D(g, 2 - m/2, Nc, m, (\psi')^c)$ and $D^*(g) := D(g\eta_{\mathrm{f}}^{-1}, 2 - m/2, Nc, m, (\psi')^c)$.

Since $D(g) \in \mathcal{M}_m(Nc, (\psi')^{-1})$ and $\theta_\xi \in \mathcal{M}_l(Nc, \psi')$, we get $D\theta_\xi \in \mathcal{M}_k(\mathcal{K}_0(Nc))$ and $D^*\theta_\xi^* \in \mathcal{M}_k(\eta_{\mathrm{f}}^{-1}\mathcal{K}_0(Nc)\eta_{\mathrm{f}})$.

For $F \in M^{\mathrm{h}}_{k,-k/2}(J^{-1}\Gamma_0^{\mathrm{h}}(Nc)J)$ define the *trace operator* $\mathrm{tr} : M^{\mathrm{h}}_{k,-k/2}(J^{-1}\Gamma_0^{\mathrm{h}}(Nc)J) \to M^{\mathrm{h}}_{k,-k/2}(U(\mathbf{Z}))$ by

$$\mathrm{tr}\, F = \sum_{\gamma \in J^{-1}\Gamma_0^{\mathrm{h}}(Nc)J \backslash U(\mathbf{Z})} F|_k \gamma.$$

Note that if F has ℓ-integral Fourier coefficients, then so does $\mathrm{tr}\, F$ by the q-expansion principle (Theorem 3.4). The form $D^*\theta_\xi^*$ corresponds to $\#\mathcal{B}$ forms $(D^*\theta_\xi^*)_{p_b} \in M_k^{\mathrm{h}}(J^{-1}\Gamma_0^{\mathrm{h}}(Nc)J)$, $b \in \mathcal{B}$. This way we can define $\mathrm{tr}\,(D^*\theta_\xi^*)$. Since both $\Psi_\beta(\mathrm{tr}\,(D^*\theta_\xi^*))$ and $f_{\phi,\chi}$ are elements of the finite-dimensional $\mathbf{C}$-vector space $\mathcal{M}_{k,-k/2}(\mathcal{K}_0(Nc))$ we can write

$$\Xi' := \Psi_\beta(\mathrm{tr}\,(D^*\theta_\xi^*)) = C f_{\phi,\chi} + g, \quad \text{with } \langle g, f_{\phi,\chi}\rangle = 0 \tag{8.1}$$

and

$$C = \frac{\langle \Xi', f_{\phi,\chi}\rangle}{\langle f_{\phi,\chi}, f_{\phi,\chi}\rangle}.$$

(For the definition of the inner product on $\mathcal{M}_{k,\nu}$ see section 7.3.)

Lemma 8.6. *For any $f \in \mathcal{M}_k$ one has*

$$\langle \Psi_\beta(f), f_{\phi,\chi}\rangle = \left\langle f, \Psi_\beta^{-1}(f_{\phi,\chi})\right\rangle.$$

Proof. We have

$$\begin{aligned}(8.2)\quad \langle \Psi_\beta(f), f_{\phi,\chi}\rangle &= (\#\mathcal{B})^{-1}\sum_{b\in\mathcal{B}}\langle \Psi_\beta(f)_b, (f_{\phi,\chi})_b\rangle\\ &= (\#\mathcal{B})^{-1}\sum_{b\in\mathcal{B}}\langle \beta(\det b) f_b, (f_{\phi,\chi})_b\rangle\\ &= (\#\mathcal{B})^{-1}\sum_{b\in\mathcal{B}}\left\langle f_b, \beta^{-1}(\det b)(f_{\phi,\chi})_b\right\rangle = \left\langle f, \Psi_\beta^{-1}(f_{\phi,\chi})\right\rangle,\end{aligned}$$

where the second and the fourth equality follow from commutativity of diagram (3.9) while the third one follows from the fact that $\beta(\det b)^{-1} = \overline{\beta(\det b)})$. □

Lemma 8.7. *One has*

$$\langle \Xi', f_{\phi,\chi}\rangle = \left\langle D\theta_\xi, \Psi_\beta^{-1}(f_{\phi,\chi})\right\rangle_{\Gamma_0^{\mathrm{h}}(Nc)}.$$

Proof. Write $x = [U(\mathbf{Z}) : \Gamma_0^{\mathrm{h}}(Nc)]^{-1}$. We have

(8.3)

$$\begin{aligned}\langle \Xi', f_{\phi,\chi}\rangle &= \left\langle \mathrm{tr}\,(D^*\theta_\xi^*), \Psi_\beta^{-1}(f_{\phi,\chi})\right\rangle\\ &= (\#\mathcal{B})^{-1}\sum_{b\in\mathcal{B}}\left\langle \mathrm{tr}\,(D^*\theta_\xi^*)_b, \Psi_\beta^{-1}(f_{\phi,\chi})_b\right\rangle\\ &= (\#\mathcal{B})^{-1}x\sum_{b\in\mathcal{B}}\left\langle \mathrm{tr}\,(D^*\theta_\xi^*)_b, \Psi_\beta^{-1}(f_{\phi,\chi})_b\right\rangle_{\Gamma_0^{\mathrm{h}}(Nc)}\\ &= (\#\mathcal{B})^{-1}x\sum_{b\in\mathcal{B}}\sum_{\gamma\in J^{-1}\Gamma_0^{\mathrm{h}}(Nc)J\backslash U(\mathbf{Z})}\left\langle (D^*\theta_\xi^*)_b|_k\gamma, \Psi_\beta^{-1}(f_{\phi,\chi})_b\right\rangle_{\Gamma_0^{\mathrm{h}}(Nc)}\\ &= (\#\mathcal{B})^{-1}x\sum_{b\in\mathcal{B}}\sum_{\gamma\in J^{-1}\Gamma_0^{\mathrm{h}}(Nc)J\backslash U(\mathbf{Z})}\left\langle (D^*\theta_\xi^*)_b, \Psi_\beta^{-1}(f_{\phi,\chi})_b|_k\gamma^{-1}\right\rangle_{\Gamma_0^{\mathrm{h}}(Nc)}\\ &= (\#\mathcal{B})^{-1}x\sum_{b\in\mathcal{B}}\sum_{\gamma\in J^{-1}\Gamma_0^{\mathrm{h}}(Nc)J\backslash U(\mathbf{Z})}\left\langle (D^*\theta_\xi^*)_b, \Psi_\beta^{-1}(f_{\phi,\chi})_b\right\rangle_{\Gamma_0^{\mathrm{h}}(Nc)}\\ &= (\#\mathcal{B})^{-1}\sum_{b\in\mathcal{B}}\cdot\left\langle (D^*\theta_\xi^*)_b, \Psi_\beta^{-1}(f_{\phi,\chi})_b\right\rangle_{\Gamma_0^{\mathrm{h}}(Nc)}\\ &= \left\langle D\theta_\xi, \Psi_\beta^{-1}(f_{\phi,\chi})\right\rangle_{\Gamma_0^{\mathrm{h}}(Nc)},\end{aligned}$$

where the first equality follows from Lemma 8.6, the fifth one and the last one from (10.9.3) in [38] and the sixth one from the fact that $\Psi_\beta^{-1}(f_{\phi,\chi}) \in \mathcal{M}_k$. □

