A LOCAL CONVERSE THEOREM FOR ARCHIMEDEAN GL(n)

MOSHE ADRIAN AND SHUICHIRO TAKEDA

Abstract. In this paper, we will prove a version of local converse theorem for GL_n over the archimedean local fields which characterizes an infinitesimal equivalence class of irreducible admissible representations of $\mathrm{GL}_n(\mathbb{R})$ (or $\mathrm{GL}_n(\mathbb{C})$) in terms of twisted local L-factors.

1. Introduction

Let F be a local field of characteristic 0, and let Irr_n be the set of (infinitesimal) equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(F)$. A so-called local converse theorem for $\mathrm{GL}_n(F)$ characterizes the set Irr_n in terms of local factors with some suitable twists. If F is non-archimedean, the first major result is the one by Henniart ([H1]) in which he shows that if two *generic* representations $\pi, \pi' \in \mathrm{Irr}_n$ are such that

$$\gamma(s, \pi \otimes \tau, \psi) = \gamma(s, \pi' \otimes \tau, \psi)$$

for all $\tau \in \mathrm{Irr}_t$ for all $t = 1, \ldots, n-1$, where the γ-factor is the one defined by Jacquet, Piatetski-Shapiro and Shalika, then $\pi = \pi'$. Later, in [Ch] Chen improved this result by requiring t be only up to $n-2$ with the extra assumption that π and π' have the same central character. It had been conjectured by Jacquet for some time that one only needs $t \leq [\frac{n}{2}]$. And very recently this conjecture has been proven by Chai in [Chai], and Jacquet and Liu in [JL] (see also [ALSX, JNS]). Let us also mention that Nien in [N] has shown an analogous result when F is a finite field.

In this paper, we will prove a local converse theorem when F is archimedean by using L-factors of Artin type (without the assumption of the genericity and the central character) with only up to GL(1)-twists for the complex case and up to GL(2)-twists for the real case. Namely, we will prove

Theorem (Complex Case). *Let $F = \mathbb{C}$. If the two representations $\pi, \pi' \in \mathrm{Irr}_n$ of $\mathrm{GL}_n(\mathbb{C})$ satisfy*

$$L(s, \pi \times \chi) = L(s, \pi' \times \chi)$$

for all characters χ on $\mathbb{C}^\times$, where the L-factors are of Artin type defined by the local Langlands correspondence, then $\pi = \pi'$.

Theorem (Real Case). *Let $F = \mathbb{R}$. If the two representations $\pi, \pi' \in \mathrm{Irr}_n$ of $\mathrm{GL}_n(\mathbb{R})$ satisfy*

$$L(s, \pi \times \tau) = L(s, \pi' \times \tau)$$

for all $\tau \in \mathrm{Irr}_t$ with $t = 1, 2$, where the L-factors are of Artin type defined by the local Langlands correspondence, then $\pi = \pi'$.

We will show that for $F = \mathbb{R}$ the bound $t \leq 2$ is sharp even if $n = 2$, namely one always needs GL(2)-twists. We will show, however, that if we assume that π and π' have the same central character and $n \leq 3$, then one only needs up to $t = 1$. However, even with the central character assumption, for $n \geq 4$ one always needs GL(2)-twists, and indeed we will give an infinite family of nonequivalent representations of $\mathrm{GL}_4(\mathbb{R})$ which cannot be distinguished by GL(1)-twists.

Let us note that the local converse theorems via gamma factors as in [H1, Ch, Chai, JL] all assume that the representations are generic. This is because gamma factors cannot distinguish the representations appearing in a same parabolically induced representation. Namely, all the constituents of a parabolically induced representation have the same gamma factor, and accordingly the genericity assumption is necessary. Our theorem should rather be considered as an archimedean analogue of the local converse theorem by Henniart in [H2] in which he characterizes the local Langlands correspondence for p-adic GL_n via L- and ϵ-factors without the genericity assumption. Of course, it makes sense to ask if it is possible to establish a local converse theorem for the archimedean case in terms of gamma factors, for generic representations. We will take up this issue in our later work.

The basic idea of our proof is that we pass to the "Galois side" via the local Langlands correspondence, so that the local L-factors, which are then essentially products of gamma functions, can be explicitly computed in terms of the data for the corresponding representations of the Weil group, and then we will compare poles of the gamma functions.

The structure of the paper is as follows: In Section 2, we will review basics of the local Langlands parameters and their local factors. In Section 3, we will introduce a certain partial order on complex numbers, which will be useful when we compare poles of local L-factors. In Sections 4 and 5, we will prove our theorems for the complex and real case, respectively. Finally in Section 6, we will discuss some of the low ranks cases.

Notations

Throughout, F is either $\mathbb{R}$ or $\mathbb{C}$. We let Irr_n be the set of infinitesimal equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(F)$. For $z \in F$, we let $|z| = \sqrt{z\bar{z}}$, so that if $F = \mathbb{R}$, this is the absolute value of z, and if $F = \mathbb{C}$, it is the usual modulus of z. We also let $\|z\| = z\bar{z} = |z|^2$. By a character, we always mean a quasi-character, and Irr_1 is the set of characters of $F^\times$. We let ψ_F be the standard choice of additive character on F; namely if $F = \mathbb{R}$, then $\psi_\mathbb{R}(r) = e^{2\pi i r}$, and if $F = \mathbb{C}$, then $\psi_\mathbb{C}(z) = \psi_\mathbb{R} \circ \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(z) = e^{2\pi i(z+\bar{z})}$. We let $\Gamma(s)$ be the gamma function. Recall that $\Gamma(s)$ satisfies

$$\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = 2^{1-s}\sqrt{\pi}\,\Gamma(s), \quad \text{(duplication formula)}.$$

Also recall that $\Gamma(s)$ has no zeroes, and has infinitely many poles, which are precisely at $s = 0, -1, -2, \ldots$ and all of which are simple.

