ON IHARA'S LEMMA FOR DEGREE ONE AND TWO COHOMOLOGY OVER IMAGINARY QUADRATIC FIELDS

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Abstract. We prove a version of Ihara's Lemma for degree $q = 1, 2$ cuspidal cohomology of the symmetric space attached to automorphic forms of arbitrary weight ($k \geq 2$) over an imaginary quadratic field with torsion (p-power) coefficients. This extends an earlier result of the author [9] which concerned the case $k = 2$, $q = 1$. Our method is different from [9] and uses results of Diamond [4] and Blasius-Franke-Grunewald [2]. We discuss the relationship of our main theorem to the problem of the existence of level-raising congruences.

1. Introduction

The classical Ihara's lemma states that the kernel of the map $\alpha : J_0(N)^2 \to J_0(Np)$ is Eisenstein if $(N, p) = 1$, where $J_0(N')$ denotes the Jacobian of the compatification of the modular curve $\Gamma_0(N') \setminus \mathbf{H}$ and α is the sum of the two standard p-degeneracy maps. In [9] the author proved an analogue of this result for degree one parabolic cohomology arising from weight 2 automorphic forms over an imaginary quadratic field F. More precisely, for $n = 0, 1$ let Y_n be the analogue over F of the modular curve $Y_0(Np^n)$ (for precise definitions cf. section 2). Write $H^q_!(Y_n, \tilde{M}_n)$ for the degree q parabolic cohomology group, where $\tilde{M}_n$ are sheaves of sections of the topological covering $\mathrm{GL}_2(F) \setminus (V \times M) \to Y_n$ with M a torsion $\mathbf{Z}[\mathrm{GL}_2(F)]$-module and $V = \mathrm{GL}_2(\mathbf{A}_F)/K_n \cdot U_2(\mathbf{C}) \cdot \mathbf{C}^\times$ for K_n a compact subgroup, which is an analogue of $\Gamma_0(Np^n)$. For a prime ideal $\mathfrak{p}$ of the ring of integers $\mathcal{O}_F$ of F we have two standard $\mathfrak{p}$-degeneracy maps whose sum $H^q_!(Y_0, \tilde{M}_0)^2 \to H^q_!(Y_1, \tilde{M}_1)$ we will call α_q. The main result of [9] (Theorem 2) then asserts that the kernel of α_1 is Eisenstein when M is a trivial GL_2-module (weight 2 case) and that in this case α_q is injective when M has exponent a power of p (cf. [loc.cit.] Remark 12).

In the current paper we prove the following the following two results:

(1) We show that α_1 is injective for all weights $k \geq 2$ if M has exponent a power of p. The method applied in [9] used modular symbols and relied on the assumption that the group action on M was trivial. To treat the case of a general weight we use an approach developed by Diamond [4] for the $\mathbf{Q}$-case and we extend it here to the case of F using some input from [12]. This is carried out in section 4.

(2) We show that α_2 is injective for all weights $k \geq 2$ if M has exponent a power of p using results of Blasius, Franke and Grunewald [2] on the vanishing

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of the restriction map from group cohomology of S-arithmetic groups to group cohomology of arithmetic groups. This is carried out in section 3.

In [10] and [4] Ihara's Lemma was used to show the existence of level-raising congruences, i.e., congruences between modular forms of level N and those of level Np. Our results however cannot be used to prove such a result. The problem is the occurrence of torsion in the degree two cohomology which prevents a 'lifting' of our result for cohomology with torsion coefficients to a statement about lattices generated by eigenforms in the spaces of automorphic forms. It is also not possible to conclude a level-raising result on the level of cohomology itself because the cup product pairing is only perfect modulo torsion hence preventing us from using the standard technique of composing α_q with its adjoint α_q^+ and relating the order of $\ker(\alpha_q^+\alpha_q)$ to the order of the congruence module. The relationship of our results to the problem of level-raising is discussed in detail in section 5.

On the other hand a level-raising result for torsion cohomology classes with trivial coefficients has recently been obtained by Calegari and Venkatesh [3]. As noted in [loc.cit.] the cohomology classes of 'raised' level constructed in [loc.cit.] do not always lift to characteristic zero hence in fact a level-raising result of the type proved in [10] and [4] is not to be expected in the context of automorphic forms over imaginary quadratic fields. We would like thank Frank Calegari and Akshay Venkatesh for sending us an early version of their book.

2. Preliminaries

2.1. **The congruence subgroups of $\mathrm{GL}_2(F)$ and symmetric spaces.** Let F be an imaginary quadratic extension of $\mathbf{Q}$ and denote by $\mathcal{O}_F$ its ring of integers. Fix once and for all an embedding $\overline{F} \hookrightarrow \mathbf{C}$. For any ideal $\mathfrak{M} \subset \mathcal{O}_F$ we will write $\Phi(\mathfrak{M})$ for the integer $N(\mathfrak{M}) \cdot \#(\mathcal{O}_F/\mathfrak{M})^\times$, where N denotes the absolute norm. Let $\mathfrak{N}$ be an ideal of $\mathcal{O}_F$ such that the $\mathbf{Z}$-ideal $\mathfrak{N} \cap \mathbf{Z}$ has a generator greater than 3. Let $\mathfrak{p}$ be a prime ideal such that $\mathfrak{p} \nmid \mathfrak{N}$. Write p for its residue characteristic. Denote by Cl_F the class group of F and choose representatives of distinct ideal classes to be prime ideals $\mathfrak{p}_j$, $j = 1, \dots, \# \mathrm{Cl}_F$, relatively prime to both $\mathfrak{N}$ and $\mathfrak{p}$. Let $\tilde{\pi}$, (resp. $\tilde{\pi}_j$) be a uniformizer of the completion $F_\mathfrak{p}$ (resp. $F_{\mathfrak{p}_j}$) of F at the prime $\mathfrak{p}$ (resp. $\mathfrak{p}_j$), and put π (resp. π_j) to be the idele $(\dots, 1, \tilde{\pi}, 1, \dots) \in \mathbf{A}_F^\times$ (resp. $(\dots, 1, \tilde{\pi}_j, 1, \dots) \in \mathbf{A}_F^\times$), where $\tilde{\pi}$ (resp. $\tilde{\pi}_j$) occurs at the $\mathfrak{p}$-th place (resp. $\mathfrak{p}_j$-th place). We will write η for $\begin{bmatrix} \pi & \\ & 1 \end{bmatrix} \in \mathrm{GL}_2(\mathbf{A}_{F,\mathrm{f}})$.

For each $n \in \mathbf{Z}_{\geq 0}$, we define compact open subgroups of $\mathrm{GL}_2(\mathbf{A}_{F,\mathrm{f}})$

$$K_n := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \prod_{\mathfrak{q} \nmid \infty} \mathrm{GL}_2(\mathcal{O}_{F,\mathfrak{q}}) \mid c \in \mathfrak{N}\mathfrak{p}^n \right\}.$$

Here $\mathbf{A}_{F,\mathrm{f}}$ denotes the finite adeles of F and $\mathcal{O}_{F,\mathfrak{q}}$ the ring of integers of $F_\mathfrak{q}$. For $n \geq 0$ we also set $K_n^\mathfrak{p} = \eta^{-1} K_n \eta$.

For any compact open subgroup K of $\mathrm{GL}_2(\mathbf{A}_{F,\mathrm{f}})$ we put $Y_K = \mathrm{GL}_2(F) \setminus \mathrm{GL}_2(\mathbf{A}_F)/K \cdot U_2(\mathbf{C}) \cdot Z_\infty$, where $Z_\infty = \mathbf{C}^\times$ is the center of $\mathrm{GL}_2(\mathbf{C})$ and $U(2) := \{M \in \mathrm{GL}_2(\mathbf{C}) \mid M\overline{M}^t = I_2\}$ (here 'bar' denotes complex conjugation and I_2 stands for the 2×2-identity matrix). If K is sufficiently large (which will be the case for all compact open subgroups we will consider) this space is a disjoint union of $\# \mathrm{Cl}_F$ connected components $Y_K = \coprod_{j=1}^{\# \mathrm{Cl}_F} (\Gamma_K)_j \setminus \mathcal{Z}$, where $\mathcal{Z} = \mathrm{GL}_2(\mathbf{C})/U_2(\mathbf{C})\mathbf{C}^\times$ and

$(\Gamma_K)_j = \mathrm{GL}_2(F) \cap \begin{bmatrix} \pi_j & \\ & 1 \end{bmatrix} K \begin{bmatrix} \pi_j & \\ & 1 \end{bmatrix}^{-1}$. To ease notation we put $Y_n := Y_{K_n}$, $Y_n^{\mathfrak{p}} := Y_{K_n^{\mathfrak{p}}}$, $\Gamma_{n,j} := (\Gamma_{K_n})_j$ and $\Gamma_{n,j}^{\mathfrak{p}} := (\Gamma_{K_n^{\mathfrak{p}}})_j$.

We have the following diagram:

$$\begin{tikzcd}
\dots \arrow[r, "\subset"] & K_{n+1} \arrow[r, "\subset"] \arrow[dr, "\subset"] & K_n \arrow[r, "\subset"] \arrow[dr, "\subset"] & K_{n-1} \arrow[r, "\subset"] & \dots \\
\dots \arrow[r, "\subset"] & K_{n+1}^{\mathfrak{p}} \arrow[r, "\subset"] \arrow[u, "\wr"'] & K_n^{\mathfrak{p}} \arrow[r, "\subset"] \arrow[u, "\wr"'] & K_{n-1}^{\mathfrak{p}} \arrow[r, "\subset"] \arrow[u, "\wr"'] & \dots
\end{tikzcd} \tag{2.1}$$

where the horizontal and diagonal arrows are inclusions and the vertical arrows are conjugation by η. Diagram (2.1) is not commutative, but it is "vertically commutative", by which we mean that given two objects in the diagram, two directed paths between those two objects define the same map if and only if the two paths contain the same number of vertical arrows.