Proposition 8.8. *Let τ be as before. Suppose there exists $r \in \mathrm{GL}_2(\mathbf{A}_{K,\mathrm{f}})$ such that $c_{f_{\phi,\chi}}(\tau, r) \neq 0$. One has*

$$\begin{aligned}(8.4)\quad \frac{\langle \Xi', f_{\phi,\chi}\rangle}{\langle f_{\phi,\chi}, f_{\phi,\chi}\rangle} &= (*)\eta^{-1}\frac{\#X_{m,Nc}\pi^3(\det\tau)^{-k}|\det r|_K^{t/2}c_{f_{\phi,\chi}}(\tau,r)}{A_N[\Gamma_0^{\mathrm{h}}(Nc):\Gamma]}\\ &\times\frac{L^{\mathrm{int}}(\mathrm{BC}(\phi),\frac{t+k}{2}+1,\beta\overline{\xi}\chi^{-1})L^{\mathrm{int}}(\mathrm{BC}(\phi),\frac{t+k}{2}+2,\beta\overline{\xi}\chi^{-1})}{L^{\mathrm{int}}(\mathrm{Symm}^2\phi,k)},\end{aligned}$$

where $\Gamma = \Gamma_1^{\rm h}(N) \cap {\rm SU}_2(\mathbf{Q})$ *(cf. section 7.3),* $(*) \in E$ *with* ${\rm val}_\ell((*)) \leq 0$*,*

$$L^{\rm int}({\rm BC}(\phi), j + (t+k)/2, \omega) := \frac{\Gamma(t+k+j)L({\rm BC}(\phi), j + \frac{t+k}{2}, \omega)}{\pi^{t+k+2j}\Omega_\phi^+\Omega_\phi^-},$$

for a Hecke character $\omega : \mathbf{A}_K^\times \to \mathbf{C}^\times$ *and*

$$L^{\rm int}({\rm Symm}^2 \phi, n) := \frac{\Gamma(n)L({\rm Symm}^2 \phi, n)}{\pi^{n+2}\Omega_\phi^+\Omega_\phi^-}$$

for any integer n*.*

Remark 8.9. The normalized L-values are algebraic (see the proof below) and are expected to be algebraic integers, but for the moment we know of no proof of the latter claim.

Proof of Proposition 8.8. By Lemma 8.7 we get

$$\langle \Xi', f_{\phi,\chi}\rangle = \left\langle D\theta_\xi, \Psi_\beta^{-1}(f_{\phi,\chi})\right\rangle_{\Gamma_0^{\rm h}(Nc)}.$$

Since the isomorphisms in diagram (3.9) are Hecke-equivariant, $\Psi_\beta^{-1}(f_{\phi,\chi})$ is still a Hecke eigenform, hence we can apply Theorem 7.8 and get

$$\begin{aligned}\left\langle D\theta_\xi, \Psi_\beta^{-1}(f_{\phi,\chi})\right\rangle_{\Gamma_0^{\rm h}(Nc)} &= \overline{C(2-m/2)} \cdot \overline{L_{\rm st}(\Psi_\beta^{-1}(f_{\phi,\chi}), 3-m/2, \xi)} \\ &= \overline{C(2-m/2)} \cdot \overline{L_{\rm st}(f_{\phi,\chi}, 3-m/2, \beta^{-1}\xi)}.\end{aligned} \tag{8.5}$$

Using Proposition 5.22 we get

$$L_{\rm st}(f_{\phi,\chi}, 3-m/2, \beta^{-1}\xi) = L({\rm BC}(\phi), \frac{t+k}{2} + 1, \beta^{-1}\xi\chi)L({\rm BC}(\phi), \frac{t+k}{2} + 2, \beta^{-1}\xi\chi).$$

Moreover, by Fact 2.1 (and the fact that $\overline{\beta} = \beta^{-1}$) we get

$$\overline{L({\rm BC}(\phi), \frac{t+k}{2} + j, \beta^{-1}\xi\chi)} = L({\rm BC}(\phi), \frac{t+k}{2} + j, \beta\overline{\xi}\chi^{-1}).$$

To ease notation write $c(\tau)$ for the τ-Fourier coefficient $c_{(f_{\phi,\chi})_r}(\tau)$ of the r-component of $f_{\phi,\chi}$. Using the formula (7.33) we get

$$C(2-m/2) = (*)\frac{(\#X_{m,Nc})\pi^{-2t-2k-3}\Gamma(t+k+2)\Gamma(t+k+1)(\det\tau)^{-k}|\det r|_K^{t/2}c(\tau)}{A_N[\Gamma_{0,n}^{\rm h}(Nc) : \Gamma]},$$

with A_N defined in section 7.4 and $(*) \in E$ with ${\rm val}_\ell((*)) \leq 0$ (note that the product $\prod_{p \in \mathbf{b}} g_p(\xi^*(p\mathcal{O}_K)p^{-2\overline{s}-n})$ in (7.33) is a finite product and g_p is a polynomial with coefficients in $\mathbf{Z}$ and constant term 1 - cf. [39], Lemma 20.5). On the other hand we have by Theorem 5.24 (note that ${\rm val}_\ell(\Gamma(k)) = 0$)

$$\langle f_{\phi,\chi}, f_{\phi,\chi}\rangle = (*)\pi^{-k-2} \cdot \langle \phi, \phi\rangle\, L({\rm Symm}^2 \phi, k)\Gamma(k),$$

where ${\rm val}_\ell((*)) = 0$. Define $L^{\rm alg}$ the same way as $L^{\rm int}$ except with $\langle\phi,\phi\rangle$ instead of $\Omega_\phi^+\Omega_\phi^-$. It follows from Remark 6.3 in [27] and from Theorem 1 on page 325 in [21] that

$$L^{\rm alg}({\rm BC}(\phi), 1 + (t+k)/2, \beta\overline{\xi}\chi^{-1}) \in \overline{\mathbf{Q}} \tag{8.6}$$

and

$$L^{\rm alg}({\rm BC}(\phi), 2 + (t+k)/2, \beta\overline{\xi}\chi^{-1}) \in \overline{\mathbf{Q}} \tag{8.7}$$

and from a result of Sturm [42] that

$$L^{\mathrm{alg}}(\mathrm{Symm}^2\,\phi, k) \in \overline{\mathbf{Q}}. \tag{8.8}$$

