ON THE JACQUET CONJECTURE ON THE LOCAL CONVERSE PROBLEM FOR p-ADIC GL_n

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Abstract. Based on previous results of Jiang, Nien and the third author, we prove that any two *minimax* unitarizable supercuspidals of GL_N that have the same depth and central character admit a *special pair* of Whittaker functions. This result gives a new reduction towards a final proof of Jacquet's conjecture on the local converse problem for GL_N. As a corollary of our result, we prove Jacquet's conjecture for GL_N, when N is prime.

1. Introduction

In the representation theory of a group G, one of the basic problems is to characterize its irreducible representations up to isomorphism. If G is the group of points of a reductive algebraic group defined over a non-archimedean local field F, there are many invariants that one can attach to a representation π of G, some of which are the central character and depth. Capturing all of these invariants, however, is a family of complex functions, invariants themselves, called the *local gamma factors* of π.

Now let $\mathrm{G}_N := \mathrm{GL}_N(F)$ and let π be an irreducible generic representation of G_N. The family of local gamma factors $\gamma(s, \pi \times \tau, \psi)$, for τ an irreducible generic representation of G_r, can be defined using Rankin-Selberg convolution [JPSS83] or the Langlands-Shahidi method [S84]. Jacquet has formulated the following conjecture on precisely which family of local gamma factors should uniquely determine π.

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Conjecture 1.1 (The Jacquet Conjecture on the Local Converse Problem)**.** *Let π_1, π_2 be irreducible generic representations of* G_N. *If*

$$\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi),$$

as functions of the complex variable s, for all irreducible generic representations τ of G_r with $r = 1, \ldots, [\frac{N}{2}]$, then $\pi_1 \cong \pi_2$.

We refer to the introductions of [Ch06] and [JNS13] for more related discussions on the previous known results on this conjecture. Moreover, in general, from the discussion in [Ch96, Sections 5.2 and 5.3], it is expected that the upper bound $[\frac{N}{2}]$ is sharp.

In [JNS13, Section 2.4], Conjecture 1.1 is shown to be equivalent to the same conjecture with the adjective "generic" replaced by "unitarizable supercuspidal" (recall that an irreducible representation is *supercuspidal* if it is not a subquotient of a properly parabolically induced representation, while all supercuspidal representations are generic). However, in the situation that π_1, π_2 are both supercuspidal, it may be that the upper bound $[\frac{N}{2}]$ is no longer sharp, at least within certain families of supercuspidals: for example, for *simple* supercuspidals (of depth $\frac{1}{N}$), the upper bound may be lowered to 1 (see [BH14, Proposition 2.2] and [AL14, Remark 3.18] in general, and [X13] in the tame case).

Thus, for $m \geq 1$ an integer, it is convenient for us to say that irreducible supercuspidal representations π_1, π_2 of G_N *satisfy hypothesis $\mathcal{H}_m$* if

($\mathcal{H}_m$) $\gamma(s, \pi_1 \times \tau, \psi) = \gamma(s, \pi_2 \times \tau, \psi)$ as functions of the complex variable s, for all irreducible supercuspidal representations τ of G_m.

For $r \geq 1$, we say that π_1, π_2 *satisfy hypothesis $\mathcal{H}_{\leq r}$* if they satisfy hypothesis $\mathcal{H}_m$, for $1 \leq m \leq r$. Then we can state a family of "conjectures".

Conjecture $\boldsymbol{\mathcal{J}(N, r)}$. *If π_1, π_2 are irreducible supercuspidal representations of G_N which satisfy hypothesis $\mathcal{H}_{\leq r}$, then $\pi_1 \simeq \pi_2$.*

Thus Jacquet's conjecture is (equivalent to) Conjecture $\mathcal{J}(N, [\frac{N}{2}])$, while Conjecture $\mathcal{J}(N, N-2)$ is a Theorem due to Chen [Ch96, Ch06] and to Cogdell and Piatetskii-Shapiro [CPS99]. On the other hand, examples in [Ch96] also show that Conjecture $\mathcal{J}(4, 1)$ is false, and similar examples show that $\mathcal{J}(N, 1)$ is false when N is composite.

Here we prove Conjecture $\mathcal{J}(N, [\frac{N}{2}])$ when N is prime (see Corollary 7.3), and make some further elementary reductions in general.

This builds on the work in [JNS13], where many cases are proved. Indeed, our work here is to tackle the "simplest" case left out in [JNS13], which is sufficient when N is prime. We hope that this will be the first step in an inductive proof allowing all N to be treated.

On the other hand, it is not clear to us whether conjecture $\mathcal{J}(N,[\frac{N}{2}])$ is optimal, when N is prime. For example, we do not know whether conjecture $\mathcal{J}(5,1)$ is true: is it possible to distinguish supercuspidal representations of $\mathrm{GL}_5(F)$ only by the local gamma factors of their twists by characters? If so, it would be tempting to suggest, more generally, that two irreducible supercuspidal representations of G_N which satisfy hypothesis $\mathcal{H}_m$, for all $m<N$ *dividing* N, should be equivalent. Our methods here do not shed light on this question.