Diagram (2.1) induces the following vertically commutative diagram of the corresponding symmetric spaces:

$$\begin{tikzcd}
\dots \arrow[r] & Y_{n+1} \arrow[r] \arrow[dr] & Y_n \arrow[r] \arrow[dr] & Y_{n-1} \arrow[r] & \dots \\
\dots \arrow[r] & Y_{n+1}^{\mathfrak{p}} \arrow[r] \arrow[u, "\wr"'] & Y_n^{\mathfrak{p}} \arrow[r] \arrow[u, "\wr"'] & Y_{n-1}^{\mathfrak{p}} \arrow[r] \arrow[u, "\wr"'] & \dots
\end{tikzcd} \tag{2.2}$$

The horizontal and diagonal arrows in diagram (2.2) are the natural projections and the vertical arrows are maps given by $(g_\infty, g_f) \mapsto (g_\infty, g_f \eta)$.

2.2. **Cohomology.** Let M be a finitely generated $\mathbf{Z}$-module with M^{tor} of exponent relatively prime to $\#\mathcal{O}_F^\times$ endowed with a $\mathrm{GL}_2(F)$-action. Denote by $\tilde{M}_K$ the sheaf of continuous sections of the topological covering $\mathrm{GL}_2(F) \setminus [(\mathrm{GL}_2(\mathbf{A}_F)/K \cdot U_2(\mathbf{C}) \cdot Z_\infty) \times M] \to Y_K$, where $\mathrm{GL}_2(F)$ acts diagonally on $(\mathrm{GL}_2(\mathbf{A}_F)/KU_2(\mathbf{C})Z_\infty) \times M$. Here M is equipped with the discrete topology. As above, we put $\tilde{M}_n := \tilde{M}_{K_n}$ and $\tilde{M}_n^{\mathfrak{p}} := \tilde{M}_{K_n^{\mathfrak{p}}}$.

Given a surjective map $\phi : Y_K \to Y_{K'}$, we get an isomorphism of sheaves $\phi^{-1}\tilde{M}_{K'} \xrightarrow{\sim} \tilde{M}_K$, which yields a map on cohomology

$$H^q(Y_{K'}, \tilde{M}_{K'}) \to H^q(Y_K, \phi^{-1}\tilde{M}_{K'}) \cong H^q(Y_K, \tilde{M}_K).$$

Hence diagram (2.2) gives rise to a vertically commutative diagram of cohomology groups:

(2.3)

$$\begin{tikzcd}
\dots & H^q(Y_{n+1}, \tilde{M}_{n+1}) \arrow[l] \arrow[d, "\wr"', "\alpha_{\mathfrak{p}}^{n+1}"] & H^q(Y_n, \tilde{M}_n) \arrow[l, "\alpha_1^{n,n+1}"'] \arrow[d, "\wr"', "\alpha_{\mathfrak{p}}^{n}"] & H^q(Y_{n-1}, \tilde{M}_{n-1}) \arrow[l, "\alpha_1^{n-1,n}"'] \arrow[d, "\wr"', "\alpha_{\mathfrak{p}}^{n-1}"] & \dots \arrow[l] \\
\dots & H^q(Y_{n+1}^{\mathfrak{p}}, \tilde{M}_{n+1}^{\mathfrak{p}}) \arrow[l] & H^q(Y_n^{\mathfrak{p}}, \tilde{M}_n^{\mathfrak{p}}) \arrow[l, "\alpha_1^{n\mathfrak{p},(n+1)\mathfrak{p}}"'] \arrow[ul, "\alpha_1^{n,(n+1)\mathfrak{p}}"'] & H^q(Y_{n-1}^{\mathfrak{p}}, \tilde{M}_{n-1}^{\mathfrak{p}}) \arrow[l, "\alpha_1^{(n-1)\mathfrak{p},n\mathfrak{p}}"'] \arrow[ul, "\alpha_1^{n-1,n\mathfrak{p}}"'] & \dots \arrow[l]
\end{tikzcd}$$

These sheaf cohomology groups can be related to the group cohomology of $\Gamma_{n,j}$ and $\Gamma_{n,j}^{\mathfrak{p}}$ with coefficients in M. In fact, for each compact open subgroup K with corresponding decomposition $Y_K = \coprod_{j=1}^{\#\mathrm{Cl}_F} (\Gamma_K)_j \setminus \mathcal{Z}$, we have the following commutative diagram in which the horizontal maps are inclusions:

$$\begin{CD} H^q_!(Y_K, \tilde{M}_K) @>>> H^q(Y_K, \tilde{M}_K) \\ @AAA @AAA \\ \bigoplus_{j=1}^{\#\mathrm{Cl}_F} H^q_P((\Gamma_K)_j, M) @>>> \bigoplus_{j=1}^{\#\mathrm{Cl}_F} H^q((\Gamma_K)_j, M) \end{CD} \tag{2.4}$$

Here $H^q_!(Y_K, \tilde{M}_K)$ denotes the image of the cohomology with compact support $H^q_c(Y_K, \tilde{M}_K)$ inside $H^q(Y_K, \tilde{M}_K)$ and H^q_P denotes the parabolic cohomology, i.e., $H^q_P((\Gamma_K)_j, M) := \ker(H^q((\Gamma_K)_j, M) \to \bigoplus_{B\in\mathcal{B}_j} H^q((\Gamma_K)_{j,B}, M))$, where $\mathcal{B}_j$ is a fixed set of representatives of $(\Gamma_K)_j$-conjugacy classes of Borel subgroups of $GL_2(F)$ and $(\Gamma_K)_{j,B} := (\Gamma_K)_j \cap B$. The vertical arrows in diagram (2.4) are isomorphisms provided that there exists a torsion-free normal subgroup of $(\Gamma_K)_j$ of finite index relatively prime to the exponent of M^{tor}. If $K = K_n$ or $K = K_n^{\mathfrak{p}}$, $n \geq 0$, this condition is satisfied because of our assumption that $\mathfrak{N} \cap \mathbf{Z}$ has a generator greater than 3 and the exponent of M^{tor} is relatively prime to $\#\mathcal{O}_F^\times$ (cf. [18], section 2.3). In what follows we may therefore identify the sheaf cohomology with the group cohomology. Note that all maps in diagram (2.3) preserve parabolic cohomology. The maps $\alpha_1^{*,*}$ are the natural restriction maps on group cohomology, so in particular they preserve the decomposition $\bigoplus_{i=1}^{\#\mathrm{Cl}_F} H^q((\Gamma_K)_j, M)$.

Let us introduce one more group:

$$K_{-1} := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(F_{\mathfrak{p}}) \times \prod_{\mathfrak{q}\nmid\mathfrak{p}\infty} \mathrm{GL}_2(\mathcal{O}_{F,\mathfrak{q}}) \mid c \in \mathfrak{N},\ ad - bc \in \prod_{\mathfrak{q}\nmid\infty} \mathcal{O}_{F,\mathfrak{q}}^\times \right\}.$$

The group K_{-1} is not compact, but we can still define

$$\Gamma_{-1,j} := \mathrm{GL}_2(F) \cap \begin{bmatrix} \tilde{\pi}_j & \\ & 1 \end{bmatrix} K_{-1} \begin{bmatrix} \tilde{\pi}_j & \\ & 1 \end{bmatrix}^{-1}$$

for $j = 1, \dots \#\mathrm{Cl}_F$. Note that $\Gamma_{-1,j}$ are not discrete subgroups of $\mathrm{SL}_2(\mathbf{C})$. They are commensurable with an S-arithmetic subgroup (in the sense of [13]) of $\mathrm{SL}_2(F)$ where $S = \{\mathfrak{p}\}$. However it still makes sense to define the group cohomology groups $H^q(\Gamma_{-1,j}, M)$ as well as the subgroups of parabolic cohomology $H^q_P(\Gamma_{-1,j}, M)$.

The sheaf and group cohomologies are in a natural way modules over the corresponding Hecke algebras. (For the definition of the Hecke action on cohomology, see [18] or [8]). Here we will only consider the subalgebra $\mathbf{T}_{n,\mathbf{Z}}$ of the full Hecke algebra which is generated over $\mathbf{Z}$ by the double cosets $T_{\mathfrak{p}'} := K \begin{bmatrix} \pi' & \\ & 1 \end{bmatrix} K$ for π' a uniformizer of $F_{\mathfrak{p}'}$ with $\mathfrak{p}'$ running over prime ideals of $\mathcal{O}_F$ such that $\mathfrak{p}' \nmid \mathfrak{N}\mathfrak{p}$. For a $\mathbf{Z}$-algebra A we set $\mathbf{T}_{n,A} := \mathbf{T}_{n,\mathbf{Z}} \otimes_{\mathbf{Z}} A$.

2.3. **The sheaf and cup product pairing.** Let $k \geq 2$. Let $\ell > k - 1$ be a prime not dividing $N(\mathfrak{N}\mathfrak{p})D_F\#\mathcal{O}_F^\times$, where D_F is the discriminant of F. Fix an isomorphism $\overline{\mathbf{Q}}_\ell \cong \mathbf{C}$. Let $E \subset \overline{\mathbf{Q}}_\ell$ be an (always sufficiently large) finite extension of $\mathbf{Q}_\ell$. In particular we assume that E contains F and all the eigenvalues of all Hecke eigenforms for a fixed k and level (this is possible since the extension of $\mathbf{Q}$ generated by these eigenvalues is a number field - cf. e.g., Theorem A in [17]).

Write $\mathcal{O}$ for the valuation ring of E, ϖ for a choice of a uniformizer and $\mathbf{F}$ for the residue field.