We note here that [42] uses a definition of the Petersson norm of ϕ which differs from ours by a factor of $\frac{3}{\pi}$, the volume of the fundamental domain for the action of $\mathrm{SL}_2(\mathbf{Z})$ on the complex upper half-plane. We assume that E contains values (8.6), (8.7), and (8.8). It follows from Proposition 6.6 that

$$\langle \phi, \phi \rangle = (*)\, \eta\, \Omega_\phi^+ \Omega_\phi^-, \tag{8.9}$$

where $(*)$ is a λ-adic unit as long as ϕ is ordinary at ℓ and $\ell > k$, which we have assumed. We also assume that E contains η. The Proposition now follows. □

8.2. **Congruence between $f_{\phi,\chi}$ and a non-Maass form.** The goal of this section is to prove the following theorem, which is the main result of the paper. To make the statement self-contained we repeat the assumptions made at the beginning of the section (the constant A_N is defined in section 7.4). In the next section we will formulate some consequences of this theorem.

Theorem 8.10. *Let $K = \mathbf{Q}(i\sqrt{D_K})$ be an imaginary quadratic field of prime discriminant $-D_K$ and class number h_K. Let k be an even positive integer divisible by $\#\mathcal{O}_K^\times$ and $\ell > k$ a rational prime such that $\ell \nmid D_K h_K$. Let $\phi \in S_{k-1}(D_K, \chi_K)$ be a newform ordinary at ℓ and such that $\overline{\rho}_\phi|_{G_K}$ is absolutely irreducible. Fix a Hecke character ξ of K such that $\mathrm{val}_\ell(\mathrm{cond}\,\xi) = 0$, $\xi_\infty(z) = \left(\frac{z}{|z|}\right)^{-t}$ for some integer $-k \leq t < -6$, $\mathrm{val}_\ell(A_N) \geq 0$ and $\ell \nmid \#(\mathcal{O}_K/N\mathcal{O}_K)^\times$, where $N = D_K h_K N_{K/\mathbf{Q}}(\mathrm{cond}\,\xi)$. Let E be a sufficiently large finite extension of $\mathbf{Q}_\ell$ with uniformizer ϖ. Fix $\chi \in \mathrm{Hom}(\mathrm{Cl}_K, \mathbf{C}^\times)$ and $\beta \in \mathrm{Char}(k/2)$. If*

$$-n := \mathrm{val}_\varpi\left(\prod_{j=1}^{2} L^{\mathrm{int}}(\mathrm{BC}(\phi), j + (t+k)/2, \beta\overline{\xi}\chi^{-1})\right) - \mathrm{val}_\varpi(L^{\mathrm{int}}(\mathrm{Symm}^2\,\phi, k)) < 0$$

then there exists $f \in \mathcal{S}^\chi_{k,-k/2}$, orthogonal to the Maass space, such that $f \equiv f_{\phi,\chi} \pmod{\varpi^n}$.

Remark 8.11. Theorem 8.10 is a generalization of Theorem 7.12 in [27], which applied to the case $K = \mathbf{Q}(i)$. In that case the character β is unique (and equals $\overline{\omega}$ in [loc.cit.]), the character χ is trivial since the class number of $\mathbf{Q}(i)$ equals 1 and the character ξ corresponds to the character which in [loc.cit.] was denoted by χ.

Proof of Theorem 8.10. Consider again equation (8.1). Note that $\Xi' = \Psi_\beta(\mathrm{tr}\,(D^*\theta^*_\chi))$ and g lie in $\mathcal{M}_{k,-k/2}$ and $f_{\phi,\chi} \in \mathcal{M}^\chi_{k,-k/2}$. We would like all of the forms to be in $\mathcal{M}^\chi_{k,-k/2}$.

Lemma 8.12. *Let Z denote the center of U. The quotient $Z(\mathbf{A})/Z(\mathbf{Q})$ is compact.*

Proof. Note that $Z(\mathbf{A}) = \bigcup_{b\in\mathcal{B}} Z(\mathbf{Q})Z(\mathbf{R})p_b Z(\hat{\mathbf{Z}})$. Since $Z(\mathbf{R})$ is compact, the lemma follows. □

Let dz be a Haar measure on $Z(\mathbf{A})/Z(\mathbf{Q})$ normalized so that $\mathrm{vol}(Z(\mathbf{A})/Z(\mathbf{Q})) = 1$. For $f \in \mathcal{M}_{k,-k/2}$ set

$$(\pi_\chi f)(g) = \int_{Z(\mathbf{A})/Z(\mathbf{Q})} f(gz)\chi^{-1}(z)dz = \frac{1}{h_K}\sum_{b\in\mathcal{B}} \chi^{-1}(p_b) f(gp_b) \in \mathcal{M}^\chi_{k,-k/2},$$

where by $\chi^{-1}(z)$, $\chi^{-1}(p_b)$ we mean $\chi^{-1}(c_K(\det z)^{1/2})$ and $\chi^{-1}(c_K(\det b))$ respectively with $c_K : \mathbf{A}_K^\times \twoheadrightarrow \mathrm{Cl}_K$ the canonical map. The last equality clearly implies that a Fourier coefficient of $\pi_\chi f$ is in $\mathcal{O}$ when the corresponding Fourier coefficient of f is in $\mathcal{O}$ (since $\ell \nmid h_K$). Note the slight abuse of terminology when we say that f (or other adelic hermitian modular form) has Fourier coefficients in $\mathcal{O}$. By saying so, we mean that for $h \in \mathcal{S}$ and $r \in \mathrm{GL}_2(\mathbf{A}_{K,\mathrm{f}})$ one has $e^{2\pi \mathrm{tr}\, h} c_f(h,r) \in \mathcal{O}$. We will continue this abuse. Apply π_χ to both sides of (8.1). Write $\Xi = \pi^{-3}\pi_\chi \Xi'$ and $g_0 = \pi^{-3}\pi_\chi g$. Then we have

$$\Xi = C_{\phi,\chi} f_{\phi,\chi} + g_0 \in \mathcal{M}^\chi_{k,-k/2} \tag{8.10}$$

with $\langle g_0, f_{\phi,\chi}\rangle = 0$ and $C_{\phi,\chi} := \pi^{-3}C$.

Combining Corollaries 7.13 and 7.15 we get that for every $b \in \mathcal{B}$ and every $h \in \mathcal{S}$, the h-Fourier coefficient $c_{\Xi_b}(h)$ of Ξ_b is in $\mathcal{O}$.