Finally we describe the contents of the paper and the scheme of the proof. In [JNS13], Jiang, Nien and the third author introduced the notion of a *special pair of Whittaker functions* for a pair of irreducible unitarizable supercuspidal representations π_1,π_2 of G_N (see Section 2). They also proved that, if there is such a pair and π_1,π_2 satisfy hypothesis $\mathcal{H}_{\le[N/2]}$, then π_1,π_2 are equivalent, as well as finding special pairs of Whittaker functions in many cases, in particular the case of depth zero representations.

The simplest case omitted in [JNS13] is a case we call *minimax*, described as follows. Each supercuspidal representation π_i is irreducibly induced from a representation of a compact-mod-centre subgroup, called an *extended maximal simple type* [BK93]; amongst the data from which this is built, is a *simple stratum* $[\mathfrak{A},n,0,\beta]$ and we say the representation is minimax if the field extension $F[\beta]/F$ has degree N and the element β is *minimal* in the sense of [BK93, (1.4.14)] (see Section 3 for recollections).

After preparing the ground in Sections 5, 6, we prove that any pair of minimax unitarizable supercuspidal representations of G_N with the same (positive) depth and central character possesses a special pair of Whittaker functions (see Proposition 7.2).

Finally, when N is prime, any irreducible supercuspidal representation is a twist by some character of either a depth zero representation or of a minimax supercuspidal representation. Then Jacquet's conjecture follows from the results in [JNS13] and a reduction to representations which are of minimal depth among their twists (see Section 4).

1.1. **Notation.** Throughout, F is a locally compact nonarchimedean local field, with ring of integers $\mathfrak{o}_F$, maximal ideal $\mathfrak{p}_F$, and residue field k_F of cardinality q_F and characteristic p; we also write ν_F for the normalized valuation on F, with image $\mathbb{Z}$, and $|\cdot|$ for the normalized absolute value on F, with image $q_F^{\mathbb{Z}}$. We use similar notation for finite extensions of F. We fix once and for all an additive character ψ_F of F which is trivial on $\mathfrak{p}_F$ and nontrivial on $\mathfrak{o}_F$.

For $r \geq 1$, we set $\mathrm{G}_r = \mathrm{GL}_r(F)$, and denote by U_r the unipotent radical of the standard Borel subgroup B_r of G_r, consisting of upper-triangular matrices. We denote by ψ_r the standard nondegenerate character of U_r, given by

$$\psi_r(u) = \psi_F\left(\sum_{i=1}^{r-1} u_{i,i+1}\right),$$

where (u_{ij}) is the matrix of $u \in \mathrm{U}_r$. We also denote by Z_r the centre of G_r, and by P_r the standard mirabolic subgroup consisting of matrices with last row equal to $(0, \ldots, 0, 1)$.

We fix an integer $N \geq 2$ and abbreviate $\mathrm{G} = \mathrm{G}_N$. We also put $V = F^N$ and $A = \mathrm{End}_F(V)$, and identify G with $\mathrm{Aut}_F(V)$ via the standard basis of F^N.

All representations considered are smooth representations with complex coefficients.

2. Special pairs of Whittaker functions

In this section, we recall the main results on special pairs of Whittaker functions. For further background, we refer to [JNS13] and the references therein.

Let π be an irreducible supercuspidal representation of G. By the existence and uniqueness of local Whittaker models, $\mathrm{Hom}_{\mathrm{G}}(\pi, \mathrm{Ind}_{\mathrm{U}_N}^{\mathrm{G}} \psi_N)$ is a one-dimensional space. A *Whittaker function* for π is any function W_π in $\mathrm{Ind}_{\mathrm{U}_N}^{\mathrm{G}} \psi_N$ which is in the image of π under a non-zero homomorphism in this Hom-space. In [JNS13], the following definitions were introduced.

Definition 2.1. *Let π be an irreducible unitarizable supercuspidal representation of* G *and let* $\mathbf{K}$ *be a compact-mod-centre open subgroup of* G*. A nonzero Whittaker function W_π for π is called* $\mathbf{K}$-special *if the support of W_π satisfies* $\mathrm{Supp}(W_\pi) \subset \mathrm{U}_N\mathbf{K}$*, and if*

$$W_\pi(k^{-1}) = \overline{W_\pi(k)} \text{ for all } k \in \mathbf{K},$$

where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

Definition 2.2. *For $i = 1, 2$, let π_i be an irreducible unitarizable supercuspidal representation of* G *and let W_{π_i} be a nonzero Whittaker function for π_i. Suppose moreover that π_1, π_2 have the same central* *character. Then* (W_{π_1}, W_{π_2}) *is called a* special pair *of Whittaker functions for the pair* (π_1, π_2) *if there exists a compact-mod-centre open subgroup* $\mathbf{K}$ *of* G *such that* W_{π_1} *and* W_{π_2} *are both* $\mathbf{K}$*-special and*

$$W_{\pi_1}(p) = W_{\pi_2}(p), \text{ for all } p \in P_N.$$

The condition in this definition that the representations have the same central character is rather mild in our situation since, by [JNS13, Corollary 2.7], if π_1, π_2 are irreducible supercuspidal representations of G which satisfy hypothesis $\mathcal{H}_1$, then they have the same central character.

The following is one of the main results in [JNS13], which provides a general approach to proving Conjecture $\mathcal{J}(N, [\frac{N}{2}])$.