Acknowledgements

This project was initiated when both authors attended the conference "The 2016 Paul J. Sally, Jr. Midwest Representation Theory Conference: In honor of the 70th Birthday of Philip Kutzko" at the University of Iowa, which was funded by the University of Iowa and the NSF. We would like to thank them for their support and also would like to thank the conference organizers for providing us with the good opportunity for the interaction.

The first named author was supported by a grant from the Simons Foundation #422638 and by a PSC-CUNY award, jointly funded by the Professional Staff Congress and The City University of New York.

The second named author was supported by Simons Foundations Collaboration Grant #35952, and NSA Young Investigator Grant H98230-16-1-0312.

2. Local Langlands parameters for GL_n

In this section, we recall the basics of the local Langland parameters for $\mathrm{GL}_n(F)$, i.e. the n-dimensional continuous complex representations of the Weil group of F, and their local factors. The most definitive reference is [T].

2.1. **Weil group.** We let W_F be the Weil group of F. So $W_{\mathbb{C}} = \mathbb{C}^\times$ and $W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times$ with $j^2 = -1$ and $jzj^{-1} = \bar{z}$. We naturally view $W_{\mathbb{C}} = \mathbb{C}^\times$ as a subgroup of $W_{\mathbb{R}}$. Note that $F^\times \cong W_F^{ab}$. This is obvious if $F = \mathbb{C}$. If $F = \mathbb{R}$, we have a surjective map

$$W_{\mathbb{R}} \longrightarrow \mathbb{R}^\times, \quad z \mapsto z\bar{z},\ j \mapsto -1, \tag{2.1}$$

whose kernel is the commutator group $[W_{\mathbb{R}}, W_{\mathbb{R}}]$, which is actually of the form $\{z \in \mathbb{C}^\times : |z| = 1\}$.

2.2. **Characters on $F^\times$.** We will describe all the characters of $F^\times$. If $F = \mathbb{C}$, each character is of the form

$$\chi_{-N,t}(z) := z^{-N}\|z\|^t,$$

where $N \in \mathbb{Z}$ and $t \in \mathbb{C}$. Let us note that if we write $z = re^{i\theta}$ with $r, \theta \in \mathbb{R}$ as usual, we have

$$\chi_{-N,t}(z) = r^{2t-N}e^{-iN\theta}.$$

But when dealing with the local factors, it seems to be more convenient to denote each character as $z^{-N}\|z\|^t$ instead of using $re^{i\theta}$, and hence we will choose this convention. Let us note that

$$\overline{\chi_{-N,t}} = \chi_{N,t-N},$$

where $\overline{\chi_{-N,t}}(z) := \overline{\chi_{-N,t}(z)} = \chi_{-N,t}(\bar{z})$ as usual.

If $F = \mathbb{R}$, each character is of the form

$$\lambda_{\varepsilon,t}(r) := r^{-\varepsilon}|r|^t = \mathrm{sign}(r)^\varepsilon|r|^{t-\varepsilon}, \quad r \in \mathbb{R}^\times,$$

where $\varepsilon \in \{0, 1\}$, $t \in \mathbb{C}$ and sign is the sign character.

2.3. **Representations of W_F.** If $F = \mathbb{C}$, an irreducible representation of $W_{\mathbb{C}} = \mathbb{C}^\times$ is 1-dimensional, namely a character on $\mathbb{C}^\times$, and hence of the form $\chi_{-N,t}$ as above. Then in general, an n-dimensional representation $\varphi : W_{\mathbb{C}} \to \mathrm{GL}_n(\mathbb{C})$ is of the form

$$\varphi = \chi_{-N_1,t_1} \oplus \cdots \oplus \chi_{-N_n,t_n}. \tag{2.2}$$

If $F = \mathbb{R}$, an irreducible representation of $W_{\mathbb{R}}$ is 1 or 2 dimensional. If it is 1-dimensional, it factors through $W_{\mathbb{R}}^{ab} \cong \mathbb{R}^\times$ and hence is identified with a character of the form $\lambda_{\varepsilon,t}$. If it is 2-dimensional, it is of the form

$$\varphi_{-N,t} := \mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \chi_{-N,t}.$$