Fix an ordinary pair (τ, b_0) and as before set $c = \det \tau$. Then $\mathrm{val}_\ell(c(\tau)) = 0$ with $c(\tau)$ as in section 8.1. Since $Nc > 1$ it follows from the proof of Lemma 11.14 in [38] together with Lemma 11.15 in [loc.cit.] and the remark following it that the order of $X_{m,Nc}$ equals the index of the group $\{x \in \mathbf{A}_K^\times \mid x_\mathfrak{p} \in \mathcal{O}_{K,\mathfrak{p}}^\times$ and $x_\mathfrak{p} - 1 \in Nc\mathcal{O}_{K,\mathfrak{p}}$ for every $\mathfrak{p} \nmid \infty\}$ inside $\mathbf{A}_K^\times$. Hence in particular the assumptions in the theorem imply that $\ell \nmid \# X_{m,Nc}$. So, from (8.10), (8.1) and Proposition 8.8 we obtain that

$$C_{\phi,\chi} = (*)\eta^{-1}\frac{L^{\mathrm{int}}(\mathrm{BC}(\phi)\frac{t+k}{2}+1, \beta\bar{\xi}\chi^{-1})L^{\mathrm{int}}(\mathrm{BC}(\phi)\frac{t+k}{2}+2, \beta\bar{\xi}\chi^{-1})}{L^{\mathrm{int}}(\mathrm{Symm}^2\phi, k)}, \tag{8.11}$$

where $\mathrm{val}_\ell((*)) \leq 0$ since $\det\tau \in \mathcal{O}$, $A_N \in \mathcal{O}$ and $[\Gamma_0^{\mathrm{h}}(Nc) : \Gamma] \in \mathbf{Z}$. Note that under our assumption on the L-function (and ignoring the factor η^{-1}), this equality (together with fact that for every $b \in \mathcal{B}$ the forms Ξ_b and $(f_{\phi,\chi})_b$ have Fourier coefficients in $\mathcal{O}$) implies that we must have a mod ϖ^n congruence between $f_{\phi,\chi}$ and $-\varpi^n g_0$. However, there is no guarantee that g_0 is orthogonal to the Maass space. So, we will now use the Hecke operator T^{h} which we constructed in section 6 to 'kill' the 'Maass' part of g_0.

Indeed, by Theorem 6.15 there exists $T^{\mathrm{h}} \in \mathcal{H}^\chi_\mathcal{O}$ such that $T^{\mathrm{h}} f_{\phi,\chi} = \eta f_{\phi,\chi}$ and $T^{\mathrm{h}} f = 0$ for any eigenform $f \in \mathcal{S}^{\mathrm{M},\chi}_{k,-k/2}$ orthogonal to $f_{\phi,\chi}$.

We apply T^{h} to both sides of (8.10). As for all $b \in \mathcal{B}$ and $h \in \mathcal{S}$, the Fourier coefficients $e^{2\pi \mathrm{tr}\, h} c_\Xi(h,b)$ of Ξ lie in $\mathcal{O}$, so do the Fourier coefficients of $T^{\mathrm{h}}\Xi$ by Propositions 4.4 and 4.5. Moreover, since θ_χ is a cusp form, so are Ξ and $T^{\mathrm{h}}\Xi$.

We thus get

$$T^{\mathrm{h}}\Xi = \eta C_{\phi,\chi} f_{\phi,\chi} + T^{\mathrm{h}} g_0 \tag{8.12}$$

with $T^{\mathrm{h}} g_0$ orthogonal to the Maass space.

As $C_{\phi,\chi} \in E \subset \mathbf{C}$ by (8.11), it makes sense to talk about its ϖ-adic valuation. Suppose $\mathrm{val}_\varpi(\eta\, C_{\phi,\chi}) = -n \in \mathbf{Z}_{<0}$. Note that since the (h,b)-Fourier coefficients of $T^{\mathrm{h}}\Xi$ and of $f_{\phi,\chi}$ lie in $\mathcal{O}$ for all $b \in \mathcal{B}$ and all $h \in \mathcal{S}$, but $\eta\, C_{f_{\phi,\chi}} \notin \mathcal{O}$, we must have that either $T^{\mathrm{h}} g_0 \neq 0$ or $e^{2\pi \mathrm{tr}\, h} c_{f_{\phi,\chi}}(h,b) \equiv 0 \bmod \varpi$ for all $b \in \mathcal{B}$ and all $h \in \mathcal{S}$.

Lemma 8.13. *There exists a pair (h,b) such that $e^{2\pi \mathrm{tr}\, h} c_{f_{\phi,\chi}}(h,b) \not\equiv 0$ mod ϖ.*

Proof. Assume on the contrary that $e^{2\pi \mathrm{tr}\, h} c_{f_{\phi,\chi}}(h,b) \equiv 0 \bmod \varpi$ for all pairs (h,b). By our choice of $\mathcal{B}$, taking $h = \left[\begin{smallmatrix} p & 0 \\ 0 & 1\end{smallmatrix}\right]$ with $p = 1$ or a prime, (5.1) implies then that $c_{b,f_{\phi,\chi}}(D_K p) \equiv 0 \bmod \varpi$ for all primes p and for $p = 1$. Using Theorem 5.16 we get that $a_\phi(pD_K) - \overline{a_\phi(pD_K)} \equiv 0 \bmod \varpi$ for all primes p and $p = 1$. Here $a_\phi(n)$ stands for the nth Fourier coefficient of ϕ. By taking $p = 1$ we conclude that $a_\phi(D_K) \equiv \overline{a_\phi(D_K)} \pmod{\varpi}$. Since $|a_\phi(D_K)| = D_K^{(k-2)/2}$ (see for example [26], formula (6.90)), we have $\mathrm{val}_\ell(a_\phi(D_K)) = 0$ since $\ell \nmid D_K$. Hence we must have $a_\phi(p) \equiv \overline{a_\phi(p)} \pmod{\varpi}$ for all primes p. This on the other hand implies that $\overline{\rho}_f|_{G_K}$ is not absolutely irreducible by Proposition 6.3. This contradicts our assumptions. □

By Lemma 8.13 we must have $T^{\mathrm{h}} g_0 \neq 0$. Write $\eta\, C_{\phi,\chi} = a\varpi^{-n}$ with $a \in \mathcal{O}^\times$. Then the Fourier coefficients of $(\varpi^n T^{\mathrm{h}} g_0)_b$ lie in $\mathcal{O}$ and one has

$$f_{\phi,\chi} \equiv -a^{-1}\varpi^n T^{\mathrm{h}} g_0 \pmod{\varpi^n}.$$

As explained above, $-a^{-1}\varpi^n T^{\mathrm{h}} g_0$ is a hermitian modular form orthogonal to the Maass space. This completes the proof of Theorem 8.10. □

Corollary 8.14. *Suppose that ξ, χ, β in Theorem 8.10 can be chosen so that*

$$\mathrm{val}_\varpi\left(\prod_{j=1}^{2} L^{\mathrm{int}}(\mathrm{BC}(\phi), j + (t+k)/2, \beta\overline{\xi}\chi^{-1})\right) = 0,$$

then n in Theorem 8.10 can be taken to be $\mathrm{val}_\varpi(L^{\mathrm{int}}(\mathrm{Symm}^2\,\phi, k))$.