Proposition 2.3 ([JNS13, Theorem 1.5])**.** *Let* π_1, π_2 *be irreducible unitarizable supercuspidal representations of* G *which have a special pair of Whittaker functions and satisfy hypothesis* $\mathcal{H}_{\leq [N/2]}$*. Then* $\pi_1 \simeq \pi_2$*.*

In [JNS13], in several cases it is proved that a pair of supercuspidal representations of G have a special pair of Whittaker functions. Here we prove another case, the simplest case left open in [JNS13]; as we will see, this is sufficient to prove Conjecture $\mathcal{J}(N, [\frac{N}{2}])$ in the case that N is prime.

3. Strata

In order to use Proposition 2.3, we need to recall some parts of the construction of supercuspidal representations in [BK93], in particular the notion of a stratum.

We begin with a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in $A = \mathrm{End}_F(V)$, with Jacobson radical $\mathfrak{P}$, and we denote by $e = e(\mathfrak{A}|\mathfrak{o}_F)$ the $\mathfrak{o}_F$-period of $\mathfrak{A}$, that is, the integer such that $\mathfrak{p}_F\mathfrak{A} = \mathfrak{P}^e$. For any such hereditary order $\mathfrak{A}$, there is an ordered basis with respect to which $\mathfrak{A}$ is in *standard form*, that is, it consists of matrices with coefficients in $\mathfrak{o}_F$ which are block upper-triangular modulo $\mathfrak{p}_F$. Such a basis can be found as follows.

Recall that an $\mathfrak{o}_F$-lattice chain in V is a set of $\mathfrak{o}_F$-lattices which is linearly ordered by inclusion and invariant under multiplication by scalars in $F^\times$. Then there is a unique $\mathfrak{o}_F$-lattice chain $\mathfrak{L} = \{L_i \mid i \in \mathbb{Z}\}$ in V such that $\mathfrak{A} = \{x \in A \mid xL_i \subseteq L_i \text{ for all } i \in \mathbb{Z}\}$. (The set $\mathfrak{L}$ is uniquely determined by $\mathfrak{A}$, though the base point L_0 for the indexing is arbitrary.) For $i = 0, \ldots, e-1$, we choose an ordered set $\mathcal{B}_i$ of vectors in L_i whose image in L_i/L_{i+1} is a basis and then the ordered basis obtained by concatenating $\mathcal{B}_0, \ldots, \mathcal{B}_{e-1}$ is as required.

A hereditary order $\mathfrak{A}$ gives rise to a parahoric subgroup $U(\mathfrak{A}) = U^0(\mathfrak{A}) = \mathfrak{A}^\times$ of G, together with a filtration by normal open subgroups $U^n(\mathfrak{A}) = 1 + \mathfrak{P}^n$, for $n \geq 1$, as well as a compact-mod-centre subgroup $\mathfrak{K}(\mathfrak{A}) = \{g \in \mathrm{G} \mid g\mathfrak{A}g^{-1} = \mathfrak{A}\}$, the normalizer of $\mathfrak{A}$ in G. We also get a "valuation" $\nu_{\mathfrak{A}}$ on A by $\nu_{\mathfrak{A}}(x) = \sup\{n \in \mathbb{Z} \mid x \in \mathfrak{P}^n\}$, and, for $x \in F$, we have $\nu_{\mathfrak{A}}(x) = e(\mathfrak{A}|\mathfrak{o}_F)\nu_F(x)$.

A *stratum* in A is a quadruple $[\mathfrak{A}, n, r, \beta]$, where $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order, $n \geq r \geq 0$ are integers, and $\beta \in \mathfrak{P}^{-n}$. Strata $[\mathfrak{A}, n, r, \beta_i]$, with $i = 1, 2$, are called *equivalent* if $\beta_1 - \beta_2 \in \mathfrak{P}^{-r}$. Thus, when $r \geq \left[\frac{n}{2}\right]$, the equivalence class of a stratum $[\mathfrak{A}, n, r, \beta]$ corresponds to a character ψ_β of $U^{r+1}(\mathfrak{A})$, trivial on $U^{n+1}(\mathfrak{A})$, via

$$\psi_\beta(x) = \psi_F \circ \mathrm{tr}_{A/F}(\beta(x-1)), \quad \text{for } x \in U^{r+1}(\mathfrak{A}).$$

The stratum $[\mathfrak{A}, n, r, \beta]$ is called *pure* if $E = F[\beta]$ is a field with $E^\times \subseteq \mathfrak{K}(\mathfrak{A})$, and $\nu_{\mathfrak{A}}(\beta) = -n$. A pure stratum $[\mathfrak{A}, n, r, \beta]$ is called *simple* if $r < -k_0(\beta, \mathfrak{A})$, where $k_0(\beta, \mathfrak{A})$ is an invariant whose definition we do not recall here (see [BK93, Definition 1.4.5]).

Of particular importance will be simple strata of the form $[\mathfrak{A}, n, n-1, \beta]$. A pure such stratum is simple if and only if the element β is *minimal* in the sense of [BK93, (1.4.14)], that is:

(i) $\nu_E(\beta)$ is coprime to the ramification index $e = e(E/F)$; and
(ii) if ϖ_F is any uniformizer of F, then the image of $\varpi_F^{-\nu_E(\beta)}\beta^e + \mathfrak{p}_E$ in k_E generates the residue class extension k_E/k_F.