Let A be an $\mathcal{O}_F$-algebra. For an integer $m \geq 0$, write $\mathrm{Sym}^m(A)$ for the ring of homogeneous polynomials in two variables of degree m with coefficients in A. For a subgroup $\Gamma \subset \mathrm{GL}_2(F)$, $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$ and $P(X,Y) \in \mathrm{Sym}^m(A)$ we set $(\gamma P)(X,Y) := P(aX+cY, bX+dY)$. If $P(aX+cY, bX+dY) \in A[X,Y]$ for every $\gamma \in \Gamma$, this defines an action of Γ on $\mathrm{Sym}^m(A)$. Set

$$L(m,A) = \mathrm{Sym}^m(A) \otimes_{\mathcal{O}_F} \mathrm{Sym}^m(A),$$

where the second factor is an $\mathcal{O}_F$-module via the non-trivial automorphism $a \mapsto \overline{a}$ of F. Set $\gamma(P \otimes Q) := \gamma P \otimes \overline{\gamma} Q$. Put

$$L(m,M) := L(m,A) \otimes_A M$$

for any A-module M. If M is an A-algebra, then $L(m,M)$ is a ring.

Now assume that A is an $\mathcal{O}_F$-algebra in which all rational primes $q \leq m$ are invertible (cf. [5], section 3.3). For a sheaf $\tilde{M}$ of A-modules we set $\tilde{M}^0 = \mathrm{Hom}(\tilde{M}, A)$ (cf. [19], p.288). Ubran ([loc.cit], p.299) shows that it is possible to view $\tilde{L}(m,A)^0$ as a subsheaf of $\tilde{L}(m,A)$. Indeed, if $\mathrm{Sym}^m(A)$ is endowed with an action of a congruence subgroup $\Gamma \subset \mathrm{GL}_2(F)$, one defines (cf. e.g. [5], section 3.3) a natural pairing $[\cdot,\cdot]_m$ on $\mathrm{Sym}^m(A) \otimes \mathrm{Sym}^m(A) \to A$ by

$$\left[\sum_{i=0}^m a_i X^i Y^{m-i}, \sum_{i=0}^m b_i X^i Y^{m-i}\right]_m := \sum_{i=0}^m (-1)^i \frac{a_i b_{m-i}}{\binom{m}{i}}.$$

Then we can define a Γ-equivariant pairing on $L(m,A) \otimes L(m,A) \to A$ by $[P \otimes P', Q \otimes Q']_m := [P,Q]_m [P',Q']_m$. Moreover, under our assumptions the pairing $[\cdot,\cdot]_m$ is perfect and we get an isomorphism $L(k-2,A)^0 \cong L(k-2,A)$ of Γ-modules. In particular in all the statements below one can replace $\tilde{L}(k-2,A)^0$ with $\tilde{L}(k-2,A)$.

Theorem 2.1 (Urban, [19], Théorème 2.5.1)**.** *There exists a pairing (induced by the cup product):*

$$J_n : H^1_!(Y_n, \tilde{L}(k-2,\mathcal{O})) \otimes H^2_!(Y_n, \tilde{L}(k-2,\mathcal{O})^0) \to H^3_c(Y_n, \mathcal{O}) \cong \mathcal{O},$$

which is perfect modulo torsion.

Lemma 2.2. *If $A \subset \mathbf{C}$, then $H^1_!(Y_n, \tilde{L}(k-2,A))$ is torsion-free.*

Proof. The exact sequence $0 \to L(k-2,A) \to L(k-2,\mathbf{C}) \to L(k-2,\mathbf{C}/A) \to 0$ induces a long exact sequence in (group) cohomology

$$\bigoplus_j H^0(\Gamma_{n,j}, L(k-2,\mathbf{C}/A)) \to \bigoplus_j H^1(\Gamma_{n,j}, L(k-2,A)) \to \bigoplus_j H^1(\Gamma_{n,j}, L(k-2,\mathbf{C})).$$

To prove the claim it is enough to show that that the sequence of H^0-groups is short exact. This follows from the assumption $\ell > k-2$ (cf. also [12], Lemma 6.2). $\square$

2.4. **Relation to automorphic forms.** For an abelian group M we set $M^{\mathrm{tf}} = M/\mathrm{torsion}$.

Let $K \subset \mathrm{GL}_2(\mathbf{A}_{F,\mathrm{f}})$ be an open compact subgroup. We will write $\mathcal{S}_k(K)$ for the $\mathbf{C}$-space of cuspidal automorphic forms on $\mathrm{GL}_2(\mathbf{A}_F)$ of (parallel) weight k and level K_n. We will not need the precise definition in what follows, but we refer the reader to [19], section 3 for details.

Theorem 2.3 (Eichler-Shimura-Harder isomorphism)**.** *There exist Hecke-equivariant isomorphisms (cf. e.g.*[19]*, section 3):*

$$\delta_q : \mathcal{S}_k(K) \xrightarrow{\sim} H^q_!(Y_K, \tilde{L}(k-2, \mathbf{C})), \quad q \in \{1, 2\}.$$

Following Urban ([19], section 4) for a fixed k and a subring $A \subset \mathbf{C}$ we define the following two sets of lattices:

$$L^1_A(K_n) = \delta_1^{-1}(\iota_1(H^1_!(Y_n, \tilde{L}(k-2, A)))^{\mathrm{tf}});$$
$$L^2_A(K_n) = \delta_2^{-1}(\iota_2(H^2_!(Y_n, \tilde{L}(k-2, A)^0)^{\mathrm{tf}})),$$

where ι_q are the canonical maps on cohomology induced by the embedding $A \hookrightarrow \mathbf{C}$.

Lemma 2.4. *Let A be a subring of $\mathbf{C}$. The lattices $L^q_A(K_n)$ satisfy the following properties:*

(i) *one has $L^q_{\mathbf{Z}}(K_n) \otimes_{\mathbf{Z}} A \cong L^q_A(K_n)$ for $q = 1, 2$;*
(ii) *the lattices $L^1_A(K_n)$ and $L^2_A(K_n)$ are free A-modules of the same rank;*
(iii) *the lattices $L^1_A(K_n)$ and $L^2_A(K_n)$ are both stable under the action of the Hecke algebra $\mathbf{T}_{n,\mathbf{Z}}$.*

Proof. The first assertion is Proposition 4.1.1 in [19]. Setting $A = \mathbf{C}$ in the (i) and using the fact that $L^q_{\mathbf{C}}(K_n) \cong H^q_!(X_n, \tilde{L}(k-2, \mathbf{C})) \cong \mathcal{S}_k(K_n)$ gives (ii). Part (iii) follows from the fact that $H^q_!(X_n, \tilde{L}(k-2, A))$ is Hecke-stable and the isomorphisms δ_q are Hecke-equivariant. □

Corollary 2.5. *The maps ι_q are injective for $q = 1, 2$.*

Proof. This follows immediately from Theorem 2.3 and Lemma 2.4(i). □

Corollary 2.6. *The pairing J_n induces a perfect pairing*

$$\langle \cdot, \cdot \rangle_n : L^1_{\mathcal{O}}(K_n) \otimes L^2_{\mathcal{O}}(K_n) \to \mathcal{O}, \quad \langle f, h \rangle_n = J_n(\delta_1(f), \delta_2(h))$$

Proof. This follows from perfectness (mod torsion) of J_n and the fact that by Corollary 2.5, the maps δ_q induce isomorphisms between $H^q_!(X_n, L(k-2, \mathcal{O}))^{\mathrm{tf}}$ and the lattices $L^q_{\mathcal{O}}(K_n)$. □

3. Ihara's lemma for H^2

In this section we will prove an analogue of Ihara's lemma for the groups $H^2_!(Y_K, \tilde{L}(k-2, E/\mathcal{O}))$. More precisely, the goal of this section is to prove the following theorem.

Theorem 3.1. *Assume $\ell \nmid \#\mathcal{O}_F^\times \Phi(\mathfrak{N}\mathfrak{p})$. The map*

$$\alpha_2 : \bigoplus_{j=1}^{\#\,\mathrm{Cl}_F} H^2_P(\Gamma_{0,j}, L(k-2, E/\mathcal{O}))^2 \to \bigoplus_{j=1}^{\#\,\mathrm{Cl}_F} H^2_P(\Gamma_{1,j}, L(k-2, E/\mathcal{O}))$$

defined as $(f, g) \mapsto \alpha_1^{0,1} f + \alpha_1^{0,1\mathfrak{p}} \alpha_{\mathfrak{p}}^0 g$ is injective.

Proof. Recall that $\alpha_1^{*,*}$ are restriction maps. Set $\Gamma'_{i,j} := \Gamma_{i,j} \cap \mathrm{SL}_2(F)$ for $i = -1, 0, 1$. We have a commutative diagram

$$\begin{tikzcd}
\bigoplus_j H^2_P(\Gamma_{0,j}, L(k-2, E/\mathcal{O}))^2 \arrow[r, "\alpha_2"] \arrow[d, "\mathrm{res}"] & \bigoplus_j H^2_P(\Gamma_{1,j}, L(k-2, E/\mathcal{O})) \arrow[d] \\
\bigoplus_j H^2_P(\Gamma'_{0,j}, L(k-2, E/\mathcal{O}))^2 \arrow[r, "\alpha_2"] & \bigoplus_j H^2_P(\Gamma'_{1,j}, L(k-2, E/\mathcal{O})),
\end{tikzcd}$$

The injectivity of the left vertical arrow follows from the fact that $\ell \nmid \#\mathcal{O}_F^\times$. So, it is enough to prove that 'bottom' α_2 is injective. The main ingredient in this proof is a result of Blasius, Franke and Grunewald which we now state in a special form pertaining to our situation.

Theorem 3.2 (Blasius, Franke, Grunewald)**.** *Let $\Gamma \subset \mathrm{SL}_2(F)$ be a congruence subgroup and let $\Gamma_S \supset \Gamma$ be an S-arithmetic subgroup of $\mathrm{SL}_2(F)$ with S a finite set of finite primes of F. Then the image of the restriction map*

$$H^*(\Gamma_S, E) \to H^*(\Gamma, E) \tag{3.1}$$

coincides with the image of the space of $\mathrm{SL}_2(\mathbf{C})$-invariant forms on the symmetric space $\mathrm{SL}_2(\mathbf{C})/K_\infty$ in the de Rham cohomology of the locally symmetric space $\Gamma \backslash \mathrm{SL}_2(\mathbf{C})/K_\infty$ (which is identified with the group cohomology of Γ). If E is replaced by a non-constant irreducible E-representation, then the map (3.1) is the zero map.