From the definition of $W_{\mathbb{R}}$, we can see that $\det(\varphi_{-N,t})(z) = \chi_{-N,t}(z\bar{z}) = \|z\|^{-N+2t}$ for $z \in \mathbb{C}^\times \subseteq W_{\mathbb{R}}$, and $\det(\varphi_{-N,t})(j) = -\chi_{-N,t}(-1) = -(-1)^N$. Hence from (2.1), as a character on $\mathbb{R}^\times$ we can identify $\det(\varphi_{-N,t})$ with $\mathrm{sign}\,\chi_{-N,t}$ where $\chi_{-N,t}$ is viewed as a character on $\mathbb{R}^\times$ via the inclusion $\mathbb{R}^\times \subseteq \mathbb{C}^\times$. Namely

$$r \mapsto \mathrm{sign}(r) r^{-N} |r|^{2t}. \tag{2.3}$$

Let us mention that if $N = 0$, the representation $\varphi_{-N,t}$ is not irreducible but we have

$$\varphi_{0,t} = \lambda_{0,t} \oplus \lambda_{1,t+1}.$$

Also since $\mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \chi_{-N,t} = \mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \overline{\chi_{-N,t}}$, we have

$$\varphi_{-N,t} = \varphi_{N,t-N}. \tag{2.4}$$

Hence we may assume $N > 0$. Namely the irreducible 2-dimensional representations of $W_{\mathbb{R}}$ are precisely the representations of the form $\varphi_{-N,t}$ with $N > 0$. In general, an n-dimensional representation $\varphi : W_{\mathbb{R}} \to \mathrm{GL}_n(\mathbb{C})$ is of the form

$$\varphi = \left(\lambda_{\varepsilon_1,t_1} \oplus \cdots \oplus \lambda_{\varepsilon_p,t_p}\right) \oplus \left(\varphi_{-N_1,u_1} \oplus \cdots \oplus \varphi_{-N_q,u_q}\right) \tag{2.5}$$

where $N_i > 0$ for all i, and $p + 2q = n$.

2.4. ***L*- and ϵ-factors.** We will recall the L- and ϵ-factors of those representations. (Though we will not use the ϵ-factors in our proofs, we will include them for completeness.) Assume $F = \mathbb{C}$. Then the L- and ϵ-factors of the character $\chi_{-N,t}$ are defined as

$$\begin{aligned} L(\chi_{-N,t}) &= 2(2\pi)^{-(t-\frac{N}{2}+\frac{|N|}{2})}\Gamma(t - \frac{N}{2} + \frac{|N|}{2}) \\ &= \begin{cases} 2(2\pi)^{-t}\Gamma(t), & \text{if } N \geq 0; \\ 2(2\pi)^{-(t-N)}\Gamma(t-N), & \text{if } N < 0, \end{cases} \end{aligned} \tag{2.6}$$

$$\epsilon(\chi_{-N,t}, \psi_{\mathbb{C}}) = i^{|N|}. \tag{2.7}$$

Let us note that

$$L(\chi_{-N,t}) = L(\overline{\chi_{-N,t}}) \quad \text{and} \quad \epsilon(\chi_{-N,t}, \psi_{\mathbb{C}}) = \epsilon(\overline{\chi_{-N,t}}, \psi_{\mathbb{C}}).$$

In general, if $\varphi : W_{\mathbb{C}} \to \mathrm{GL}_n(\mathbb{C})$ is an n-dimensional representation as in (2.2), we define the local factors multiplicatively as

$$L(\varphi) = L(\chi_{-N_1,t_1}) \cdots L(\chi_{-N_n,t_n}),$$
$$\epsilon(\varphi, \psi_{\mathbb{C}}) = \epsilon(\chi_{-N_1,t_1}, \psi_{\mathbb{C}}) \cdots \epsilon(\chi_{-N_n,t_n}, \psi_{\mathbb{C}}).$$

Assume $F = \mathbb{R}$. For the 1-dimensional $\lambda_{\varepsilon,t}$,

$$L(\lambda_{\varepsilon,t}) = \pi^{-\frac{t}{2}} \Gamma\left(\frac{t}{2}\right), \tag{2.8}$$

$$\epsilon(\lambda_{\varepsilon,t}, \psi_{\mathbb{R}}) = (-i)^{\varepsilon}. \tag{2.9}$$

For the 2-dimensional representation $\varphi_{-N,t}$,

$$L(\varphi_{-N,t}) = L(\chi_{-N,t}) = \begin{cases} 2(2\pi)^{-t}\Gamma(t), & \text{if } N \geq 0; \\ 2(2\pi)^{-(t-N)}\Gamma(t-N), & \text{if } N < 0, \end{cases} \tag{2.10}$$

$$\epsilon(\varphi_{-N,t}, \psi_{\mathbb{R}}) = -i \cdot \epsilon(\chi_{-N,t}, \psi_{\mathbb{C}}) = -i^{|N|+1}. \tag{2.11}$$

In general, if $\varphi : W_{\mathbb{R}} \to \mathrm{GL}_n(\mathbb{C})$ is an n-dimensional representation as in (2.5), we again define the local factors multiplicatively as

$$L(\varphi) = L(\lambda_{\varepsilon_1,t_1}) \cdots L(\lambda_{\varepsilon_p,t_p}) \cdot L(\varphi_{-N_1,u_1}) \cdots L(\chi_{-N_q,u_q})$$
$$\epsilon(\varphi, \psi_{\mathbb{R}}) = \epsilon(\lambda_{\varepsilon_1,t_1}, \psi_{\mathbb{R}}) \cdots \epsilon(\lambda_{\varepsilon_p,t_p}, \psi_{\mathbb{R}}) \cdot \epsilon(\varphi_{-N_1,u_1}, \psi_{\mathbb{R}}) \cdots \epsilon(\chi_{-N_q,u_q}, \psi_{\mathbb{R}}).$$