We call a simple stratum of the form $[\mathfrak{A}, n, n-1, \beta]$ a *minimal* stratum. A particular case occurs when in fact $\beta \in F$; in this case we call the stratum $[\mathfrak{A}, n, n-1, \beta]$ a *scalar* stratum

Finally, we call a pure stratum $[\mathfrak{A}, n, r, \beta]$ a *max* stratum if the extension $E = F[\beta]/F$ is maximal in A (that is, of degree N), in which case $\nu_{\mathfrak{A}}$ coincides with ν_E on E and $e(\mathfrak{A}|\mathfrak{o}_F) = e(E/F)$. We call a max simple stratum $[\mathfrak{A}, n, n-1, \beta]$ a *minimax* stratum, in which case $n, e(\mathfrak{A}|\mathfrak{o}_F)$ are coprime.

The first step in the construction and classification of the positive depth supercuspidal representations of G in [BK93] is to prove that any such representation π *contains* a simple stratum $[\mathfrak{A}, n, n-1, \beta]$, in the sense that $\mathrm{Hom}_{U^n(\mathfrak{A})}(\psi_\beta, \pi) \neq 0$. The *depth* $\ell(\pi)$ of π is then the depth $n/e(\mathfrak{A}|\mathfrak{o}_F)$ of any such simple stratum; this is independent of any choices, as is the degree of the extension $E = F[\beta]/F$. In particular, if π contains a minimax stratum $[\mathfrak{A}, n, n-1, \beta]$, then n and $e = e(E/F)$ are determined by the depth $\ell(\pi) = n/e$, since they are coprime, as is the residue class degree $f = f(E/F) = N/e$. For more details, see for example [KM90, Proposition 1.14].

4. Twisting by characters

In this section, we give a modest reduction of Conjecture $\mathcal{J}(N,r)$ to supercuspidal representations which are of minimal depth amongst all representations obtained from them by twisting by a character.

For π an irreducible representation of G and χ a character of $F^\times$, we write $\pi\chi$ for the representation $\pi \otimes \chi \circ \det$ of G. We say that an irreducible supercuspidal representation π of G is *of minimal depth in its twist class* if

$$\ell(\pi\chi) \geq \ell(\pi), \text{ for all characters } \chi \text{ of } F^\times.$$

The representation π is of minimal depth in its twist class if and only if it does *not* contain a scalar minimal stratum; that is, if $[\mathfrak{A}, n, n-1, \beta]$ is a minimal stratum contained in π then $F[\beta]$ is a proper extension of F (see [KM90, Remark 1.15] and [BK93, Lemma 2.4.11]). Note that, in the case that N is prime, this implies that any minimal stratum contained in π is a minimax stratum. (One can also see this directly from the classification of Carayol [C84] – see also [B87, p209].)

We will not recall here the definitions of local factors of pairs of supercuspidal representations from [JPSS83]. However, from the definitions (see also [JPSS83, Theorem 2.7]), a straightforward check shows the following, which is surely well-known.

Lemma 4.1. *Let r be a natural number with $r < N$, let π, τ be generic irreducible representations of G, G_r respectively, let χ be a character of $F^\times$, and let $s \in \mathbb{C}$. Then*

$$\begin{aligned} L(s, \pi\chi \times \tau) &= L(s, \pi \times \tau\chi), \\ \varepsilon(s, \pi\chi \times \tau, \psi) &= \varepsilon(s, \pi \times \tau\chi, \psi), \\ \gamma(s, \pi\chi \times \tau, \psi) &= \gamma(s, \pi \times \tau\chi, \psi). \end{aligned}$$

We also recall that the depth $\ell(\pi)$ of an irreducible supercuspidal representation can be determined from the conductor of the standard epsilon factor $\varepsilon(s, \pi, \psi) = \varepsilon(s, \pi \times \mathbf{1}, \psi)$, where $\mathbf{1}$ is the trivial representation of G_1 (see [B87]); indeed the same is true for an arbitrary discrete series representation π, by [LR03, Theorem 3.1].

Now we can reduce Conjecture $\mathcal{J}(N,r)$ to the following special case:

Conjecture $\boldsymbol{\mathcal{J}_0(N,r)}$. *If π_1, π_2 are irreducible supercuspidal representations of G of minimal depth in their twist class which satisfy hypothesis $\mathcal{H}_{\leq r}$, then $\pi_1 \simeq \pi_2$.*

Proposition 4.2. *For $1 \leq r < N$, Conjecture $\mathcal{J}_0(N,r)$ is equivalent to Conjecture $\mathcal{J}(N,r)$.*

Proof. It is clear that Conjecture $\mathcal{J}(N,r)$ implies Conjecture $\mathcal{J}_0(N,r)$. For the converse, we assume that Conjecture $\mathcal{J}_0(N,r)$ is true, and let π_1, π_2 be irreducible supercuspidal representations of G which satisfy hypothesis $\mathcal{H}_{\leq r}$. Then, for χ any character of $F^\times$ and τ any supercuspidal representation of G_m with $1 \leq m \leq r$, Lemma 4.1 and property $\mathcal{H}_{\leq r}$ imply that

$$\begin{aligned} \gamma(s, \pi_1\chi \times \tau, \psi) &= \gamma(s, \pi_1 \times \tau\chi, \psi) \\ &= \gamma(s, \pi_2 \times \tau\chi, \psi) \;=\; \gamma(s, \pi_2\chi \times \tau, \psi). \end{aligned} \tag{4.3}$$

In particular, using the case $m = 1$ with $\tau = \mathbf{1}$ the trivial representation, this implies that $\ell(\pi_1\chi) = \ell(\pi_2\chi)$.