Remark 8.15. As already discussed in Remark 7.15 of [27], the existence of character ξ as in Corollary 8.14 is not known in general. Note that one needs to "control" two L-values at the same time to ensure that their product is a ϖ-adic unit. However, if the class number of K is larger than one, we now have (slightly) more flexibility as we also get to choose the character β (or equivalently $\chi^{-1}\beta$). While we still do not have a proof for this fact it seems very unlikely that for all the possible combinations of the characters ξ and β the product of L-values should always involve non-zero powers of ϖ.

Remark 8.16. The ordinarity assumption on ϕ in Theorem 8.10 is used in section 6 to construct the Hecke operator T^{h} annihilating the Maass part of g_0 as above as well as to ensure that $(*)$ in (8.9) is a ϖ-adic unit. Note that the operator T^{h} is not necessary provided that ϕ is not congruent (mod ϖ) to any other $\phi' \in S_{k-1}(D_K, \chi_K)$. Indeed, then there cannot be any 'Maass part' of $g_0 \in \mathcal{S}^{\chi}_{k,-k/2}$ that is congruent to $f_{\phi,\chi}$. One expects that the set of primes ℓ of $\mathbf{Q}$ such that a given (non-CM) form ϕ is ordinary at ℓ has Dirichlet density one, but for now no proof of this fact is known. An analogous statement for elliptic curves was proved by Serre [35].

8.3. **The Maass ideal.**

Corollary 8.17. *Under the assumptions of Theorem 8.10 there a is cuspidal Hecke eigenform $g \in \mathcal{S}^{\chi}_{k,-k/2}$ such that $\mathrm{val}_\varpi(\lambda_{f_{\phi,\chi}}(T) - \lambda_g(T)) > 0$ for all Hecke operators $T \in \mathcal{H}^{\chi}_{\mathcal{O}}$. Here the homomorphism λ sends the Hecke operator to its eigenvalue.*

Proof. Let f be as in Theorem 8.10. Using the decomposition (6.4), we see that there exists a Hecke operator $T_0 \in \mathcal{H}^{\chi}_{\mathcal{O}}$ such that $T_0 f_{\phi,\chi} = f_{\phi,\chi}$ and $T_0 f' = 0$ for each $f' \in \mathcal{S}^{\chi}_{k,-k/2}$ which is orthogonal to all Hecke eigenforms whose eigenvalues are congruent to those of $f_{\phi,\chi}$ (mod ϖ). Let $\mathcal{N}^{\mathrm{h}}$ be a basis of eigenforms for

$\mathcal{H}^{\chi}_{k,-k/2}$ such that $f_{\phi,\chi} \in \mathcal{N}^{\mathrm{h}}$. Suppose all the elements of $\mathcal{N}^{\mathrm{h}}$ whose eigenvalues are congruent to those of $f_{\phi,\chi}$ (mod ϖ) lie in the Maass space. Then $T_0 f = 0$. However, since $f_{\phi,\chi} \equiv f$ (mod ϖ), this implies that with respect to some base $\mathcal{B}$ for all $b \in \mathcal{B}$ all the Fourier coefficients of the b-component of $f_{\phi,\chi}$ are in $\varpi\mathcal{O}$. By Theorem 5.16 and (5.1) this is only possible if $\phi \equiv \phi^{\rho}$ (mod ϖ). This however leads to a contradiction by Proposition 6.3. □

Recall that we have a Hecke-stable decomposition

$$\mathcal{S}^{\chi}_{k,-k/2} = \mathcal{S}^{\mathrm{M},\chi}_{k,-k/2} \oplus \mathcal{S}^{\mathrm{NM},\chi}_{k,-k/2},$$

where $\mathcal{S}^{\mathrm{NM},\chi}_{k,-k/2}$, denotes the orthogonal complement of $\mathcal{S}^{\mathrm{M},\chi}_{k,-k/2}$ inside $\mathcal{S}^{\chi}_{k,-k/2}$. Denote by $\mathcal{H}^{\mathrm{NM},\chi}_{\mathcal{O}}$ the image of $\mathcal{H}^{\chi}_{\mathcal{O}}$ inside $\mathrm{End}_{\mathbf{C}}(\mathcal{S}^{\mathrm{NM},\chi}_{k,-k/2})$ and let $\Phi : \mathcal{H}^{\chi}_{\mathcal{O}} \twoheadrightarrow \mathcal{H}^{\mathrm{NM},\chi}_{\mathcal{O}}$ be the canonical $\mathcal{O}$-algebra epimorphism. Let $\mathrm{Ann}(f_{\phi,\chi}) \subset \mathcal{H}^{\chi}_{\mathcal{O}}$ denote the annihilator of $f_{\phi,\chi}$. It is a prime ideal of $\mathcal{H}^{\chi}_{\mathcal{O}}$ and $\lambda_{f_{\phi,\chi}} : \mathcal{H}^{\chi}_{\mathcal{O}} \twoheadrightarrow \mathcal{O}$ induces an $\mathcal{O}$-algebra isomorphism $\mathcal{H}^{\chi}_{\mathcal{O}}/\mathrm{Ann}(f_{\phi,\chi}) \xrightarrow{\sim} \mathcal{O}$.

Definition 8.18. As Φ is surjective, $\Phi(\mathrm{Ann}(f_{\phi,\chi}))$ is an ideal of $\mathcal{H}^{\mathrm{NM},\chi}_{\mathcal{O}}$. We call it the *Maass ideal associated to* $f_{\phi,\chi}$.

There exists a non-negative integer r for which the diagram

$$\begin{array}{ccc}
\mathcal{H}^{\chi}_{\mathcal{O}} & \xrightarrow{\Phi} & \mathcal{H}^{\mathrm{NM},\chi}_{\mathcal{O}} \\
\downarrow & & \downarrow \\
\mathcal{H}^{\chi}_{\mathcal{O}}/\mathrm{Ann}(f_{\phi,\chi}) & \xrightarrow{\Phi} & \mathcal{H}^{\mathrm{NM},\chi}_{\mathcal{O}}/\Phi(\mathrm{Ann}(f_{\phi,\chi})) \\
\lambda_{f_{\phi,\chi}} \downarrow \wr & & \downarrow \wr \\
\mathcal{O} & \longrightarrow & \mathcal{O}/\varpi^{r}\mathcal{O}
\end{array} \tag{8.13}$$

all of whose arrows are $\mathcal{O}$-algebra epimorphisms, commutes.