Let us note that for the parameter $\varphi_{0,t}$, we can check that $L(\varphi_{0,t}) = L(\chi_{0,t}) = 2(2\pi)^{-t}\Gamma(t)$, which is indeed equal to $L(\lambda_{0,t})L(\lambda_{1,t+1}) = \pi^{-t/2}\Gamma(\frac{t}{2}) \cdot \pi^{-(t+1)/2}\Gamma(\frac{t+1}{2})$ by the duplication formula. Namely, we have

$$L(\varphi_{0,t}) = L(\lambda_{0,t})L(\lambda_{1,t+1}) \tag{2.12}$$

Also one can check that $\epsilon(\varphi_{0,t}, \psi_{\mathbb{R}}) = -i \cdot \epsilon(\chi_{0,t}, \psi_{\mathbb{C}}) = -i$, which is indeed equal to $\epsilon(\lambda_{0,t}, \psi_{\mathbb{R}})\epsilon(\lambda_{1,t+1}, \psi_{\mathbb{R}}) = -i$. Namely, we have

$$\epsilon(\varphi_{0,t}, \psi_{\mathbb{R}}) = \epsilon(\lambda_{0,t}, \psi_{\mathbb{R}})\epsilon(\lambda_{1,t+1}, \psi_{\mathbb{R}}). \tag{2.13}$$

2.5. **GL(1)-twist.** Assume $F = \mathbb{C}$. Let $\chi_{-M,s}$ be a character on $\mathbb{C}^{\times}$, and let φ be an n-dimensional representation of $W_{\mathbb{C}}$ as in (2.2). Then the twist $\varphi \otimes \chi_{-M,s}$ by $\chi_{-M,s}$ is given by

$$\varphi \otimes \chi_{-M,s} = \chi_{-(N_1+M),t_1+s} \oplus \cdots \oplus \chi_{-(N_n+M),t_n+s}. \tag{2.14}$$

Assume $F = \mathbb{R}$. Let $\lambda_{\delta,s}$ be a character on $\mathbb{R}^{\times}$. For the 1-dimensional parameter $\lambda_{\varepsilon,t}$, the twist by $\lambda_{\delta,s}$ is given by

$$\lambda_{\varepsilon,t} \otimes \lambda_{\delta,s} = \lambda_{\varepsilon+\delta\,(\mathrm{mod}\,2),\,t+s-\gamma}, \quad \text{where } \gamma = \begin{cases} 2 & \text{if } \varepsilon = \delta = 1; \\ 0, & \text{otherwise.} \end{cases}$$

For the 2-dimensional parameter $\varphi_{-N,t} = \mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}} \chi_{-N,t}$, the twisted parameter $\varphi_{-N,t}\otimes\lambda_{\delta,s}$ is computed as

$$\begin{aligned}\varphi_{-N,t} \otimes \lambda_{\delta,s} &= \mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}}(\chi_{-N,t} \otimes (\lambda_{\delta,s} \circ N_{\mathbb{C}/\mathbb{R}}))\\ &= \mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}}(\chi_{-N,t} \otimes \chi_{0,s-\delta})\\ &= \mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}} \chi_{-N,t+s-\delta},\end{aligned}$$

namely

$$\varphi_{-N,t} \otimes \lambda_{\delta,s} = \varphi_{-N,t+s-\delta}.$$

Accordingly, we have

$$L(\varphi_{-N,t} \otimes \lambda_{\delta,s}) = \begin{cases} 2(2\pi)^{-(t+s-\delta)}\Gamma(t+s-\delta), & \text{if } N \geq 0;\\ 2(2\pi)^{-(t+s-\delta-N)}\Gamma(t+s-\delta-N) & \text{if } N < 0;\end{cases} \tag{2.15}$$

$$\epsilon(\varphi_{-N,t} \otimes \lambda_{\delta,s}, \psi_\mathbb{R}) = -i^{|N|+1}. \tag{2.16}$$

If $\varphi : W_\mathbb{R} \to \mathrm{GL}_n(\mathbb{C})$ is an n-dimensional representation as in (2.5), we have

$$\varphi \otimes \lambda_{\delta,s} = \left(\lambda_{\delta_1,t_1+s-\gamma_1} \oplus \cdots \oplus \lambda_{\delta_p,t_p+s-\gamma_p}\right) \oplus \left(\varphi_{-N_1,u_1+s-\delta} \oplus \cdots \oplus \varphi_{-N_q,u_q+s-\delta}\right),$$

where $\delta_i = \varepsilon_i + \delta \pmod 2$, and $\gamma_i = 2$ if $\varepsilon_i = \delta = 1$ and $\gamma_i = 0$ otherwise.