Now we pick a character χ of $F^\times$ such that $\pi_1\chi$ is of minimal depth in its twist class, that is, such that $\ell(\pi_1\chi)$ is minimal in $\{\ell(\pi_1\chi) \mid \chi \text{ a character of } F^\times\}$. Then $\pi_2\chi$ is also of minimal depth in its twist class and (4.3) now implies that the representations $\pi_1\chi, \pi_2\chi$ satisfy hypothesis $\mathcal{H}_{\leq r}$. Thus, by the assumption that Conjecture $\mathcal{J}_0(N,r)$ is true, we deduce that $\pi_1\chi \simeq \pi_2\chi$, whence $\pi_1 \simeq \pi_2$ as required. □

5. Unipotent and mirabolic subgroups

Although we have fixed standard mirabolic and maximal unipotent subgroups, it will be convenient in the sequel to allow these to vary, working in the basis-free setting of Section 3. Thus, in this section, we gather some notation for arbitrary mirabolic and maximal unipotent subgroups.

A maximal flag in V

$$\mathcal{F} = \{0 = V_0 \subset V_1 \subset \cdots \subset V_{N-1} \subset V_N = V\},$$

with $\dim_F(V_i) = i$, determines both a maximal unipotent subgroup $U_\mathcal{F}$ and a mirabolic subgroup $P_\mathcal{F}$ by

$$\begin{aligned} U_\mathcal{F} &= \{g \in \mathrm{G} \mid (g-1)V_i \subseteq V_{i-1}, \text{ for } 1 \leq i \leq N\}, \\ P_\mathcal{F} &= \{g \in \mathrm{G} \mid (g-1)V \subseteq V_{N-1}\}. \end{aligned}$$

Of course, $P_\mathcal{F}$ does not depend on the whole flag $\mathcal{F}$, but $U_\mathcal{F}$ does: there is a bijection between maximal flags in V and maximal unipotent subgroups of G.

Given now an ordered basis $\mathcal{B} = (v_1, \ldots, v_N)$ of V we get a decomposition $V = \bigoplus_{i=1}^N W_i$, where $W_i = \langle v_i \rangle_F$ is the F-linear span of v_i. We set $A_{ij} = \mathrm{Hom}_F(W_j, W_i)$ so that $A = \bigoplus_{1\leq i,j\leq N} A_{ij}$, and define $\mathbf{1}_{ij} \in A_{ij}$ by $\mathbf{1}_{ij}(v_j) = v_i$. Thus saying that $a = (a_{ij})$ is the matrix of some $a \in A$ with respect to $\mathcal{B}$, is the same as saying

$$a = \sum_{1 \le i,j \le N} a_{ij} \mathbf{1}_{ij}.$$

We also get a maximal flag $\mathcal{F}_{\mathcal{B}}$ by setting

$$V_i = \bigoplus_{j=1}^{i} W_j = \langle v_1, \dots, v_i \rangle_F.$$

We denote the corresponding unipotent subgroup and mirabolic by $U_{\mathcal{B}}$ and $P_{\mathcal{B}}$ respectively. Finally, we get a nondegenerate character $\psi_{\mathcal{B}}$ of $U_{\mathcal{B}}$, given by

$$\psi_{\mathcal{B}}(u) = \psi_F \left(\sum_{i=1}^{N-1} u_{i,i+1} \right),$$

where $u \in U_{\mathcal{B}}$ and (u_{ij}) is the matrix of u with respect to the basis $\mathcal{B}$. The same formula also defines a function $\psi_{\mathcal{B}}$ on $P_{\mathcal{B}}$ (though it is not a character).

The standard mirabolic subgroup, maximal unipotent subgroup, and nondegenerate character, are given by choosing $\mathcal{B}$ to be the standard basis of $V = F^N$.

6. Minimax strata

In this section, for a minimax stratum $[\mathfrak{A}, n, n-1, \beta]$, we examine the relationship between the basis with respect to which β is in companion form (and the associated mirabolic subgroup) and the order $\mathfrak{A}$.

We begin in the max setting (but not necessarily minimax). Suppose $\beta \in A$ is such that $E = F[\beta]$ is a field extension of F of maximal degree N. We define the *function* ψ_β of A by

$$\psi_\beta(x) = \psi_F \circ \mathrm{tr}_{A/F}(\beta(x-1)), \qquad \text{for } x \in A.$$