Proof. This is Theorem 4 in [2]. □

It follows from a theorem of Serre (cf. [9], Theorem 8) that $\Gamma'_{-1,j}$ is the amalgamated product of $\Gamma'_{0,j}$ and $(\Gamma'_{0,j})^\mathfrak{p}$ along $\Gamma'_{1,j} := \Gamma'_{0,j} \cap (\Gamma'_{0,j})^\mathfrak{p}$. Hence using the exact cohomology sequence of Lyndon (cf. [15], p. 127) one gets that the top row in the following diagram is exact (for any coefficients M which we suppress):

(3.2)

$$\begin{tikzcd}[column sep=large]
\bigoplus_j H^2(\Gamma'_{-1,j}) \arrow[r, "f \mapsto (f|_{\Gamma'_{0,j}}, f|_{(\Gamma'_{0,j})^\mathfrak{p}})"] \arrow[d, equal] & \bigoplus_j (H^2(\Gamma'_{0,j}) \oplus H^2((\Gamma'_{0,j})^\mathfrak{p})) \arrow[r, "(f,g) \mapsto f|_{\Gamma'_{1,j}} - g|_{\Gamma'_{1,j}}"] \arrow[d, "\wr"', "(f,g) \mapsto (f, -\left[\begin{smallmatrix}\pi & \\ & 1\end{smallmatrix}\right]^* g)"] & \bigoplus_j H^2(\Gamma'_{1,j}) \arrow[d, equal] \\
\bigoplus_j H^2(\Gamma'_{-1,j}) \arrow[r, "\beta_M", "f \mapsto (f|_{\Gamma'_{0,j}}, -\left[\begin{smallmatrix}\pi & \\ & 1\end{smallmatrix}\right]^* f|_{\Gamma'_{0,j}})"'] & \bigoplus_j H^2(\Gamma'_{0,j})^2 \arrow[r, "\alpha_M", "(f,g) \mapsto (f|_{\Gamma'_{1,j}} + \left[\begin{smallmatrix}\pi^{-1} & \\ & 1\end{smallmatrix}\right]^* g|_{\Gamma'_{1,j}})"'] & \bigoplus_j H^2(\Gamma'_{1,j}),
\end{tikzcd}$$

where the map $\left[\begin{smallmatrix}\pi & \\ & 1\end{smallmatrix}\right]^*$ is induced from the map $H^2(Y_0^\mathfrak{p}, \tilde{M}_0^\mathfrak{p}) \to H^2(Y_0, \tilde{M}_0)$ arising from the isomorphism $K_0^\mathfrak{p} \xrightarrow{\sim} K_0$ given by conjugation by $\left[\begin{smallmatrix}\pi & \\ & 1\end{smallmatrix}\right]$ (i.e., equals the inverse of $\alpha_\mathfrak{p}^0$). Note that the maps in the bottom row and the map represented by the middle vertical arrow do not necessarily preserve the direct summands (they do if $\mathfrak{p}$ is principal), but the maps in the top row do.

It is clear that this diagram commutes (essentially by the definitions of the maps involved). Let $(f, g) \in \ker \alpha_M$. Then the corresponding element in $\bigoplus_j (H^2(\Gamma'_{0,j}) \oplus H^2((\Gamma'_{0,j})^\mathfrak{p}))$ is in the image of the top left arrow. Hence by commutativity (f, g) is in the image of β_M, so, to prove the theorem it is enough to show that $\beta_{L(k-2,E/\mathcal{O})} = 0$. For this it suffices to prove that both of the restriction maps $f \mapsto f|_{\Gamma'_{0,j}}$ and $f \mapsto f|_{(\Gamma'_{0,j})^\mathfrak{p}}$ are the zero maps (when $M = L(k-2, E/\mathcal{O})$). We will show that the first restriction map is zero, the proof of the vanishing of the second map being essentially identical.

Consider the following commutative diagram with exact columns and horizontal arrows being restriction maps:

$$\begin{CD} H^2(\Gamma'_{-1,j}, L(k-2,E)) @>>> H^2(\Gamma'_{0,j}, L(k-2,E)) \\ @VVV @VVV \\ H^2(\Gamma'_{-1,j}, L(k-2,E/\mathcal{O})) @>>> H^2(\Gamma'_{0,j}, L(k-2,E/\mathcal{O})) \\ @VVV @VVV \\ H^3(\Gamma'_{-1,j}, L(k-2,\mathcal{O})) @>>> H^3(\Gamma'_{0,j}, L(k-2,\mathcal{O})) \end{CD} \tag{3.3}$$

By Theorem 3.2 (with $S = \{\mathfrak{p}\}$) the top horizontal arrow in (3.3) is the zero map (for $k > 2$ the representation $L(k-2, E)$ is irreducible (cf. e.g. [7], p.45) and for $k = 2$ one checks that the space of $\mathrm{SL}_2(\mathbf{C})$-invariant forms is zero). Since $\Gamma_{0,j}$ is a torsion-free discrete subgroup of $\mathrm{GL}_2(F)$, it follows from Proposition 18(b) of [14], that the cohomological dimension of $\Gamma_{0,j}$ is no greater than two (because the real dimension of the symmetric space for $\mathrm{GL}_2(F)$ is 3). As the index of $\Gamma'_{0,j}$ in $\Gamma_{0,j}$ is finite, Proposition 5(b) in [loc.cit.] implies that the cohomological dimension of $\Gamma'_{0,j}$ is also no greater than two. Hence we must have $H^3(\Gamma'_{0,j}, L(k-2,\mathcal{O})) = 0$. Thus the bottom horizontal arrow in diagram (3.3) is the zero map. We will now show that the middle horizontal arrow is also zero.

Write A for the image of the top-left vertical arrow, B for $H^2(\Gamma'_{-1,j}, L(k-2, E/\mathcal{O}))$ and C for the image of the bottom-left vertical arrow. We have a short exact sequence

$$0 \to A \to B \to C \to 0. \tag{3.4}$$

It is enough to show that (3.4) splits as a sequence of $\mathcal{O}$-modules. By taking the Pontryagin duals we obtain an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}}(C, E/\mathcal{O}) \to \mathrm{Hom}_{\mathcal{O}}(B, E/\mathcal{O}) \to \mathrm{Hom}_{\mathcal{O}}(A, E/\mathcal{O}). \tag{3.5}$$

Let us first show that all of the Hom-groups in (3.5) are finitely generated $\mathcal{O}$-modules.

As noted before, $\Gamma'_{-1,j}$ is the amalgamated product of $\Gamma'_{0,j}$ and $(\Gamma'_{0,j})^{\mathfrak{p}}$ along $\Gamma'_{1,j}$. By the same argument as in the case of $\Gamma'_{0,j}$ one sees that $(\Gamma'_{0,j})^{\mathfrak{p}}$ and $\Gamma'_{1,j}$ all have cohomological dimension no greater than 2. Hence, by Proposition 7 of [14], the group $\Gamma'_{-1,j}$ has cohomological dimension no greater than 3. Thus, one can apply Proposition 4 of [loc.cit.] to conclude that $H^3(\Gamma'_{-1,j}, L(k-2,\mathcal{O}))$ is a finitely generated $\mathcal{O}$-module. Thus C and $\mathrm{Hom}(C, E/\mathcal{O})$ are finite groups. Moreover, by Remarque 1 of [loc.cit.] we also get that $H^2(\Gamma'_{-1,j}, L(k-2,E))$ is a finite dimensional vector space over E. It follows that $A \cong (E/\mathcal{O})^m$ for some m. Thus, $\mathrm{Hom}_{\mathcal{O}}(A, E/\mathcal{O}) \cong \mathcal{O}^m$. Hence all the Hom-groups in (3.5) are finitely generated $\mathcal{O}$-modules. So, in particular $\mathrm{Hom}_{\mathcal{O}}(B, E/\mathcal{O}) \cong \mathcal{O}^m \oplus G$, where G is a finite group. We conclude that $B \cong (E/\mathcal{O})^m \oplus G$. Then it is clear that (3.4) splits. □

4. Ihara's lemma for H^1

In this section we will prove an analogue of Ihara's lemma for the groups $H^1_!(Y_K, \tilde{L}(k-2, E/\mathcal{O}))$. More precisely, the goal of this section is to prove the following theorem.

Theorem 4.1. *Suppose $\ell > k-2$ and $\ell \nmid \#\mathcal{O}_F^\times N(\mathfrak{N})$. The map*

$$\alpha_1 : \bigoplus_{j=1}^{\#\,\mathrm{Cl}_F} H^1_P(\Gamma_{0,j}, L(k-2, E/\mathcal{O}))^2 \to \bigoplus_{j=1}^{\#\,\mathrm{Cl}_F} H^1_P(\Gamma_{1,j}, L(k-2, E/\mathcal{O}))$$

defined as $(f,g) \mapsto \alpha_1^{0,1} f + \alpha_1^{0,1\mathfrak{p}} \alpha_{\mathfrak{p}}^0 g$ is injective.

Remark 4.2. In [9] the author proved Theorem 4.1 for the case $k=2$, so we will assume below that $k>2$.

Remark 4.3. If one knew that $H^2_P(\Gamma_{0,j}, L(k-2, \mathcal{O}))$ was torsion-free then the proof of Theorem 3.1 would carry over verbatim to this case (with n-degree cohomology replaced by $(n-1)$-degree cohomology) as then the bottom horizontal arrow in diagram (3.3) would have to be zero on the maximal torsion subgroup which in turn contains the image of the bottom-left vertical arrow. In this case the assumption that $\ell > k-2$ is unnecessary.