2.6. GL(2)**-twist.** Assume $F = \mathbb{R}$. Let $\varphi_{-N,t}$ and $\varphi_{-M,s}$ be 2-dimensional representations of $W_\mathbb{R}$ as above with $N, M > 0$. Then one can see

$$\begin{aligned}\varphi_{-N,t} \otimes \varphi_{-M,s} &= \left(\mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}} \chi_{-N,t}\right) \otimes \left(\mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}} \chi_{-M,s}\right)\\ &= \left(\mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}} \chi_{-N,t} \cdot \chi_{-M,s}\right) \oplus \left(\mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}} \chi_{-N,t} \cdot \overline{\chi_{-M,s}}\right)\\ &= \left(\mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}} \chi_{-(N+M),t+s}\right) \oplus \left(\mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}} \chi_{-N,t} \cdot \chi_{M,s-M}\right)\\ &= \varphi_{-(N+M),t+s} \oplus \left(\mathrm{Ind}_{W_\mathbb{C}}^{W_\mathbb{R}} \chi_{-(N-M),t+s-M}\right)\\ &= \varphi_{-(N+M),t+s} \oplus \varphi_{-(N-M),t+s-M}.\end{aligned}$$

Then, coupled together with the identity

$$\varphi_{-(N-M),t+s-M} = \varphi_{-(M-N),t+s-N}$$

by (2.4), we obtain

$$L(\varphi_{-N,t} \otimes \varphi_{-M,s}) = 4(2\pi)^{-2(t+s)+\min\{N,M\}} \cdot \Gamma(t+s) \cdot \Gamma(t+s-\min\{N,M\}) \tag{2.17}$$

$$\epsilon(\varphi_{-N,t} \otimes \varphi_{-M,s}) = (-1)^{\max\{N,M\}}. \tag{2.18}$$

Let us note that if $N = M$, then $\varphi_{-(N-M),t+s-M}$ is reducible as mentioned earlier. But even in this case, one can check that those two formulas hold thanks to (2.12) and (2.13).

2.7. **Local Langlands correspondence for** $\mathrm{GL}_n(F)$**.** By the archimedean local Langlands correspondence, originally established by Langlands ([L]), there is a one-to-one correspondence between the set Irr_n and the set Φ_n of all continuous n-dimensional representations of W_F. If $\pi_1 \in \mathrm{Irr}_n$ and $\pi_2 \in \mathrm{Irr}_t$ correspond to $\varphi_1 \in \Phi_n$ and $\varphi_2 \in \Phi_t$, respectively, then the local factors are defined to be

$$L(s, \pi_1 \times \pi_2) = L(\varphi_1 \otimes \varphi_2 |\cdot|_F^s)$$
$$\epsilon(s, \pi_1 \times \pi_2, \psi_F) = \epsilon(\varphi_1 \otimes \varphi_2 |\cdot|_F^s, \psi_F),$$

where $|\cdot|_{\mathbb{C}} = \|\cdot\|$ and $|\cdot|_{\mathbb{R}} = |\cdot|$. Further if $\varphi \in \mathrm{Irr}_n$ corresponds to $\varphi_\pi \in \Phi_n$ and if ω_π is the central character of π, we have

$$\omega_\pi = \det(\varphi_\pi).$$

In particular, if π is a representation of $\mathrm{GL}_2(\mathbb{R})$ such that $\varphi_\pi = \varphi_{-N,t}$, then ω_π is as in (2.3).

3. A partial order

In this section we will define a certain partial order on complex numbers, which will play a crucial role in our proof.

3.1. **Definition.** We define a partial order $\preceq$ on the set $\mathbb{C}$ of complex numbers as follows: For $t_1, t_2 \in \mathbb{C}$,

$$t_1 \preceq t_2 \quad \text{if} \quad t_2 - t_1 \in \mathbb{Z}^{\geq 0},$$

and otherwise they are incomparable. We use the symbol $\prec$ for strict inequality, namely

$$t_1 \prec t_2 \quad \text{if} \quad t_1 \preceq t_2 \text{ but } t_1 \neq t_2.$$

Then one can check that $t_1 \prec t_2$ if and only if

$$\{\text{poles of } \Gamma(s+t_1)\} \supsetneq \{\text{poles of } \Gamma(s+t_2)\},$$

and t_1 and t_2 are incomparable if and only if

$$\{\text{poles of } \Gamma(s+t_1)\} \cap \{\text{poles of } \Gamma(s+t_2)\} = \emptyset.$$

If A is a finite set of complex numbers and $t \in A$ is a minimal element in A (i.e. whenever $t' \in A$ is such that $t' \preceq t$, we have $t' = t$), we say t is *minimal in* A. Of course, a minimal element might not be unique.

Let $F(s)$ be a function on $\mathbb{C}$. If $F(s)$ has a pole at $s = t$ and if it is maximal among the poles of $F(s)$ with respect to $\preceq$ (i.e. if $F(s)$ has a pole at $s = t'$ and $t \preceq t'$, then $t = t'$), then we call the pole at $s = t$ a *maximal pole* of $F(s)$. In particular, for a fixed $t \in \mathbb{C}$, the gamma function $\Gamma(s+t)$ has a unique maximal pole at $s = -t$. More generally, a product $\prod_{i=1}^n \Gamma(s+t_i)$ of gamma functions has a maximal pole precisely at $s = -t_i$ where t_i is a minimal element in the set $\{t_1, \dots, t_n\}$. Of course, again, a maximal pole is not necessarily unique in general.

3.2. **A lemma.** The following lemma will be repeatedly used throughout.