There is an ordered basis $\mathcal{B} = (v_1, \dots, v_N)$ for V with respect to which

$$\psi_\beta | U_{\mathcal{B}} = \psi_{\mathcal{B}}$$

is the nondegenerate character associated to $\mathcal{B}$. Indeed, there is, up to $E^\times$-conjugacy, a unique maximal unipotent subgroup U such that ψ_β is trivial on the derived group U^{der}, and we have $U_{\mathcal{B}} = U$. More explicitly, if $v_1 \in V$ is arbitrary, then putting

$$v_j = \beta^{j-1} v_1, \text{ for } 2 \le j \le N,$$

gives a basis as required, and every such basis arises in this way. With respect to the basis $\mathcal{B}$, the matrix of β is the companion matrix of the minimum polynomial of β. (See [BH98, Section 2] for all of this.) The crucial (though trivial) observation is that we also have an equality of functions (not characters)

$$\psi_\beta | P_{\mathcal{B}} = \psi_{\mathcal{B}}. \tag{6.1}$$

Since E/F is maximal, there is a unique hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in A normalized by $E^\times$; more precisely, it is given by the $\mathfrak{o}_F$-lattice chain $\{\mathfrak{p}_E^i \mid i \in \mathbb{Z}\}$ so we have

$$\mathfrak{A} = \{x \in A \mid x\mathfrak{p}_E^i \subseteq \mathfrak{p}_E^i, \text{ for all } i \in \mathbb{Z}\}.$$

It has $\mathfrak{o}_F$-period $e(E/F)$, and consequently $\nu_{\mathfrak{A}}(\beta) = \nu_E(\beta)$. We assume $n = -\nu_{\mathfrak{A}}(\beta) > 0$ so that $[\mathfrak{A}, n, 0, \beta]$ is a pure stratum, and then the restriction of the function ψ_β defines a character of $U^{[\frac{n}{2}]+1}(\mathfrak{A})$; moreover, by (6.1), we have an equality of characters

$$\psi_\beta | U^{[\frac{n}{2}]+1}(\mathfrak{A}) \cap P_{\mathcal{B}} = \psi_{\mathcal{B}}. \tag{6.2}$$

Now we have the following:

Lemma 6.3. *Suppose $[\mathfrak{A}, n, 0, \beta]$ is a max pure stratum and put $B = C_A(E)$ and $\mathfrak{B} = \mathfrak{A} \cap B$, an $\mathfrak{o}_E$-hereditary order in B. Let P be* any *mirabolic subgroup of* G*. Then, for any integer $m \geq 1$, we have*

$$(U^m(\mathfrak{B})U^n(\mathfrak{A})) \cap P = U^n(\mathfrak{A}) \cap P.$$

Proof. Notice that actually $B = E$ and $\mathfrak{B} = \mathfrak{o}_E$ in this situation. We pick an arbitrary uniformizer ϖ_E for E. We prove that, for any $m \geq 1$,

$$\left(U^m(\mathfrak{B})U^{m+1}(\mathfrak{A})\right) \cap P = U^{m+1}(\mathfrak{A}) \cap P,$$

and the result follows by iteration. This claim is equivalent to the following additive statement: setting $\mathcal{P} = \{p - 1 \mid p \in P\}$, we have

$$\left(\mathfrak{p}_E^m + \mathfrak{P}^{m+1}\right) \cap \mathcal{P} = \mathfrak{P}^{m+1} \cap \mathcal{P},$$

where $\mathfrak{P} = \mathrm{rad}(\mathfrak{A})$ as usual. So suppose $x \in \mathfrak{p}_E^m$ and $y \in \mathfrak{P}^{m+1}$ are such that $z := x + y \in \mathcal{P}$. In particular, z has eigenvalue 0, and the same is then true of $\varpi_E^{-m} z \in \mathfrak{O}_E + \mathfrak{P}$, and of its image in $\mathfrak{A}/\mathfrak{P}$. However, this image is in $k_E \hookrightarrow \mathfrak{A}/\mathfrak{P}$, and the only element of k_E with eigenvalue 0 is 0 itself. Thus $\varpi_E^{-m} z \in \mathfrak{P}$ and $z \in \mathfrak{P}^{m+1}$, as required. $\square$

If $\mathcal{B} = (v_1, \ldots, v_N)$ is an ordered basis, we put $W_i = \langle v_i \rangle_F$ and $A_{ij} = \mathrm{End}_F(W_j, W_i)$, as before. We say that $\mathcal{B}$ is a *splitting basis for* $\mathfrak{A}$ if

$$\mathfrak{A} = \bigoplus_{1 \leq i,j \leq N} (\mathfrak{A} \cap A_{ij}).$$

In particular, any basis with respect to which $\mathfrak{A}$ is in standard form is a splitting basis. Any permutation of a splitting basis $\mathcal{B}$ is also a splitting basis; more generally, any basis obtained by a monomial change of basis from $\mathcal{B}$ is a splitting basis.

Now we specialize to the case of a minimax stratum $[\mathfrak{A}, n, n-1, \beta]$; in this case, we prove that any basis $\mathcal{B}$ with respect to which ψ_β defines a character of $U_{\mathcal{B}}$ is also a splitting basis for $\mathfrak{A}$.

Lemma 6.4. *Suppose $[\mathfrak{A}, n, n-1, \beta]$ is a minimax stratum and let $\mathcal{B} = (v_1, \ldots, v_N)$ be an ordered basis for V such that ψ_β is trivial on $U_{\mathcal{B}}^{\mathrm{der}}$. Then $\mathcal{B}$ is a splitting basis for $\mathfrak{A}$. Moreover, writing $W_i = \langle v_i \rangle_F$ and $A_{ij} = \mathrm{End}_F(W_j, W_i)$, for each $1 \leq i, j \leq N$ the lattice $\mathfrak{A} \cap A_{ij}$ depends only on the depth $n/e(\mathfrak{A}|\mathfrak{o}_F)$ of the stratum.*

Proof. Note that the result is unaffected by multiplying the v_i by scalars so, identifying V with E via $v_1 \mapsto 1$, we may assume $\mathcal{B} = (1, \beta, \ldots, \beta^{N-1})$.

