Congruences among automorphic forms on unitary groups and the Bloch-Kato conjecture

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The Bloch-Kato philosophy

M = motive

$\swarrow \qquad \searrow$

$L(M) \qquad \mathrm{Sel}(M)$

The Bloch-Kato Conjecture (1990) predicts a relationship between the two bottom objects.

Examples:

- Analytic class number formula
- BSD conjecture

Ribet (1976) - Converse to Herbrand's Theorem

Let $\chi : (\mathbf{Z}/p\mathbf{Z})^\times \to \mu_{p-1}$ be an even Dirichlet character.

Theorem (Ribet, 1976)

If $p \mid L(-1, \chi)$, then $p \mid \# \mathrm{Cl}_{\mathbf{Q}(\mu_p)}(\chi^{-1}\omega^{-1})$.

Ribet (1976) - Converse to Herbrand's Theorem

Proof.

1. χ gives rise to an Eisenstein series E_χ.
2. If $p \mid L(-1,\chi)$, then $E_\chi \equiv$ cusp form f mod p.
3. Thus the mod p Galois representation $\overline{\rho}_f$ attached to f is reducible but not semisimple (since f is a cusp form).
4. So it gives rise to a non-trivial extension

$$0 \to \mathbf{F} \to \overline{\rho}_f \to \mathbf{F}(\chi\omega) \to 0,$$

which corresponds to a non-trivial element of the $\chi^{-1}\omega^{-1}$-part of $\mathrm{Cl}_{\mathbf{Q}(\mu_p)}$.

□

Ribet's approach revisited

1. $M_{/\mathbf{Q}}$=algebraic group, π=automorphic representation of $M(\mathbf{A})$.
2. Realize M as a Levi in some algebraic group $G_{/\mathbf{Q}}$.
3. Lift π to Π=aut. repr. of $G(\mathbf{A})$. The Galois repr. attached to Π will be reducible, semisimple.
4. Assuming that p divides the relevant L-value(s) of π, construct an aut. repr. Π' of $G(\mathbf{A})$ such that
    - its Hecke eigenvalues are congruent to those of Π mod p
    - the p-adic Galois representation of Π' is irreducible.
5. Get a non-trivial extension, hence elements in Selmer group.

Examples of such results

- **K. Ribet** ($M = \mathrm{GL}_1$, $G = \mathrm{GL}_2$, $\Pi = E_\chi$) (1976)
- **A. Wiles** (same as Ribet, MC of Iwasawa Theory) (1990)
- **Skinner-Urban** ($M = \mathrm{R}_{K/\mathbf{Q}}\,\mathrm{GL}_2$, $G = \mathrm{U}(2,2)$, $\Pi =$ Eisenstein series) (2006)
- **T. Berger** ($M = \mathrm{R}_{K/\mathbf{Q}}\,\mathrm{GL}_1$, $G = \mathrm{R}_{K/\mathbf{Q}}\,\mathrm{GL}_2$, Π=Eisenstein series) (2005)
- **J. Brown** ($M = \mathrm{GL}_2$, $G = \mathrm{Sp}_4$, $\Pi =$ Saito-Kurokawa lift) (2005)
- **K.** ($M = \mathrm{R}_{K/\mathbf{Q}}\,\mathrm{GL}_2$, $G = \mathrm{U}(2,2)$, $\Pi =$ CAP representation) (2006/07)

Key difficulty: Construct Π' with irreducible Galois representation, "congruent" to Π. Very different techniques are used.

Main result (rough version)

- K=imaginary quadratic field of discriminant D_K
- f=modular form of weight $k-1$, level D_K
- ρ_f=p-adic Galois representation attached to f.

Theorem (Main theorem (rough version))

$$p^n \mid L^{\mathrm{alg}}(Sym^2 f, k) \implies p^n \mid \# \mathrm{Sel}(K, \mathrm{ad}^0 \rho_f)^\vee.$$

Setup

- $K/\mathbf{Q}$ = imaginary quadratic, D_K=discriminant
- χ_K= Dirichlet character associated to $K/\mathbf{Q}$
- $G = \mathrm{U}(2,2) = \{M \in \mathrm{R}_{K/\mathbf{Q}}\,\mathrm{GL}_4 \mid MJ\overline{M}^t = J\}$, $J = \begin{bmatrix} & I_2 \\ -I_2 & \end{bmatrix}$.
- $\mathcal{H} = \{Z \in M_2(\mathbf{C}) \mid -i(Z - \overline{Z}^t) > 0\}$
- If $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in G(\mathbf{R})$, then

$$gZ = (AZ + B)(CZ + D)^{-1}.$$

- $\mathcal{S}_k := \{F : \mathcal{H} \to \mathbf{C} \mid F(gZ) = \det(CZ + D)^k F(Z), g \in G(\mathbf{Z}), F = \text{holomorphic, cuspidal}\}$

Maass lifting

Theorem (Kojima, Gritsenko, Krieg (1982-1991))

There exists a **C**-*linear lift*

$$S_{k-1}(D_K, \chi_K) \to \mathcal{S}_k, \quad f \mapsto F_f$$

which is Hecke-equivariant if $\# \mathrm{Cl}_K = 1$.

If $\# \mathrm{Cl}_K > 1$, no way to define Hecke operators at split primes $\to$ need an adelic theory.