Lemma 3.1. *Let* $t_1, \ldots, t_m, t'_1, \ldots, t'_{m'} \in \mathbb{C}$. *Suppose*

$$F(s) \cdot \prod_{i=1}^{m} \Gamma(s + t_i) = \prod_{j=1}^{m'} \Gamma(s + t'_j) \tag{3.2}$$

as functions in s*, where* $F(s)$ *is a function on* $\mathbb{C}$ *whose zeros and poles do not interfere with the poles of the* $\Gamma(s + t_i)$*'s and* $\Gamma(s + t'_i)$*'s. Then* $m = m'$*, and further for each* $i \in \{1, \ldots, m\}$*, there is a corresponding* $j \in \{1, \ldots, m'\}$ *such that* $t_i = t'_j$*, namely*

$$\{t_1, \ldots, t_m\} = \{t'_1, \ldots, t'_{m'}\}$$

as multisets. Accordingly, $F \equiv 1$.

Proof. This is almost immediate if one looks at the maximal poles of both sides. Namely, if t_k is minimal in $\{t_1, \cdots, t_m\}$, the left hand side has a maximal pole at $s = -t_k$, and hence the right hand side has a maximal pole at $s = -t_k$. But each maximal pole of the right hand side occurs at $s = -t'_l$ for some t'_l which is minimal in $\{t'_1, \ldots, t'_{m'}\}$. Hence $t_k = t'_l$ for some l. After reordering the indices, if necessary, we may assume $t_1 = t'_1$. Hence (3.2) is written as

$$F(s) \cdot \prod_{i=2}^{m} \Gamma(s + t_i) = \prod_{j=2}^{m'} \Gamma(s + t'_j).$$

By arguing inductively, one proves the lemma. □

4. Proof for complex case

We will prove our main theorem for $F = \mathbb{C}$. Let π and π' be infinitesimal equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(\mathbb{C})$, and assume that their corresponding Langlands parameters φ and φ' are given by

$$\varphi = \chi_{-N_1,t_1} \oplus \cdots \oplus \chi_{-N_n,t_n}$$
$$\varphi' = \chi_{-N'_1,t'_1} \oplus \cdots \oplus \chi_{-N'_n,t'_n}.$$

The assertion $L(s, \pi \times \tau) = L(s, \pi' \times \tau)$ for all $\tau \in \mathrm{Irr}_1$ is the same as

$$L(\varphi \otimes \chi_{-M,s}) = L(\varphi' \otimes \chi_{-M,s}) \tag{4.1}$$

for all $M \in \mathbb{Z}$ and $s \in \mathbb{C}$. As in (2.14), we have

$$\varphi \otimes \chi_{-M,s} = \chi_{-(N_1+M),t_1+s} \oplus \cdots \oplus \chi_{-(N_n+M),t_n+s}$$
$$\varphi' \otimes \chi_{-M,s} = \chi_{-(N'_1+M),t'_1+s} \oplus \cdots \oplus \chi_{-(N'_n+M),t'_n+s},$$

and hence the equality (4.1) implies

$$\prod_{i=1}^{n} L(\chi_{-(N_i+M),t_i+s}) = \prod_{j=1}^{n} L(\chi_{-(N'_j+M),t'_j+s}) \tag{4.2}$$

for all $M \in \mathbb{Z}$ and $s \in \mathbb{C}$.

We will show (4.2) implies $\varphi = \varphi'$. But once we show that φ and φ' have a common constituent, the theorem follows by induction. Namely, it suffices to prove

$$\chi_{-N_i,t_i} = \chi_{-N'_j,t'_j} \tag{4.3}$$

for some $i, j \in \{1, \dots, n\}$.

The basic idea of our proof is that we will examine "maximal poles" of both sides of (4.2) for different choices of M. Our proof has three major steps, which is outlined as follows. In Step 1, we will show that $\{t_1, \dots, t_n\} = \{t'_1, \dots, t'_n\}$ as multisets. In Step 2, we will show that $\min\{N_1, \dots, N_n\} = \min\{N'_1, \dots, N'_n\}$. Finally in Step 3, we will finish up the proof of the theorem. In each of the steps we will choose an appropriate M.

4.1. **Step 1.** Let us choose M large enough so that $N_i + M > 0$ and $N'_j + M > 0$ for all $i, j \in \{1, \cdots, n\}$. By (2.6), we can write (4.2) as

$$F(s)\prod_{i=1}^{n}\Gamma(s+t_i) = \prod_{j=1}^{n}\Gamma(s+t'_j)$$

for some function $F(s)$ without a pole or zero. Hence by Lemma 3.1, we have

$$\{t_1, \dots, t_n\} = \{t'_1, \dots, t'_n\}$$

as multisets.

4.2. **Step 2.** Let us first note that we can now reorder the indices i and j in such a way that

$$t_1 = t'_1, t_2 = t'_2, \dots, t_n = t'_n. \tag{4.4}$$

Let

$$\begin{aligned} N_{\min} &:= \min\{N_1, \dots, N_n\} \\ N'_{\min} &:= \min\{N'_1, \dots, N'_n\}. \end{aligned}$$