Corollary 8.19. *If r is the integer from diagram (8.13), and n is as in Theorem 8.10, then $r \geq n$.*

Proof. Set $\mathcal{N}^{\mathrm{NM}} := \{f \in \mathcal{N}^{\mathrm{h}} \mid f \in \mathcal{S}^{\mathrm{NM},\chi}_{k,-k/2}\}$. Choose any $T \in \Phi^{-1}(\varpi^{r}) \subset \mathcal{H}^{\chi}_{\mathcal{O}}$. Suppose that $r < n$, and let f be as in Theorem 8.10. We have

$$f_{\phi,\chi} \equiv f \pmod{\varpi^{n}}. \tag{8.14}$$

and $Tf = \varpi^{r} f$ and $T f_{\phi,\chi} = 0$. Hence applying T to both sides of (8.14), we see that for some base $\mathcal{B}$ and every $b \in \mathcal{B}$ all the Fourier coefficients of the b-component of $\varpi^{r} f$ lie in $\varpi^{n}\mathcal{O}$. Since $r < n$ this along with (8.14) implies that all the Fourier coefficients of the b-component of $f_{\phi,\chi}$ lie $\varpi\mathcal{O}$, which is impossible as shown in the proof of Corollary 8.17. □

Remark 8.20. The Maass ideal plays a role similar to the classical Eisenstein ideal. Its index inside the Hecke algebra is a measure of the congruences between $f_{\phi,\chi}$ and eigenforms in $\mathcal{S}^{\chi}_{k,-k/2}$ which are orthogonal to the Maass space. While Corollary 8.17 only guarantees a Hecke eigenform f orthogonal to the Maass space congruent to $f_{\phi,\chi}$ modulo ϖ, the quotient $\mathcal{H}^{\mathrm{NM},\chi}_{\mathcal{O}}/\Phi(\mathrm{Ann}(f_{\phi,\chi}))$ takes into account all such eigenforms f at the same time and hence gives a better idea of how much congruence there is between $f_{\phi,\chi}$ and eigenforms orthogonal to the Maass space. Also, it is exactly the index of the Maass ideal inside $\mathcal{H}_{\mathcal{O}}^{\mathrm{NM},\chi}$ that bounds the order of the appropriate Selmer group from below, as we discuss in the next section.

8.4. **Unitary analogue of Harder's conjecture.** Let E be a sufficiently large finite extension of $\mathbf{Q}_\ell$, with ring of integers $\mathcal{O}$ and uniformizer ϖ. The original Harder's conjecture states that if ℓ is "large" and $\varpi \mid L^{\mathrm{alg}}(f, j+k)$ (the appropriately normalized algebraic part of the special value of the standard L-function of f) for a cuspidal elliptic eigenform $f = \sum_n a_n(f)e(z) \in S_r(\mathrm{SL}_2(\mathbf{Z}))$, then there exists a cuspidal (vector-valued) Siegel modular eigenform F of full level, whose eigenvalues for the Hecke operators $T(p)$ (for all primes p) are congruent to $p^{k-2}+p^{j+k-1}+a_p(f)$ modulo ϖ. Here $r = j+2k-2$, and $T(p)$ is the Hecke operator acting on the space of Siegel modular forms given by the double coset $\mathrm{Sp}_4(\mathbf{Z})\,\mathrm{diag}(1,1,p,p)\,\mathrm{Sp}_4(\mathbf{Z})$. For details see [19] or [45].

Recently Dummigan [13] formulated an analogue of this conjecture for the group $\mathrm{U}(2,2)$. (We are grateful to him for sending us his preprint). Let $\phi \in S_{k-1}(D_K, \chi_K)$ be as before. Let j be an integer such that $0 \le j \le (k-4)/2$ (note that our k differs from Dummigan's k by 1). Suppose

$$\mathrm{val}_\varpi(L^{\mathrm{alg}}(\mathrm{Symm}^2\,\phi, 2k-4-2j)) > 0.$$

Write π_ϕ for the automorphic representation of $\mathrm{GL}_2(\mathbf{A})$ associated with ϕ. Let $\Pi(\phi)$ denote the representation $\mathrm{Ind}_{P(\mathbf{A})}^{U(\mathbf{A})}(\mathrm{BC}_{K/\mathbf{Q}}(\pi_\phi)\cdot|\det|^{k-(3/2)-j})$ of $U(\mathbf{A})$. Then the unitary analogue of Harder's conjecture asserts that if $0 \le j < (k-4)/2$ then there exists a cuspidal automorphic representation Π of $U(\mathbf{A})$ (whose finite part contributes to the cuspidal cohomology of degree 4 - for details cf. [13]), unramified away from D_K, such that

$$\lambda_{\Pi(\phi)}(T) \equiv \lambda_\Pi(T) \pmod{\varpi} \tag{8.15}$$

for all Hecke operators T in the local Hecke algebras away from D_K. Here λ denotes the appropriate Hecke eigenvalue.

Let us now briefly explain the relation of Corollary 8.17 to this conjecture. First, assume that ϕ is not congruent to any other $\psi \in S_{k-1}(D_K, \chi_K) \bmod \varpi$. This implies that the Hida congruence module of ϕ is a ϖ-adic unit, so that $\mathrm{val}_\varpi(L^{\mathrm{int}}(\mathrm{Symm}^2\,\phi, k)) = \mathrm{val}_\varpi(L^{\mathrm{alg}}(\mathrm{Symm}^2\,\phi, k))$ (cf. Proposition 6.6). Secondly, when $j = (k-4)/2$, the automorphic representation Π_ϕ attached to the Maass lift $f_{\phi,\mathbf{1}}$ is associated (in the sense of Piatetski-Shapiro [32]) with $\Pi(\phi)$. In fact the local representations are isomorphic at all finite places, hence Π_ϕ shares the Hecke eigenvalues with $\Pi(\phi)$. Let $g \in \mathcal{S}^{\mathbf{1}}_{k,-k/2}$ be a Hecke eigenform congruent to $f_{\phi,\chi}$ as in Corollary 8.17. Then the automorphic representation Π of $U(\mathbf{A})$ associated with g is cuspidal and unramified everywhere and its eigenvalues satisfy (8.15).

However, note that Dummigan's conjecture specifically excludes the case $j = (k-4)/2$ hence our result should be viewed as complementary to that conjecture rather than as a case of it. Indeed, the case $j = (k-4)/2$ is special because of the existence of the CAP representation Π_ϕ of $U(\mathbf{A})$ associated with $\Pi(\phi)$ which has a holomorphic vector $f_{\phi,\chi}$ in it. The holomorphicity of $f_{\phi,\chi}$ in particular allows us to use a holomorphic Eisenstein series to define the form Ξ and for such Eisenstein series we know the integrality of their Fourier coefficients thanks to results of Shimura (cf. section 7.5). The main point of Dummigan's conjecture is the prediction of the congruence (8.15) in the absence of a CAP representation.