Proof of Theorem 4.1. In this proof we mostly follow Diamond [4], proof of Lemma 3.2, but indicate where the arguments of [loc.cit.] need to be modified. Let $\mathcal{O}_{(\mathfrak{p})}$ denote the ring of $\mathfrak{p}$-integers in F, i.e., the elements of F, whose $\mathfrak{q}$-adic valuation is non-negative for every prime $\mathfrak{q} \neq \mathfrak{p}$. For $j = 1, 2, \ldots, \#\,\mathrm{Cl}_F$ put

$$\mathrm{SL}_2(\mathcal{O}_{(\mathfrak{p})})_j := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(F) \mid a, d \in \mathcal{O}_{(\mathfrak{p})}, b \in \mathfrak{p}_j \mathcal{O}_{(\mathfrak{p})}, c \in \mathfrak{p}_j^{-1} \mathcal{O}_{(\mathfrak{p})} \right\}.$$

Define the j-th principal congruence subgroup of level $\mathfrak{N}$ by

$$\Gamma_{\mathfrak{N},j} := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathcal{O}_{(\mathfrak{p})})_j \mid b, c \in \mathfrak{N}\mathcal{O}_{(\mathfrak{p})}, a \equiv d \equiv 1 \mod \mathfrak{N}\mathcal{O}_{(\mathfrak{p})} \right\}.$$

By Lemma 10 in [9], we may assume that the ideals $\mathfrak{p}_j$ satisfy $(N\mathfrak{p}_j - 1, \ell) = 1$. We have a commutative diagram

$$\begin{array}{ccc} \bigoplus_j H^1_P(\Gamma_{0,j}, L(k-2, E/\mathcal{O}))^2 & \xrightarrow{\alpha} & \bigoplus_j H^1_P(\Gamma_{1,j}, L(k-2, E/\mathcal{O})) \\ \downarrow{\scriptstyle \mathrm{res}} & & \downarrow \\ \bigoplus_j H^1_P(\Gamma_j(\mathfrak{N}), L(k-2, E/\mathcal{O}))^2 & \xrightarrow{\alpha} & \bigoplus_j H^1_P(\Gamma_\cap, L(k-2, E/\mathcal{O})), \end{array}$$

where the group $\Gamma_j(\mathfrak{N}) \subset \Gamma_{0,j} \cap \mathrm{SL}_2(\mathcal{O}_F)_j$ is defined as the principal congruence subgroup of level $\mathfrak{N}$, $\Gamma_\cap := \Gamma_j(\mathfrak{N}) \cap \Gamma_j(\mathfrak{N})^{\mathfrak{p}}$ and $\mathrm{SL}_2(\mathcal{O}_F)_j$ is defined in the same way as $\mathrm{SL}_2(\mathcal{O}_{(\mathfrak{p})})_j$ but with $\mathcal{O}_{(\mathfrak{p})}$ replaced by $\mathcal{O}_F$. The injectivity of the left vertical arrow follows from the fact that $\ell \nmid \#\mathcal{O}_F^\times N(\mathfrak{N})$.

As in [4] we will reduce the problem to showing that $H^1_{S_j}(\Gamma_{\mathfrak{N},j}, L(k-2, E/\mathcal{O})) = 0$ for all j, where $S_j \subset \Gamma_{\mathfrak{N},j}$ is the subset of elements conjugate in $\mathrm{SL}_2(\mathcal{O}_{(\mathfrak{p})})_j$ to $\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix}$. Here for a group G, a subset $Q \subset G$ and a G-module A we write $H^1_Q(G, A)$ for the subgroup of $H^1(G, A)$ whose classes are represented by the cocycles u satisfying $u(\gamma) \in (\gamma - 1)A$ for all $\gamma \in Q$ (by which we mean that for every $\gamma \in G$ there exists $a \in A$ such that $u(\gamma) = \gamma a - a$). Note that in the case when $G = \Gamma_j(\mathfrak{N})$ and Q is the subset of parabolic elements one clearly has $H^1_Q(G, A) \supset H^1_P(G, A)$.

By arguments analogous to the ones used in the proof of Theorem 3.1 one obtains a commutative diagram

(4.1)

$$\begin{tikzcd}[column sep=large]
\bigoplus_j H^1(\Gamma_{\mathfrak{N},j}) \arrow[r, "f\mapsto(f|_{\Gamma_j(\mathfrak{N})},f|_{\Gamma_j(\mathfrak{N})^{\mathfrak{p}}})"] \arrow[d, equal] & \bigoplus_j (H^1(\Gamma_j(\mathfrak{N})) \oplus H^1(\Gamma_j(\mathfrak{N})^{\mathfrak{p}})) \arrow[r, "(f,g)\mapsto f|_{\Gamma_\cap}-g|_{\Gamma_\cap}"] \arrow[d, "\wr"', "(f,g)\mapsto(f,-\left[\begin{smallmatrix}\pi & \\ & 1\end{smallmatrix}\right]^* g)"] & \bigoplus_j H^1(\Gamma_\cap) \arrow[d, equal] \\
\bigoplus_j H^1(\Gamma_{\mathfrak{N},j}) \arrow[r, "\beta", "f\mapsto(f|_{\Gamma_j(\mathfrak{N})},-\left[\begin{smallmatrix}\pi & \\ & 1\end{smallmatrix}\right]^* g|_{\Gamma_j(\mathfrak{N})})"'] & \bigoplus_j H^1(\Gamma_j(\mathfrak{N}))^2 \arrow[r, "\alpha", "(f,g)\mapsto(f|_{\Gamma_\cap}+\left[\begin{smallmatrix}\pi^{-1} & \\ & 1\end{smallmatrix}\right]^* g|_{\Gamma_\cap})"'] & \bigoplus_j H^1(\Gamma_\cap)
\end{tikzcd}$$

with exact rows.

Lemma 4.4. *Every cocycle u in $Z^1(\Gamma_{\mathfrak{N},j}, L(k-2, E/\mathcal{O}))$, which is mapped by β into $\bigoplus_j H^1_P(\Gamma_j(\mathfrak{N}))^2$ represents a class in $H^1_{S_j}(\Gamma_{\mathfrak{N},j}, L(k-2, E/\mathcal{O}))$.*

Proof. This is proved as in [4], p.211 using the fact that $H^1_P \subset H^1_Q$. $\square$

So, it is enough to prove that $H^1_{S_j}(\Gamma_{\mathfrak{N},j}, L(k-2, E/\mathcal{O})) = 0$. In fact one can easily see (cf. [4], p. 211) that it is enough to prove that $H^1_{S_j}(\Gamma_{\mathfrak{N},j}, L(k-2, \mathbf{F})) = 0$, where $\mathbf{F} = \mathcal{O}/\varpi\mathcal{O}$. The inflation-restriction exact sequence

$$0 \to H^1(\mathrm{SL}_2(\mathcal{O}_F/\ell), L(k-2, \mathbf{F})) \to H^1(\Gamma_{\mathfrak{N},j}, L(k-2, \mathbf{F})) \to H^1(\Gamma_{\mathfrak{N}\ell,j}, L(k-2, \mathbf{F}))$$

gives rise to an exact sequence

(4.2)

$$0 \to H^1_Q(\mathrm{SL}_2(\mathcal{O}_F/\ell), L(k-2, \mathbf{F})) \to H^1_{S_j}(\Gamma_{\mathfrak{N},j}, L(k-2, \mathbf{F})) \to H^1_{S_j \cap \Gamma_{\mathfrak{N}\ell,j}}(\Gamma_{\mathfrak{N}\ell,j}, L(k-2, \mathbf{F})),$$

where Q is an ℓ-Sylow subgroup of $\mathrm{SL}_2(\mathcal{O}_F/\ell)$.

Lemma 4.5. *One has $H^1_Q(\mathrm{SL}_2(\mathcal{O}_F/\ell), L(k-2, \mathbf{F})) = 0$.*

Proof. This is just an adaptation of the proof of Lemma 3.1 in [4]. Let us outline the arguments for the case of an inert ℓ - the split case is proved similarly. Set $G = \mathrm{SL}_2(\mathcal{O}_F/\ell)$ and write I for the representation $\mathrm{Ind}^G_Q(\mathbf{F})$, where we treat $\mathbf{F}$ as a trivial Q-module. We can regard elements of I as $\mathbf{F}$-valued functions on $\mathbf{F}_{\ell^2} \times \mathbf{F}_{\ell^2} \setminus \{(0,0)\}$. Set $\Sigma := \{\sigma : \mathbf{F}_{\ell^2} \hookrightarrow \mathbf{F}\}$ and write $\mathrm{Sym}^{k-2}_\sigma(\mathbf{F})$ for the $\mathbf{F}_{\ell^2}[G]$-module of homogeneous polynomials in two variables with coefficients in $\mathbf{F}$ where the actions of both $\mathbf{F}_{\ell^2}$ and G are via the embedding σ. While the action of G on I is canonical we can also add an $\mathbf{F}_{\ell^2}$-module structure via σ and denote the resulting $\mathbf{F}_{\ell^2}[G]$-module by I_σ. For every $\sigma \in \Sigma$ we then have a natural $\mathbf{F}_{\ell^2}[G]$-equivariant injection $\phi_\sigma : \mathrm{Sym}^{k-2}_\sigma(\mathbf{F}) \to I_\sigma$ given by $\phi_\sigma(P)(a,b) = P(\sigma(a), \sigma(b))$. We then define an $\mathbf{F}_{\ell^2}[G]$-equivariant injection $\phi : L(k-2, \mathbf{F}) \to \bigotimes_{\sigma\in\Sigma} I_\sigma$ by $\phi := \bigotimes_{\sigma\in\Sigma} \phi_\sigma$. Let us fix an ordering of the elements of Σ and denote them by σ_1 and σ_2. We will introduce a convention that for $\mathbf{F}$-modules M, N in tensor products $M \otimes_{\mathbf{F}_{\ell^2}} N$ the $\mathbf{F}^2_\ell$-action on the module on the left of $\otimes_{\mathbf{F}^2_\ell}$ is via σ_1 and on the module on the right of $\otimes_{\mathbf{F}^2_\ell}$ the action is via σ_2. In particular we will just write $I \otimes I$, instead of $\bigotimes_{\sigma\in\Sigma} I_\sigma$. Note that $I \otimes I \cong \mathrm{Ind}^G_Q(\mathbf{F} \otimes_{\mathbf{F}_{\ell^2}} \mathrm{Res}^G_Q I)$, where