As we mentioned above, we will show $N_{\min} = N'_{\min}$ in this step. First assume $N'_{\min} < N_{\min}$. Choose $M = -N'_{\min} - 1$. Then (4.2) is written as

$$F(s)\prod_{i=1}^{n}\Gamma(s+t_i) = \prod_{j=1}^{n}\Gamma(s+t'_j+\varepsilon_j), \tag{4.5}$$

where

$$\varepsilon_j = \begin{cases} 1, & \text{if } N'_j = N'_{\min}; \\ 0, & \text{if } N'_j > N'_{\min}, \end{cases}$$

and $\varepsilon_j \neq 0$ for at least one j. Now the left hand side of (4.5) has a maximal pole at $s = -t_k$ for some t_k which is minimal in $\{t_1, \dots, t_n\}$. Then $t_k = t'_l + \varepsilon_l$ for some l and $t'_l + \varepsilon_l$ is minimal in $\{t'_1 + \varepsilon_1, \dots, t'_n + \varepsilon_n\}$. Assume $\varepsilon_l = 1$. Then we have $t'_l + \varepsilon_l = t_l + 1$ by (4.4), which implies $t_l + 1 = t_k$. But then $t_l \prec t_k$, which contradicts the minimality of t_k. Hence $\varepsilon_l = 0$, namely $t_k = t'_l$. Then (4.5) is now written as

$$F(s)\prod_{i\neq k}\Gamma(s+t_i) = \prod_{j\neq l}\Gamma(s+t'_j+\varepsilon_j) \tag{4.6}$$

by cancelling $\Gamma(s+t_k)$ from the left and $\Gamma(s+t'_l)$ from the right. By arguing inductively, one can show that $\varepsilon_j = 0$ for all $j \in \{1, \ldots, n\}$ which is a contradiction. Hence we must have $N'_{\min} \geq N_{\min}$. By symmetry, we can show that $N'_{\min} \leq N_{\min}$. Hence $N_{\min} = N'_{\min}$.

4.3. **Step 3.** Finally, we will show (4.3) for some i and j, which will complete the proof of the theorem. For this purpose, let us reorder the indices so that

$$N_1 = N_2 = \cdots = N_d < N_{d+1} \leq \cdots \leq N_n, \tag{4.7}$$

while still keeping the relation (4.4). Such indexing is certainly possible. Now let us twist φ and φ' by $\chi_{-M,s}$ with $M = -N_1 - 1 = -N'_{\min} - 1$. Then (4.2) is now written as

$$F(s) \prod_{i=1}^{d} \Gamma(s + t_i + 1) \cdot \prod_{i=d+1}^{n} \Gamma(s + t_i) = \prod_{j=1}^{n} \Gamma(s + t'_j + \varepsilon_j) \tag{4.8}$$

where ε_j is as before and $\varepsilon_j = 1$ for at least one j. Let

$$A := A_1 \cup A_2, \quad A_1 := \{t_i + 1 : i \in \{1, \ldots, d\}\}, \quad A_2 := \{t_i : i \in \{d+1, \ldots, n\}\}$$
$$B := \{t'_j + \varepsilon_j : j \in \{1, \ldots, n\}\}.$$

Of course by Lemma 3.1, we know $A = B$ as multisets. We will again look at "maximal poles" of (4.8), which correspond to minimal elements of $A = B$. We consider the following two cases, depending on whether a minimal element is in A_1 or A_2.

First assume $t_k + 1 \in A_1$ with $k \leq d$ is minmal in A. By $A = B$, we have $t_k + 1 = t'_l + \varepsilon_l$ for some l, which is minimal in B. Assume $\varepsilon_l = 1$. Then this means $N'_l = N_1 = N_k$ and $t_k = t'_l$. Hence we have (4.3) with $i = k$ and $j = l$. Assume $\varepsilon_l = 0$. First assume $\varepsilon_k = 0$. Then $t'_k + \varepsilon_k = t_k = t'_l + \varepsilon_l - 1$, namely $t'_k + \varepsilon_k \prec t'_l + \varepsilon_l$, which contradicts the minimality of $t'_l + \varepsilon_l$. Hence $\varepsilon_k = 1$, which implies $N'_k = N_1 = N_k$. Since $t_k = t'_k$ by (4.4), the equality (4.3) is satisfied with $i = j = k$.

Next assume that $t_k \in A_2$ with $k > d$ is minmal in A. By $A = B$, we know that $t_k = t'_l + \varepsilon_l$ for some l, which is minimal in B. Assume $\varepsilon_l = 1$, so $t_k = t'_l + 1 = t_l + 1$. Then $t_l \prec t_k$. Since t_k is minimal in A, we must have $l \leq d$. Hence $N_l = N_1$. Also since $\varepsilon_l = 1$ implies $N'_l = N_1$, we have $N'_l = N_l$. Hence, we have (4.3) with $i = j = l$. Finally assume $\varepsilon_l = 0$, i.e. $t_k = t'_l$. Then we can reorder the indices j for the multiset B by swapping k and l without changing the relations (4.4) and (4.7). So we can cancel $\Gamma(s + t_k)$ from both sides of (4.8). We can now argue inductively by repeating the above arguments until we obtain (4.3).

The proof is complete.

5. Proof for real case

We will prove our main theorem for $F = \mathbb{R}$. Assume $\pi, \pi' \in \mathrm{Irr}_n$ are such that $L(s, \pi \times \tau) = L(s, \pi' \times \tau)$ for all $\tau \in \mathrm{Irr}_t$ with $t = 1, 2$. Namely if φ and φ' are the corresponding Langlands parameters, we have

$$L(\varphi \otimes \lambda_{\varepsilon,s}) = L(\varphi' \otimes \lambda_{\varepsilon,s}) \tag{5.1}$$

for all $\varepsilon \in \{0, 1\}$ and $s \in \mathbb{C}$, and

$$L(\varphi \otimes \varphi_{-M,s}) = L(\varphi' \otimes \varphi_{-M,s}) \tag{5.2}$$

for all $M \in \mathbb{Z}$ and $s \in \mathbb{C}$.