We put $n = -\nu_E(\beta)$, $e = e(E/F) = e(\mathfrak{A}|\mathfrak{o}_F)$ and $f = f(E/F)$. Multiplication by n induces a bijection $\mathbb{Z}/e\mathbb{Z} \to \mathbb{Z}/e\mathbb{Z}$. Thus, for each $i = 0, \ldots, e-1$, there is a unique integer r_i, with $0 \leq r_i < e$, such that $nr_i \equiv -i \pmod{e}$. We write $nr_i = d_i e - i$; then the fact that β is minimal implies that, for each $i = 0, \ldots, e-1$, the set

$$\mathcal{B}'_i := \{\varpi_F^{nk+d_i} \beta^{ek+r_i} \mid 0 \leq k \leq f-1\}$$

reduces to a basis for $\mathfrak{p}_E^i/\mathfrak{p}_E^{i+1}$. Thus the ordered basis $\mathcal{B}'$, obtained by ordering each $\mathcal{B}'_i$ arbitrarily and then concatenating $\mathcal{B}'_0, \ldots, \mathcal{B}'_{e-1}$, is a splitting basis for $\mathfrak{A}$ with respect to which $\mathfrak{A}$ is in standard form. The change of basis matrix from $\mathcal{B}'$ to $\mathcal{B} = (1, \ldots, \beta^{ef-1})$ is monomial with entries from $\varpi_F^{\mathbb{Z}}$. Moreover, the (i, j) entry depends only on (i, j, n, e), and the result follows since n, e are determined by n/e as they are coprime. □

7. Jacquet's conjecture

Finally, we prove that if π_1, π_2 are a pair of supercuspidal representations of G containing minimax strata, then they have a special pair of Whittaker functions. When N is prime, any positive depth supercuspidal of minimal depth in its twist class contains a minimax stratum, so Jacquet's conjecture will follow from Propositions 2.3 and 4.2, together with the depth zero case from [JNS13, Corollary 1.7].

At this point, we need to recall a little more on the construction of the supercuspidal representations of G. Since it is all we will need, we only give definitions for representations which contain a minimax stratum. Thus, in a slightly different language, we are recounting the constructions of Carayol [C84].

Let $[\mathfrak{A}, n, n-1, \beta]$ be a minimax stratum, with $E = F[\beta]$. Associated to the simple stratum $[\mathfrak{A}, n, 0, \beta]$, we have the following compact open subgroups of G, contained in the normalizer $\mathfrak{K}(\mathfrak{A})$:

$$\begin{aligned} H^1 &= H^1(\beta, \mathfrak{A}) = U^1(\mathfrak{o}_E)U^{\left[\frac{n}{2}\right]+1}(\mathfrak{A}), \\ J^1 &= J^1(\beta, \mathfrak{A}) = U^1(\mathfrak{o}_E)U^{\left[\frac{n+1}{2}\right]}(\mathfrak{A}), \\ \mathbf{J} &= \mathbf{J}(\beta, \mathfrak{A}) = E^\times U^{\left[\frac{n+1}{2}\right]}(\mathfrak{A}). \end{aligned}$$

A *simple character* is then a character of H^1 which extends the character ψ_β of $U^{\left[\frac{n}{2}\right]+1}(\mathfrak{A})$. Given such a simple character θ, there is a unique irreducible representation η of J^1 which contains θ on restriction to H^1 (indeed, it is a multiple). An *extended maximal simple type* is then an irreducible representation Λ of $\mathbf{J}$ which extends η. Given such a maximal simple type, the representation

$$\operatorname{ind}_{\mathbf{J}}^{\mathrm{G}} \Lambda$$

is irreducible and supercuspidal and, moreover, every irreducible supercuspidal representation containing θ arises in this way.

Any irreducible supercuspidal representation π containing a minimax stratum $[\mathfrak{A}, n, n-1, \beta]$ contains some simple character θ associated to a simple stratum $[\mathfrak{A}, n, 0, \beta']$, with $[\mathfrak{A}, n, n-1, \beta']$ minimax and equivalent to $[\mathfrak{A}, n, n-1, \beta]$. Thus π also contains $[\mathfrak{A}, n, n-1, \beta']$ and we may assume $\beta' = \beta$.

To prove the main result of this paper, recall that we need to exhibit a special pair of Whittaker functions for two supercuspidal representations of a specific form. In [PS08], Whittaker functions are constructed which carry the properties that we need. We record the result in a form that does not require additional background. Recall first (see §6) that to β we associate a basis $\mathcal{B}$ and a unipotent subgroup $U_{\mathcal{B}}$ such that

$$\psi_\beta | U_{\mathcal{B}} = \psi_{\mathcal{B}}$$

is the nondegenerate character associated to $\mathcal{B}$.