Res^G_Q denotes the restriction to Q functor. It is then a consequence of Shapiro's lemma that

$$H^1_Q(G, I\otimes I) = H^1_Q(Q, \mathbf{F}\otimes_{\mathbf{F}_{\ell^2}} \mathrm{Res}^G_Q I).$$

Since Q is solvable, an easy calculation shows that the latter cohomology group is zero. Thus we have reduced the problem to showing that the sequence:

$$0 \to L(k-2,\mathbf{F})^G \to (I\otimes I)^G \to \left(\frac{I\otimes I}{L(k-2,\mathbf{F})}\right)^G \to 0 \tag{4.3}$$

is exact. The rest of the proof consists of decomposing the module $I\otimes I$ and analyzing its G-fixed points. Here we follow closely the strategy of the proof of Lemma 3.1 in [4] with modifications needed to account for dealing with $I\otimes I$ instead of just I. In this we use some results of [12] which dealt with a very similar situation.

We have $I_\sigma = \bigoplus_{n=0}^{\ell^2-2} I_{n,\sigma}$, where

$$I_{n,\sigma} = \{f : \mathbf{F}_{\ell^2}\times\mathbf{F}_{\ell^2}\setminus\{(0,0)\} \to \mathbf{F} \mid f((xa,xb)) = \sigma(x)^n f((a,b))\}.$$

Observe that every $I_{n,\sigma}$ is an $\mathbf{F}$-vector space of dimension ℓ^2+1 and that $L(k-2,\mathbf{F}) \subset I_{k-2}\otimes I_{k-2} := I_{\kappa-2,\sigma_1}\otimes_{\mathbf{F}_{\ell^2}} I_{k-2,\sigma_2}$. Moreover, note that the action of G on $I\otimes I$ preserves every summand $I_m\otimes I_n$, so all we need to prove is that the map

$$(I_{k-2}\otimes I_{k-2})^G \to \left(\frac{I_{k-2}\otimes I_{k-2}}{L(k-2,\mathbf{F})}\right)^G$$

is surjective. In fact we will prove that the module on the right is zero. To do this we decompose $I_{k-2}\otimes I_{k-2}$ as in Lemma 5.5 of [12] (with obvious modifications), and thus it suffices to show that

$$(\mathrm{Sym}^{\ell^2-1-(k-2)}_{\sigma_1}(\mathbf{F}))\otimes I_{k-2})^G \oplus (I_{k-2}\otimes \mathrm{Sym}^{\ell^2-1-(k-2)}_{\sigma_2}(\mathbf{F}))^G = 0. \tag{4.4}$$

Write $\mathbf{F}^{r,\sigma}$ for the one-dimensional $\mathbf{F}$-vector space on which $\begin{bmatrix} a & b\\ c & d\end{bmatrix} \in G$ acts via $\sigma(d)^r$. Note that we have an $\mathbf{F}_{\ell^2}[G]$-module isomorphism:

$$\begin{aligned} &I_{r,\sigma} \cong \mathrm{Ind}^G_P(\mathbf{F}^{r,\sigma}) \\ &= \left\{ f : G\to\mathbf{F} \mid f\left(\begin{bmatrix} d^{-1} & b\\ 0 & d\end{bmatrix} g\right) = \sigma(d)^r f(g) \text{ for every } g\in G, \begin{bmatrix} d^{-1} & b\\ 0 & d\end{bmatrix}\in P\right\}, \end{aligned} \tag{4.5}$$

where $P\subset G$ is the upper-triangular Borel subgroup of G. One then has

$$I_{k-2,\sigma_1}\otimes \mathrm{Sym}^{\ell^2-1-(k-2)}_{\sigma_2}(\mathbf{F}) \cong \mathrm{Ind}^G_P(\mathbf{F}^{k-2,\sigma_1}\otimes \mathrm{Res}^G_P \mathrm{Sym}^{\ell^2-1-(k-2)}_{\sigma_2}(\mathbf{F})). \tag{4.6}$$

It is easy to check that $(\mathbf{F}^{k-2,\sigma_1}\otimes \mathrm{Sym}^{\ell^2-1-(k-2)}_{\sigma_2}(\mathbf{F}))^P = 0$ or our range of k. Then by Shapiro's Lemma and (4.6) we obtain that the second direct summand in (4.4) is zero. We prove that the first one is zero in an analogous way. $\square$

It now suffices to prove the vanishing of $H^1_{S_j\cap\Gamma_{\mathfrak{N}\ell,j}}(\Gamma_{\mathfrak{N}\ell,j}, L(k-2,\mathbf{F}))$. For this first note that $L(k-2,\mathbf{F})$ is a trivial $\Gamma_{\mathfrak{N}\ell,j}$-module, so one has

$$H^1_{S_j\cap\Gamma_{\mathfrak{N}\ell,j}}(\Gamma_{\mathfrak{N}\ell,j}, L(k-2,\mathbf{F})) = \ker(\mathrm{Hom}(\Gamma_{\mathfrak{N}\ell,j}, L(k-2,\mathbf{F})) \to \mathrm{Hom}(U_{\mathfrak{N}\ell,j}, L(k-2,\mathbf{F}))),$$

where $U_{\mathfrak{N}\ell,j}$ is the smallest normal subgroup of $\mathrm{SL}_2(\mathcal{O}_{(\mathfrak{p})})_j$ containing the matrices of the form $\begin{bmatrix} 1 & x\\ & 1\end{bmatrix} \in \Gamma_{\mathfrak{N}\ell,j}$. This means that $x\in \mathfrak{N}\ell\mathfrak{p}_j\mathcal{O}_{(\mathfrak{p})}$. Note that since $\Gamma_{\mathfrak{N}\ell,j}$ is normal in $\mathrm{SL}_2(\mathcal{O}_{(\mathfrak{p})})$, the group $U_{\mathfrak{N}\ell}$ is indeed contained in $\Gamma_{\mathfrak{N}\ell,j}$.

By a result of Serre (cf. [13], Theoreme 2(b), p.498 or [9], Theorem 11) we have that $U_{\mathfrak{N}\ell,j} = \Gamma_{\mathfrak{N}\ell\mathfrak{p}_j\mathcal{O}_{(\mathfrak{p})}}$. We now argue as in the proof of Proposition 4 of [9]. Put

$$\Gamma'_{\mathfrak{N}\ell\mathfrak{p}_j} := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathcal{O}_{(\mathfrak{p})})_j \mid b, c \in \mathfrak{N}\ell\mathfrak{p}_j\mathcal{O}_{(\mathfrak{p})} \right\}.$$

Let $f \in \mathrm{Hom}(\Gamma_{\mathfrak{N}\ell\mathfrak{p}_j\mathcal{O}_{(\mathfrak{p})}}, L(k-2, \mathbf{F}))$. Since $\Gamma_{\mathfrak{N}\ell\mathfrak{p}_j\mathcal{O}_{(\mathfrak{p})}}$ is a normal subgroup of $\Gamma'_{\mathfrak{N}\ell\mathfrak{p}_j}$ of index $N\mathfrak{p}_j - 1$ we have $f\left(\Gamma'_{\mathfrak{N}\ell\mathfrak{p}_i}\right) = 0$ by our choice of the ideals $\mathfrak{p}_j$. On one hand f is zero on the elements of the form $\begin{bmatrix} 1 & \\ c & 1 \end{bmatrix}$, $c \in \mathfrak{N}\ell\mathfrak{p}_i^{-1}\mathcal{O}_{(\mathfrak{p})}$, and on the other hand elements of this form together with $\Gamma'_{\mathfrak{N}\ell\mathfrak{p}_j}$ generate $\Gamma_{\mathfrak{N}\ell,j}$, so $f(\Gamma_{\mathfrak{N}\ell,j}) = 0$, as asserted. This implies that $H^1_{S_j \cap \Gamma_{\mathfrak{N}\ell,j}}(\Gamma_{\mathfrak{N}\ell,j}, L(k-2, \mathbf{F}))$ vanishes. The claim now follows from exactness of (4.2). $\square$

5. The module of congruences

In this section we will explain the relationship of Theorems 3.1 and 4.1 to the problem of the existence of level-raising congruences for the modules of automorphic forms over F and the cohomology.

In this section we keep the assumptions from section 2.3. Let M be a finitely generated free $\mathcal{O}$-module. For a submodule N of M we define

$$N^{\mathrm{sat}} = (N \otimes_{\mathcal{O}} E) \cap M.$$

Lemma 5.1 (Bellaïche - Graftieaux, Lemme 4.1.1)**.** *Let $u : N \to M$ be an injective homomorphism. For a sufficiently large positive integer s one has an isomorphism of $\mathcal{O}$-modules:*

$$u(N)^{\mathrm{sat}}/u(N) \cong \ker u_s, \quad \text{where } u_s := u \otimes 1 : N \otimes \mathcal{O}/\varpi^s \to M \otimes \mathcal{O}/\varpi^s.$$

Let A and B be two submodules of M such that $A \cap B = 0$, $A^{\mathrm{sat}} = A$ and $B^{\mathrm{sat}} = B$. Following Bellaïche and Graftieaux we define the module of congruences between A and B by $(A \oplus B)^{\mathrm{sat}}/(A \oplus B)$.