Adelic Maass lifting

Theorem (K. 2007)

If $2 \nmid \# \mathrm{Cl}_K$ then there exists a $\mathbf{C}$*-linear map*

$$S_{k-1}(D_K, \chi_K)^{\# \mathrm{Cl}_K} \to \mathcal{A}_0(G(\mathbf{A}))$$

which is Hecke equivariant (for all local Hecke algebras).

The p-adic Galois representation attached to a Hecke eigenform lying in the Maass space is reducible and semisimple.

CAP representations

- $f \to \pi_f$ on $\mathrm{GL}_2(\mathbf{A})$
- $F_f \to \Pi_f=$ CAP associated to π_f on $G(\mathbf{A})$, i.e.,

$$\Pi_{f,p} \cong \mathrm{Ind}_{M(\mathbf{Q}_p)}^{G(\mathbf{Q}_p)} (\mathrm{BC}_{K/\mathbf{Q}}\, \pi_f)_p$$

for almost every p, where $M \cong \mathrm{R}_{K/\mathbf{Q}}\, \mathrm{GL}_2$=Levi.

Jacquet-Shalika (1981), Moeglin-Waldspurger (1990): There are no CAP representations on GL_n.

CAP ideal

$$\mathcal{S}_k = \mathcal{S}_k^{\mathrm{M}} \oplus \mathcal{S}_k^{\mathrm{NM}}$$

$\mathcal{O}$ = ring of integers in a finite extension of $\mathbf{Q}_p$,
$\mathbf{T}$ = the $\mathcal{O}$-Hecke algebra of $\mathcal{S}_k$,
$\mathbf{T}^{\mathrm{NM}}$ = quotient of $\mathbf{T}$ acting on $\mathcal{S}_k^{\mathrm{NM}}$,
$\phi : \mathbf{T} \twoheadrightarrow \mathbf{T}^{\mathrm{NM}}$ canonical epimorphism.

Definition (CAP ideal)

Set I_f = CAP ideal = ideal of $\mathbf{T}^{\mathrm{NM}}$ generated by $\phi(\mathrm{Ann}(F_f))$.

$\mathbf{T}^{\mathrm{NM}}/I_f$ measures congruences between F_f and non-Maass forms.

Conjectures

Conjecture (On the size of the CAP ideal)

Let $\#\mathcal{O}_K^\times \mid k$, $f \in S_{k-1}(D_K, \chi_K)$ *a newform,* $p > k$ *a prime not dividing some explicit constants. Then*

$$\mathrm{val}_p(\#\mathbf{T}^{\mathrm{NM}}/I_f) \geq \mathrm{val}_p(\#\mathcal{O}/L^{\mathrm{alg}}(Sym^2 f, k)).$$

Conjecture (On the existence of Galois representations)

If F *is a hermitian cuspidal eigenform, there exists a continuous representation*

$$\rho_F : G_K \to \mathrm{GL}_4(\overline{\mathbf{Q}}_p)$$

having the expected properties.

Let

- $\rho_f : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ be the Galois representation attached to f by Deligne;
- $T = \mathrm{ad}^0 \rho_f|_{G_K}(-1)$,
- $W = T \otimes \mathrm{Frac}(\mathcal{O})/\mathcal{O}$.

Corollary

Assume the above two conjectures. Under some mild hypotheses on ρ_f one has

$$\mathrm{val}_p(\# \,\mathrm{Sel}(K, W)^\vee) \geq \mathrm{val}_p(\#\mathcal{O}/L^{\mathrm{alg}}(Sym^2 f, k)).$$

Main result

Theorem (K. 2007)

Suppose $\# \mathrm{Cl}_K = 1$ *($\# \mathrm{Cl}_K$=odd - work in progress). If*

- *f is ordinary at p with $\overline{\rho}_f|_{G_K}$ absolutely irreducible*
- *and there exists a Hecke character ψ of K of conductor prime to p with $\psi_\infty(z) = z^{-t}\overline{z}^t$ and*

$$\mathrm{val}_p \left(\prod_{j=1}^{2} L^{\mathrm{alg}}(\mathrm{BC}(f), j + t + k/2, \psi) \right) = 0,$$

then the conjecture on the size of the CAP ideal is true.

Beyond $U(2,2)$ (work in progress...)

- Ikeda lifts: lifting modular forms on GL_2 to modular forms on $U(n,n)$ ($f \mapsto F_f$).

Conjecture (On the size of the CAP ideal of an Ikeda lift to $U(n,n)$)

Let n be even, $f \in S_{k-1}(D_K, \chi_K)$ a newform, $p > k$ a prime not dividing some explicit constants. Then

$$\mathrm{val}_p(\#\mathbf{T}^{\mathrm{NM}}/I_f) \geq \mathrm{val}_p\left(\#\mathcal{O}/\prod_{j=2}^{n} L^{\mathrm{alg}}(Sym^2 f, k+j-2, \chi_K^j)\right).$$

- Similar conjecture can be phrased for Ikeda lifts to $SU(n,n)$.

Constructing the non-Maass cusp form F with $F \equiv F_f$.

Ingredients used in the proof:

- Properties of Siegel Eisenstein series due to Shimura;
- Inner product relations between F_f and other hermitian modular forms;
- Hida theory + deformation theory of Galois representations.

Thank you.

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