9. The Bloch-Kato conjecture

In section 9.1 we will discuss how the results of the previous sections can be applied to give evidence for the Bloch-Kato conjecture for a twist of the adjoint motive of an elliptic modular form ϕ. Since these results (and proofs) are completely analogous to the case considered in [27], we will just give the relevant statements and refer the reader to [loc. cit.] for details.

9.1. Selmer groups.

We begin by defining the Selmer group. For a profinite group $\mathcal{G}$ and a $\mathcal{G}$-module M (where we assume the action of $\mathcal{G}$ on M to be continuous) we will consider the group $H^1_{\text{cont}}(\mathcal{G}, M)$ of cohomology classes of continuous cocycles $\mathcal{G} \to M$. To shorten notation we will suppress the subscript 'cont' and simply write $H^1(\mathcal{G}, M)$. For a field L, and a $\text{Gal}(\overline{L}/L)$-module M (with a continuous action of $\text{Gal}(\overline{L}/L)$) we sometimes write $H^1(L, M)$ instead of $H^1_{\text{cont}}(\text{Gal}(\overline{L}/L), M)$.

Let L be a number field. For a rational prime p denote by Σ_p the set of primes of L lying over p. Let $\Sigma \supset \Sigma_\ell$ be a finite set of primes of L and denote by G_Σ the Galois group of the maximal Galois extension L_Σ of L unramified outside of Σ. Let E be a (sufficiently large) finite extension of $\mathbf{Q}_\ell$ with ring of integer $\mathcal{O}$ and a fixed uniformizer ϖ. Let V be a finite dimensional E-vector space with a continuous G_Σ-action. Let $T \subset V$ be a G_Σ-stable $\mathcal{O}$-lattice. Set $W := V/T$.

We begin by defining local Selmer groups. For every $\mathfrak{p} \in \Sigma$ set

$$H^1_{\text{un}}(L_\mathfrak{p}, M) := \ker\{H^1(L_\mathfrak{p}, M) \xrightarrow{\text{res}} H^1(I_\mathfrak{p}, M)\}.$$

Define the local $\mathfrak{p}$-Selmer group (for V) by

$$H^1_f(L_\mathfrak{p}, V) := \begin{cases} H^1_{\text{un}}(L_\mathfrak{p}, V) & \mathfrak{p} \in \Sigma \setminus \Sigma_\ell \\ \ker\{H^1(L_\mathfrak{p}, V) \to H^1(L_\mathfrak{p}, V \otimes B_{\text{crys}})\} & \mathfrak{p} \in \Sigma_\ell. \end{cases}$$

Here B_{crys} denotes Fontaine's ring of ℓ-adic periods (cf. [14]).

For $\mathfrak{p} \in \Sigma_\ell$, we call the $D_\mathfrak{p}$-module V *crystalline* (or the G_L-module V *crystalline at* $\mathfrak{p}$) if $\dim_{\mathbf{Q}_\ell} V = \dim_{\mathbf{Q}_\ell} H^0(L_\mathfrak{p}, V \otimes B_{\text{crys}})$. When we refer to a Galois representation $\rho : G_L \to GL(V)$ as being crystalline at $\mathfrak{p}$, we mean that V with the G_L-module structure defined by ρ is crystalline at $\mathfrak{p}$.

For every $\mathfrak{p}$, define $H^1_f(L_\mathfrak{p}, W)$ to be the image of $H^1_f(L_\mathfrak{p}, V)$ under the natural map $H^1(L_\mathfrak{p}, V) \to H^1(L_\mathfrak{p}, W)$. Using the fact that $\text{Gal}(\overline{\kappa}_\mathfrak{p} : \kappa_\mathfrak{p}) = \hat{\mathbf{Z}}$ has cohomological dimension 1, one easily sees that if W is unramified at $\mathfrak{p}$ and $\mathfrak{p} \notin \Sigma_\ell$, then $H^1_f(L_\mathfrak{p}, W) = H^1_{\text{un}}(L_\mathfrak{p}, W)$. Here $\kappa_\mathfrak{p}$ denotes the residue field of $L_\mathfrak{p}$.

For a $\mathbf{Z}_\ell$-module M, we write $M^\vee$ for its Pontryagin dual defined as

$$M^\vee = \text{Hom}_{\text{cont}}(M, \mathbf{Q}_\ell/\mathbf{Z}_\ell).$$

Moreover, if M is a Galois module, we denote by $M(n) := M \otimes \epsilon^n$ its n-th Tate twist.

Definition 9.1. The group

$$H^1_f(L, W) := \ker \left\{ H^1(G_\Sigma, W) \xrightarrow{\text{res}} \bigoplus_{\mathfrak{p} \in \Sigma} \frac{H^1(L_\mathfrak{p}, W)}{H^1_f(L_\mathfrak{p}, W)} \right\}$$

is called the (global) *Selmer group of* W.

For $L = \mathbf{Q}$, the group $H^1_f(\mathbf{Q}, W)$ is the Selmer group defined by Bloch and Kato [6], section 5. Let $\rho : G_\Sigma \to \mathrm{GL}_E(V)$ denote the representation giving the action of G_Σ on V. The following two lemmas are easy (cf. [34], Lemma 1.5.7 and [40]).

Lemma 9.2. *$H^1_f(L, W)^\vee$ is a finitely generated $\mathcal{O}$-module.*

Lemma 9.3. *If the mod ϖ reduction $\overline{\rho}$ of ρ is absolutely irreducible, then the length of $H^1_f(L, W)^\vee$ as an $\mathcal{O}$-module is independent of the choice of the lattice T.*

Remark 9.4. For an $\mathcal{O}$-module M, $\mathrm{val}_\ell(\#M) = [\mathcal{O}/\varpi : \mathbf{F}_\ell]\,\mathrm{length}_\mathcal{O}(M)$.