The maps α_q $(q = 1, 2)$ induce corresponding (injective) maps

$$\alpha_q : L^q_{\mathcal{O}}(K_0)^2 \to L^q_{\mathcal{O}}(K_1) \quad (f, g) \mapsto [K_0 1 K_1]f + [K_0 \eta K_1]g,$$

where the double cosets act on f and g in the usual way. Define $\alpha_1^+ : L^2_{\mathcal{O}}(K_1) \to L^2_{\mathcal{O}}(K_0)^2$ to be the adjoint of α_1 and $\alpha_2^+ : L^1_{\mathcal{O}}(K_1) \to L^1_{\mathcal{O}}(K_0)^2$ to be the adjoint of α_2 with respect to the pairings $\langle\langle \cdot, \cdot \rangle\rangle$ and $\langle \cdot, \cdot \rangle_1$, where $\langle\langle \cdot, \cdot \rangle\rangle$ is the pairing $L^1_{\mathcal{O}}(K_0)^2 \otimes L^2_{\mathcal{O}}(K_0)^2 \to \mathcal{O}$ defined by $\langle\langle (f, g), (f', g') \rangle\rangle = \langle f, f' \rangle_0 + \langle g, g' \rangle_0$.

Set $M_q = L^q_{\mathcal{O}}(K_1)$, $A_q = \mathrm{Im}(\alpha_q)$, $B_q = A_q^{\perp} \subset M_q$. For brevity write $\mathbf{T}_{\mathcal{O}}$ for $\mathbf{T}_{1,\mathcal{O}}$. One clearly has $\mathcal{S}_k(K_1) = M_q \otimes_{\mathcal{O}} \mathbf{C} = X \oplus Y$, where $X := (A_1 \otimes_{\mathcal{O}} \mathbf{C}) = (A_2 \otimes_{\mathcal{O}} \mathbf{C})$ and $Y := (B_1 \otimes_{\mathbf{Z}} \mathbf{C}) = (B_2 \otimes_{\mathbf{Z}} \mathbf{C})$ with X and Y stable under the action of $\mathbf{T}_{\mathcal{O}}$. Define $\mathbf{T}_X$ (resp. $\mathbf{T}_Y$) to be the image of $\mathbf{T}_{\mathcal{O}}$ inside $\mathrm{End}_{\mathbf{C}}(X)$ (resp. $\mathrm{End}_{\mathbf{C}}(Y)$). Then one has the following canonical inclusion (given by restrictions): $\mathbf{T}_{\mathcal{O}} \hookrightarrow \mathbf{T}_X \oplus \mathbf{T}_Y$.

Let the (finite) quotient $C(\mathbf{T}_{\mathcal{O}}) := (\mathbf{T}_X \oplus \mathbf{T}_Y)/\mathbf{T}_{\mathcal{O}}$ be the Hecke congruence module between X and Y. Note that if $C(\mathbf{T}_{\mathcal{O}}) \neq 0$ there exists Hecke eigenforms $f \in A_q$ and $g \in B_q$ whose eigenvalues (for all Hecke operators in $\mathbf{T}_{\mathcal{O}}$) are congruent modulo ϖ.

Lemma 5.2. *Let $q \in \{1, 2\}$. Suppose that $C(\mathbf{T}_{\mathcal{O}}) = 0$ and that $A_q = A_q^{\mathrm{sat}}$. Then $M_q^{\mathrm{sat}} = M_q = A_q \oplus B_q$, i.e., the module of congruences between A_q and B_q is zero.*

Proof. This follows from [6], Lemma 4 combined with Lemma 1 together with the easy facts that $B_q = B_q^{\rm sat}$ and $M_q = M_q^{\rm sat}$. $\square$

Assume for the moment that $A_q = A_q^{\rm sat}$ for $q = 1, 2$ (this equality is sometimes referred to as 'Ihara's lemma' - cf. [1]). It implies that there exists an $\mathcal{O}$-submodule $P_q \subset L^q_{\mathcal{O}}(K_1)$ such that $L^q_{\mathcal{O}}(K_1) = A_q \oplus P_q$. Given this, for $n = 0, 1$ we can construct $\mathcal{O}$-module isomorphisms $\Psi_n : L^1_{\mathcal{O}}(K_n) \xrightarrow{\sim} L^2_{\mathcal{O}}(K_n)$ such that $\Psi_1 = \Psi_A \oplus \Psi_P$ with $\Psi_A : A_1 \xrightarrow{\sim} A_2$ and $\Psi_P : P_1 \xrightarrow{\sim} P_2$, where $L^q_{\mathcal{O}}(K_1) = A_q \oplus P_q$. Furthermore when we consider Ψ_n as an automorphism of $L^q_{\mathbf{C}}(K_n)$ and Ψ_A as an automorphism of X, then $\det \Psi_n$ and $\det \Psi_A$ are independent of the choice of Ψ_n and Ψ_A up to a unit in $\mathcal{O}$.

Suppose now in addition that $C(\mathbf{T}_{\mathcal{O}}) = 0$ (i.e., that there are no level raising congruences). Then Lemma 5.2 implies that $M_q = A_q \oplus B_q$ for $q = 1, 2$, i.e., we can write α_q as $(\alpha_A, 0)$. Consider the following sequence of isomorphisms:

$$L^1_{\mathcal{O}}(K_0)^2 \xrightarrow{\alpha_A} A_1 \xrightarrow{\Psi_A} A_2 \xrightarrow{\alpha_A^+} L^2_{\mathcal{O}}(K_0)^2 \xrightarrow{(\Psi_0,\Psi_0)^{-1}} L^1_{\mathcal{O}}(K_0)^2 \tag{5.1}$$

where surjectivity of α_A^+ follows from perfectness of the pairings involved (Theorem 2.1). For $q = 2$ we get an analogous diagram with L^1's and A_1's interchanged with L^2's and A_2's and the isomorphisms Ψ_A and (Ψ_0, Ψ_0) replaced by their inverses. Using self-adjointness of the Hecke operators in $\mathbf{T}_{n,\mathcal{O}}$ and arguing as in the proof of Lemma 2 of [16] one can show that $\alpha^+_{q,\mathbf{C}}\alpha_{q,\mathbf{C}} = \begin{bmatrix} N\mathfrak{p}+1 & T_{\mathfrak{p}} \\ T_{\mathfrak{p}} & (N\mathfrak{p})^{k-2}(N\mathfrak{p}+1)\end{bmatrix}$, where $\alpha_{q,\mathbf{C}}$ (resp. $\alpha^+_{q,\mathbf{C}}$) denotes the complexification of α_q (resp. α_q^+) and the map $\alpha^+_{q,\mathbf{C}}\alpha_{q,\mathbf{C}}$ makes sense as an automorphism of $L^1_{\mathbf{C}}(K_0)^2$. It follows from this and the fact that the composite in (5.1) is an $\mathcal{O}$-module isomorphism that $\det(\alpha^+_{q,\mathbf{C}}\alpha_{q,\mathbf{C}}) \in \mathcal{O}^\times$. Inverting the logic we conclude that there exist level-raising congruences (i.e., $C(\mathbf{T}_{\mathcal{O}}) \neq 0$) if $\det(\alpha^+_{q,\mathbf{C}}\alpha_{q,\mathbf{C}}) \notin \mathcal{O}^\times$. We thus arrive at a conclusion that for a cuspidal Hecke eigenform f on $\mathrm{GL}_2(\mathbf{A}_F)$ of (parallel) weight k and level $\mathfrak{N}$ such that $a_{\mathfrak{p}}^2 \equiv (N\mathfrak{p})^{k-2}(N\mathfrak{p}+1)^2 \pmod{\varpi}$ (where we write $a_{\mathfrak{p}}$ for the f-eigenvalue of $T_{\mathfrak{p}}$) there exists a cuspidal Hecke eigenform g on $\mathrm{GL}_2(\mathbf{A}_F)$ of weight k, level $\mathfrak{N}\mathfrak{p}$ (new at $\mathfrak{p}$) with $f \equiv g \pmod{\varpi}$ - the congruence being between the Hecke eigenvalues for $T_{\mathfrak{a}}$ for $\mathfrak{a}$ a prime with $\mathfrak{a} \nmid \mathfrak{N}\mathfrak{p}$. This would be an analogue for F of classical level-raising congruences of Ribet and Diamond [10], [4].

However, the problem is that A_2 is not necessarily saturated due to the presence of torsion in the degree two cohomology (see e.g. [11] for some examples where this group contains torsion). Let us discuss it in some more details and offer a weaker result (Theorem 5.3) on the level of cohomology.

By Lemma 5.1 there exists an integer s such that $A_q^{\mathrm{sat}}/A_q \cong \ker(\alpha_{q,s})$. Consider the following commutative diagram:

(5.2)

$$\begin{tikzcd}
L^q_{\mathcal{O}}(K_0)^2 \arrow[r, hook, "\alpha_q"] \arrow[d, two heads] & L^q_{\mathcal{O}}(K_1) \arrow[d, two heads] \\
L^q_{\mathcal{O}}(K_0)^2 \otimes \mathcal{O}/\varpi^s \arrow[r, "\alpha_{q,s}:=\alpha_q\otimes 1"] \arrow[d, equal] & L^q_{\mathcal{O}}(K_1) \otimes \mathcal{O}/\varpi^s \arrow[d, equal] \\
\bigoplus_j (H^q_P(\Gamma_{0,j}, L(k-2,\mathcal{O}))^{\mathrm{tf}})^2 \otimes \mathcal{O}/\varpi^s \arrow[r, "\alpha_q\otimes 1"] & \bigoplus_j (H^q_P(\Gamma_{1,j}, L(k-2,\mathcal{O}))^{\mathrm{tf}}) \otimes \mathcal{O}/\varpi^s.
\end{tikzcd}$$

Theorem 5.3. *For every positive integer s the map*

$$\alpha_q \otimes 1 : \bigoplus_j (H^q_P(\Gamma_{0,j}, L(k-2,\mathcal{O})))^2 \otimes \mathcal{O}/\varpi^s \to \bigoplus_j (H^q_P(\Gamma_{1,j}, L(k-2,\mathcal{O}))) \otimes \mathcal{O}/\varpi^s$$

is injective.