Proposition 7.1 ([PS08, Theorem 5.8]). *There exists a Whittaker function W for π such that $\operatorname{Supp}(W) \subset U_{\mathcal{B}}\mathbf{J}$ and such that, for $g \in P_{\mathcal{B}}$,*

$$W(g) = \begin{cases} \psi_{\mathcal{B}}(u)\theta(h) & \text{if } g = uh \in (J_i^1 \cap U_{\mathcal{B}})H_i^1, \\ 0 & \text{otherwise.} \end{cases}$$

Our main result is:

Proposition 7.2. *For $i = 1, 2$, let π_i be a (positive depth) unitarizable supercuspidal representation of* G *containing a minimax stratum. Suppose that π_1, π_2 have the same depth and the same central character. Then π_1, π_2 have a special pair of Whittaker functions.*

Proof. For $i = 1, 2$, let $[\mathfrak{A}_i, n_i, n_i - 1, \beta_i]$ be a minimax stratum of period $e_i = e(\mathfrak{A}_i|\mathfrak{o}_F)$ contained in π_i, with the property that π_i also contains a simple character θ_i of $H_i^1 = H^1(\beta_i, \mathfrak{A}_i)$. Since the representations have the same depth we have $n_1/e_1 = n_2/e_2$; since they are minimax, we have $\gcd(n_i, e_i) = 1$ and we may write $n = n_1 = n_2$ and $e = e_1 = e_2$.

Fix $v \in V$ and let $g \in \mathrm{G}$ be the change of basis matrix from the basis $\mathcal{B} = (v, \beta_1 v, \ldots, \beta_1^{N-1} v)$ to $(v, \beta_2 v, \ldots, \beta_2^{N-1} v)$. Then, replacing the stratum $[\mathfrak{A}_2, n, n-1, \beta_2]$ and the simple character θ_2 by their conjugates by g, we can assume that *both* β_i are in companion matrix form with respect to $\mathcal{B}$, i.e. $\beta_1^{j-1} v = \beta_2^{j-1} v$, for $1 \leq j \leq N$. By Lemma 6.4, the hereditary orders coincide: $\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{A}$. By Lemma 6.3, we have $H_i^1 \cap P_{\mathcal{B}} = U^{[\frac{n}{2}]+1}(\mathfrak{A}) \cap P_{\mathcal{B}}$ so, by (6.2), we have

$$\theta_i | H_i^1 \cap P_{\mathcal{B}} = \psi_{\mathcal{B}},$$

independent of i. Moreover, we then have $\mathrm{Hom}_{H_i^1 \cap U_{\mathcal{B}}}(\theta_i, \psi_{\mathcal{B}}) \neq 0$.

We abbreviate $J_i^1 = J^1(\beta_i, \mathfrak{A})$ and $\mathbf{J}_i = \mathbf{J}(\beta_i, \mathfrak{A})$, and denote by η_i the unique irreducible representation of J_i^1 containing θ_i. Then we also have $\mathrm{Hom}_{J_i^1 \cap U_{\mathcal{B}}}(\eta_i, \psi_{\mathcal{B}}) \neq 0$, by [PS08, Theorem 2.6]. Writing $\pi_i = \mathrm{ind}_{\mathbf{J}_i}^{\mathrm{G}} \Lambda_i$, with Λ_i extending η_i, we have $\mathbf{J}_i \cap U_{\mathcal{B}} = J_i^1 \cap U_{\mathcal{B}}$ and we see that $\mathrm{Hom}_{\mathbf{J}_i \cap U_{\mathcal{B}}}(\Lambda_i, \psi_{\mathcal{B}}) \neq 0$. Thus we have a Whittaker function W_i as constructed in Proposition 7.1 relative to the pair $(U_{\mathcal{B}}, \psi_{\mathcal{B}})$ and these coincide on $P_{\mathcal{B}}$ since, for $g \in P_{\mathcal{B}}$,

$$W_i(g) = \begin{cases} \psi_{\mathcal{B}}(g) & \text{if } g \in (J_i^1 \cap U_{\mathcal{B}})(H_i^1 \cap P_{\mathcal{B}}), \\ 0 & \text{otherwise.} \end{cases}$$

Putting $\mathbf{K} = \mathfrak{K}(\mathfrak{A})$, both W_i are $\mathbf{K}$-special (see [JNS13, Lemma 4.2]) so we have found a special pair of Whittaker functions. □

Corollary 7.3. *For N prime, Conjecture $\mathcal{J}(N, [\frac{N}{2}])$ is true.*

Proof. Let π_1, π_2 be unitarizable supercuspidal representations of G satisfying hypothesis $\mathcal{H}_{\leq [N/2]}$, and of minimal depth in their twist classes. In particular, from hypothesis $\mathcal{H}_1$, the representations π_1, π_2 have the same central character and depth. If both have depth zero then they are equivalent by [JNS13, Corollary 1.7], so we assume they are of positive depth. Since N is prime, π_i contains a minimax stratum $[\mathfrak{A}_i, n, n_i - 1, \beta_i]$, for $i = 1, 2$. Since the representations π_1, π_2 have the same depth, we have $n_1/e_1 = n_2/e_2$ and, since n_i, e_i are coprime, in fact $n_1 = n_2$. But now Proposition 7.2 implies that π_1, π_2 have a special pair of Whittaker functions, and Proposition 2.3 implies that $\pi_1 \simeq \pi_2$. Thus Conjecture $\mathcal{J}_0(N, [\frac{N}{2}])$ is true, and the result now follows from Proposition 4.2. □

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