Let K be an imaginary quadratic field of prime discriminant $-D_K$. Let $\phi = \sum_{n=1}^\infty a(n)q^n \in \mathcal{N}$ be such that $\overline{\rho}_\phi|_{G_K}$ is absolutely irreducible. Then by Proposition 6.3, $f_{\phi,\chi} \neq 0$. From now on we also assume that $\mathrm{ad}^0\,\overline{\rho}_\phi|_{G_K}$, the trace-0-endomorphisms of the representation space of $\overline{\rho}_\phi|_{G_K}$ with the usual G_K-action, is absolutely irreducible. Finally, to be able to show that the cohomology classes we produce are unramified at the prime D_K we assume that (under the chosen embedding $\overline{\mathbf{Q}}_\ell \hookrightarrow \mathbf{C}$) the Fourier coefficient $a(D_K)$ is neither congruent to D_K^k nor to D_K^{k-4} modulo ϖ (see [27], Lemma 9.23 for how this assumption is used). By (6.9) we have

$$\rho_{f_{\phi,\chi}} \cong (\rho_\phi|_{G_K} \oplus (\rho_\phi \otimes \epsilon)|_{G_K}) \otimes \chi\epsilon^{2-k/2}.$$

Let V denote the representation space of

$$\mathrm{ad}^0\,\rho_\phi|_{G_K}(-1) = \mathrm{ad}^0\,\rho_\phi|_{G_K} \otimes \epsilon^{-1} \subset \mathrm{Hom}_E((\rho_\phi \otimes \epsilon)|_{G_K}, \rho_\phi|_{G_K})$$

of G_K. Let $T \subset V$ be some choice of a G_K-stable lattice. Set $W = V/T$. Note that the action of G_K on V factors through G_{Σ_ℓ}. Since the mod ϖ reduction of $\mathrm{ad}^0\,\rho_\phi|_{G_K} \otimes \epsilon^{-1}$ is absolutely irreducible by assumption, $\mathrm{val}_\ell(H^1_f(K, W)^\vee)$ is independent of the choice of T.

Let $\mathcal{N}^{\mathrm{NM}}$, $\mathcal{H}^{\mathrm{NM},\chi}_\mathcal{O}$ and Φ be as in section 8.3. Let $\mathfrak{m}_\phi$ be the maximal ideal of $\mathcal{H}^{\mathrm{NM},\chi}_\mathcal{O}$ corresponding to $f_{\phi,\chi}$ and write $\mathcal{H}^{\mathrm{NM},\chi}_{\mathfrak{m}_\phi}$ for the localization of $\mathcal{H}^{\mathrm{NM},\chi}_\mathcal{O}$ at $\mathfrak{m}_\phi$ and $\Phi_{\mathfrak{m}_\phi}$ for the corresponding "local" component of Φ. Write $\mathcal{N}^{\mathrm{NM}}_{f_{\phi,\chi}}$ for the subset of $\mathcal{N}^{\mathrm{NM}}$ consisting of eigenforms whose corresponding maximal ideal of $\mathcal{H}^{\mathrm{NM},\chi}_\mathcal{O}$ is $\mathfrak{m}_\phi$.

The main result of this section is the following theorem.

Theorem 9.5. *Let W be as above. Suppose that for each $f \in \mathcal{N}^{\mathrm{NM}}_{f_{\phi,\chi}}$, the representation $\rho_f : G_K \to \mathrm{GL}_4(E)$ is absolutely irreducible. Then*

$$\mathrm{val}_\ell(\#H^1_f(K, W)^\vee) \geq \mathrm{val}_\ell(\#\mathcal{H}^{\mathrm{NM},\chi}_{\mathfrak{m}_\phi}/\Phi_{\mathfrak{m}_\phi}(\mathrm{Ann}(f_{\phi,\chi}))).$$

Proof. This is proved in the same way as Theorem 9.10 in [27] and we will not reproduce the proof here. The key point is that eigenforms $f \in \mathcal{S}^{\mathrm{NM},\chi}_{k,-k/2}$ congruent to $f_{\phi,\chi}$ modulo powers of ϖ give rise to non-split extensions of $\rho_\phi(1)|_{G_K}$ by $\rho_\phi|_{G_K}$. These extensions are checked to satisfy the local conditions defining the Selmer group and can be put together to generate a submodule of $H^1_f(K, W)$ of order no smaller than the index of the Maass ideal inside the local Hecke algebra. In this one mostly follows Urban [44]. □

Remark 9.6. The irreducibility assumption in Theorem 9.5 is presumably unnecessary. If one assumes multiplicity one for the Maass forms in the sense that the only eigenforms in $\mathcal{M}^\chi_{k,-k/2}$ (i.e., in particular holomorphic) sharing all Hecke eigenvalues with $f_{\phi,\chi}$ are multiples of $f_{\phi,\chi}$ then one needs to show that all non-CAP cuspidal automorphic representations have irreducible Galois representations. This is expected to be the case, but we know of no proof of this fact.

Corollary 9.7. *With the same assumptions and notation as in Theorem 8.10 and Theorem 9.5 we have*

$$\mathrm{val}_\ell(\# H^1_f(K,W)) \geq n[\mathcal{O}/\varpi : \mathbf{F}].$$

If in addition the characters ξ, β, χ in Theorem 8.10 can be taken as in Corollary 8.14, then

$$\mathrm{val}_\ell(\# H^1_f(K,W)) \geq \mathrm{val}_\ell(\# \mathcal{O}/L^{\mathrm{int}}(\mathrm{Symm}^2\,\phi, k)).$$

Proof. The corollary follows immediately from Theorem 9.5 and Corollary 8.19. □

With the assumptions as in Corollary 9.7 we have thus the following inclusion of the fractional ideals of $\mathcal{O}$:

$$\# H^1_f(K,W)\cdot\mathcal{O} \subset L^{\mathrm{int}}(\mathrm{Symm}^2\,\phi, k)\cdot\mathcal{O}. \tag{9.1}$$

Note that since χ_K is the nebentypus of ϕ one has

$$\mathrm{ad}^0\,\rho_\phi(-1)\chi_K = \mathrm{Symm}^2\,\rho_\phi(k-3).$$

Here we treat χ_K as a Galois character via class field theory. Hence assuming that a certain technicality concerning the Tamagawa factors at the prime D_K can be proved (cf. section 9.3 in [27]), the Bloch-Kato conjecture can be formulated as follows:

Conjecture 9.8 (Bloch-Kato)**.** *One has the following equality of fractional ideals of $\mathcal{O}$:*

(9.2)

$$\# H^1_f(\mathbf{Q}, \mathrm{Symm}^2\,\rho_\phi(k-3))\cdot\mathcal{O} = \# H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\rho_\phi(-1)\chi_K)\cdot\mathcal{O} = L^{\mathrm{int}}(\mathrm{Symm}^2\,\phi, k)\cdot\mathcal{O}.$$

Thus Corollary 9.7 provides evidence for Conjecture 9.8, but falls short of proving that the left-hand side of (9.2) is contained in the right-hand side, because the module $H^1_f(K,W) = H^1_f(K, \mathrm{ad}^0\,\rho_\phi|_{G_K}(-1))$ can potentially be larger than $H^1_f(\mathbf{Q}, \mathrm{ad}^0\,\rho_\phi(-1)\chi_K)$. For a more detailed discussion see [27], section 9.3.

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