It follows from Theorem 5.3 that the map

$$\alpha_q : \bigoplus_j (H^q_P(\Gamma_{0,j}, L(k-2,\mathcal{O})))^2 \to \bigoplus_j (H^q_P(\Gamma_{1,j}, L(k-2,\mathcal{O})))$$

is injective and that the image of the first module is saturated in the second one. However, Theorem 5.3 implies injectivity of the bottom map in diagram (5.2) only in the case of H^1 which is torsion-free by Lemma 2.2. Thus Theorem 5.3 falls short of proving that A_2 is saturated and we only obtain the following corollary:

Corollary 5.4. *One has $A_1^{\mathrm{sat}} = A_1$. Moreover if $\bigoplus_j H^2_P(\Gamma_{i,j}, L(k-2,\mathcal{O}))$ is torsion-free for $i = 0, 1$, then $A_2^{\mathrm{sat}} = A_2$.*

Finally, let us note that one cannot use Theorem 5.3 to prove the existence of level-raising congruences on the level of cohomology groups instead of the lattices L because the pairings J_n are only perfect modulo torsion (Theorem 2.1).

Proof of Theorem 5.3. We need to prove that the map $\alpha_q \otimes 1$ in Theorem 5.3 is injective. To shorten formulas for a $\mathbf{Z}$-module A we put $\tilde{A} := L(k-2, A)$. The short exact sequence $0 \to \tilde{\mathcal{O}} \xrightarrow{\cdot\varpi^s} \tilde{\mathcal{O}} \to \widetilde{\mathcal{O}/\varpi^s\mathcal{O}} \to 0$ gives rise to the following commutative diagram where the bottom two rows are exact sequences and the objects in the top row are by definition the kernels of the vertical maps.

(5.3)

$$\begin{tikzcd}
\bigoplus_j H^q_P(\Gamma_{0,j}, \tilde{\mathcal{O}}) \arrow[r, "\cdot\varpi^s"] \arrow[d] & \bigoplus_j H^q_P(\Gamma_{0,j}, \tilde{\mathcal{O}}) \arrow[r] \arrow[d] & \bigoplus_j H^q_P(\Gamma_{0,j}, \widetilde{\mathcal{O}/\varpi^s}) \arrow[d] \\
\bigoplus_j H^q(\Gamma_{0,j}, \tilde{\mathcal{O}}) \arrow[r, "\cdot\varpi^s"] \arrow[d] & \bigoplus_j H^q(\Gamma_{0,j}, \tilde{\mathcal{O}}) \arrow[r] \arrow[d] & \bigoplus_j H^q(\Gamma_{0,j}, \widetilde{\mathcal{O}/\varpi^s}) \arrow[d] \\
\bigoplus_j \bigoplus_{B\in\mathcal{B}_j} H^q(\Gamma_{0,j,B}, \tilde{\mathcal{O}}) \arrow[r, "\cdot\varpi^s"] & \bigoplus_j \bigoplus_{B\in\mathcal{B}_j} H^q(\Gamma_{0,j,B}, \tilde{\mathcal{O}}) \arrow[r] & \bigoplus_j \bigoplus_{B\in\mathcal{B}_j} H^q(\Gamma_{0,j,B}, \widetilde{\mathcal{O}/\varpi^s}).
\end{tikzcd}$$

One gets an identical diagram for the groups $\Gamma_{1,j}$. By performing a subset of the proof of snake lemma, one can see that the top row is also exact provided that the bottom-left horizontal arrow is injective. For $q = 1$ this follows from the exactness of the sequence of H^0's. For $q = 2$ this is a consequence of the following lemma:

Lemma 5.5. *The groups $H^2(\Gamma_{0,j,B}, \tilde{\mathcal{O}})$ and $H^2(\Gamma_{1,j,B}, \tilde{\mathcal{O}})$ are torsion-free.*

Proof. Let $B \subset \mathrm{SL}_2(F)$ be a Borel subgroup. Write $B = MN$ for its Levi decomposition. Note that $M \cap \Gamma_{0,j}$ is a finite group of order dividing $\#\mathcal{O}_F^\times$, hence we have $H^2(\Gamma_{0,j,B}, \tilde{\mathcal{O}}) \hookrightarrow H^2(\Gamma_{0,j} \cap N, \tilde{\mathcal{O}})$, so it is enough to show that the latter group is torsion-free.

It is easy to see that $\Gamma_{0,j} \cap N \cong \mathbf{Z}^2$ as abelian groups. Write G for $\Gamma_{0,j} \cap N \cong \mathbf{Z}^2$ and choose a subgroup $H \subset G$ such that $H \cong \mathbf{Z}$. From the Hochshild-Serre spectral sequence one concludes that the sequence

$$H^2(G/H, \tilde{\mathcal{O}}^H) \to H^2(G, \tilde{\mathcal{O}})^* \to H^1(G/H, H^1(H, \tilde{\mathcal{O}})), \tag{5.4}$$

where $H^2(G, M)^* = \ker(\mathrm{res}(H^2(G, M) \to H^2(H, M))$ is exact. Since H has cohomological dimension one we obtain that $H^2(G, \tilde{\mathcal{O}})^* = H^2(G, \tilde{\mathcal{O}})$ and that $H^2(G/H, \tilde{\mathcal{O}}^H) = 0$. Hence we conclude from (5.4) that $H^2(G, \tilde{\mathcal{O}})$ injects into $H^1(G/H, H^1(H, \tilde{\mathcal{O}}))$ and thus it is enough to prove that the latter group is torsion-free.

Note that since the sequence of H-invariants

$$0 \to \tilde{\mathcal{O}}^H \to (\tilde{\mathcal{O}} \otimes E)^H \to (\tilde{\mathcal{O}} \otimes E/\mathcal{O})^H \to 0$$

is exact and H has cohomological dimension one, we obtain a short exact sequence of cohomology groups

$$0 \to H^1(H, \tilde{\mathcal{O}}) \to H^1(H, \tilde{\mathcal{O}} \otimes E) \to H^1(H, \tilde{\mathcal{O}} \otimes E/\mathcal{O}) \to 0. \tag{5.5}$$

Since all the groups in (5.5) are naturally G/H-modules we can compute their long exact sequence of cohomology for the group G/H and obtain:

$$\begin{aligned} 0 \to H^1(H, \tilde{\mathcal{O}})^{G/H} \to H^1(H, \tilde{\mathcal{O}} \otimes E)^{G/H} \xrightarrow{\phi} H^1(H, \tilde{\mathcal{O}} \otimes E/\mathcal{O})^{G/H} \to \\ \to H^1(G/H, H^1(H, \tilde{\mathcal{O}})) \to H^1(G/H, H^1(H, \tilde{\mathcal{O}} \otimes E)) \end{aligned} \tag{5.6}$$

Since the last group in (5.6) is an E-vector space it is clearly torsion-free, so it suffices to prove that ϕ is surjective.

To do this, first note that since G is abelian the group G/H acts on $f \in H^1(H, M)$ by $(\gamma \cdot f)(h) = \gamma \cdot f(h)$. Moreover, since $H \cong \mathbf{Z}$ one has $H^1(H, M) \cong \frac{M}{(\sigma-1)M}$, where σ is a generator of H and 1 denotes the identity in H (hence zero in $\mathbf{Z}$). So, our claim follows from exactness of the following sequence (which is easy to show):

(5.7)

$$0 \to \left(\frac{\tilde{\mathcal{O}}}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tilde{\mathcal{O}}} \right)^{G/H} \to \left(\frac{\tilde{\mathcal{O}} \otimes E}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tilde{\mathcal{O}} \otimes E} \right)^{G/H} \to \left(\frac{\tilde{\mathcal{O}} \otimes E/\mathcal{O}}{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tilde{\mathcal{O}} \otimes E/\mathcal{O}} \right)^{G/H} \to 0$$

The same proof works for $\Gamma_{1,j,B}$. $\square$

Using the exactness of the top row in (5.3) we get the following commutative diagram

$$\begin{tikzcd}
(\bigoplus_j H^q_P(\Gamma_{0,j},\tilde{\mathcal{O}}))^2 \otimes \mathcal{O}/\varpi^s \arrow[r, equal] \arrow[d, "\alpha_q\otimes 1"] & \frac{(\bigoplus_j H^q_P(\Gamma_{0,j},\tilde{\mathcal{O}}))^2}{\varpi^s(\bigoplus_j H^q_P(\Gamma_{0,j},\tilde{\mathcal{O}}))^2} \arrow[r, hook] \arrow[d] & (\bigoplus_j H^q_P(\Gamma_{0,j},\widetilde{\mathcal{O}/\varpi^s}))^2 \arrow[d, "\alpha_{q,\mathcal{O}/\varpi^s}"] \\
(\bigoplus_j H^q_P(\Gamma_{1,j},\tilde{\mathcal{O}})) \otimes \mathcal{O}/\varpi^s \arrow[r, equal] & \frac{(\bigoplus_j H^q_P(\Gamma_{1,j},\tilde{\mathcal{O}}))}{\varpi^s(\bigoplus_j H^q_P(\Gamma_{1,j},\tilde{\mathcal{O}}))} \arrow[r, hook] & (\bigoplus_j H^q_P(\Gamma_{1,j},\widetilde{\mathcal{O}/\varpi^s})),
\end{tikzcd}$$

where $\alpha_{q,\mathcal{O}/\varpi^s}$ is induced from α_q. The map $\alpha_{q,\mathcal{O}/\varpi^s}$ is injective, by Theorem 4.1 when $q=1$ and by Theorem 3.1 when $q=2$. Hence the first vertical arrow is injective. $\square$

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