Let us write

$$\varphi = \left(\lambda_{\varepsilon_1,t_1} \oplus \cdots \oplus \lambda_{\varepsilon_p,t_p}\right) \oplus \left(\varphi_{-N_1,u_1} \oplus \cdots \oplus \varphi_{-N_q,u_q}\right)$$
$$\varphi' = \left(\lambda_{\varepsilon'_1,t'_1} \oplus \cdots \oplus \lambda_{\varepsilon'_{p'},t'_{p'}}\right) \oplus \left(\varphi_{-N'_1,u'_1} \oplus \cdots \oplus \varphi_{-N'_{q'}u'_{q'}}\right)$$

where we may assume $N_i > 0$ and $N'_j > 0$ for all i, j. Note that the numbers of 1-dimensional constituents for φ and φ' are, respectively, p and p', and those for 2-dimensional ones are, respectively, q and q', so $n = p + 2q = p' + 2q'$.

The proof has three steps, and the basic philosophy is the same as the complex case in that we choose appropriate twists and examine "maximal poles". In Step 1, we will show $\{u_1, ..., u_q\} = \{u'_1, ..., u'_{q'}\}$ and $\{t_1 - \varepsilon_1, ..., t_p - \varepsilon_p\} = \{t'_1 - \varepsilon'_1, ..., t'_{p'} - \varepsilon'_{p'}\}$ as multisets, which also implies $p = p'$ and $q = q'$. In Step 2, we will show the 1-dimensional constituents are equal. Finally in Step 3, we will show the 2-dimensional constituents are equal.

5.1. **Step 1.** Consider the twist by $\varphi_{-M,s}$ with $M \geq 0$. Since all the N_i's and N'_j's are positive, (2.15) and (2.17) give

$$L(\varphi \otimes \varphi_{-M,s}) = F(s) \prod_{i=1}^{p} \Gamma(s + t_i - \varepsilon_i) \cdot \prod_{i=1}^{q} \Gamma(s + u_i) \cdot \Gamma(s + u_i - \min\{M, N_i\})$$
$$L(\varphi' \otimes \varphi_{-M,s}) = F'(s) \prod_{j=1}^{p'} \Gamma(s + t'_j - \varepsilon'_j) \cdot \prod_{j=1}^{q'} \Gamma(s + u'_j) \cdot \Gamma(s + u'_j - \min\{M, N'_j\}),$$

where $F(s)$ and $F'(s)$ are functions without a zero or pole. And the equality (5.2) implies those two are equal. For the choices $M = 0$ (so $\min\{M, N_i\} = \min\{M, N'_j\} = 0$) and $M = 1$ (so $\min\{M, N_i\} = \min\{M, N'_j\} = 1$), Lemma 3.1 implies

(5.3)
$$\begin{aligned}&\{t_1 - \varepsilon_1, \ldots, t_p - \varepsilon_p\} \cup \{u_1, \ldots, u_q\} \cup \{u_1, \ldots, u_q\}\\ =&\{t'_1 - \varepsilon'_1, \ldots, t'_{p'} - \varepsilon'_{p'}\} \cup \{u'_1, \ldots, u'_{q'}\} \cup \{u'_1, \ldots, u'_{q'}\},\end{aligned}$$

and

(5.4)
$$\begin{aligned}&\{t_1 - \varepsilon_1, \ldots, t_p - \varepsilon_p\} \cup \{u_1, \ldots, u_q\} \cup \{u_1 - 1, \ldots, u_q - 1\}\\ =&\{t'_1 - \varepsilon'_1, \ldots, t'_{p'} - \varepsilon'_{p'}\} \cup \{u'_1, \ldots, u'_{q'}\} \cup \{u'_1 - 1, \ldots, u'_{q'} - 1\}\end{aligned}$$

as multisets. In what follows, by using (5.3) and (5.4), we will show

(5.5)
$$\{t_1 - \varepsilon_1, \ldots, t_p - \varepsilon_p\} = \{t'_1 - \varepsilon'_1, \ldots, t'_{p'} - \varepsilon'_{p'}\}$$

(5.6)
$$\{u_1, \ldots, u_q\} = \{u'_1, \ldots, u'_{q'}\}$$

as multisets, which will also imply $p = p'$ and $q = q'$.

The basic idea is to argue inductively by comparing a minimal element in both sides of (5.3), and after each step we will shrink the multisets in both (5.3) and (5.4) by eliminating the minimal element.

First assume $t_k - \varepsilon_k$ is minimal in the left hand side of (5.3). Then either $t_k - \varepsilon_k = t'_l - \varepsilon'_{l'}$ for some l, or $t_k - \varepsilon_k = u'_l$. If the former is the case, i.e.

$$t_k - \varepsilon_k = t'_l - \varepsilon'_l,$$

we can eliminate $t_k - \varepsilon_k$ from the left hand sides of both (5.3) and (5.4) and eliminate $t'_l - \varepsilon'_l$ from the right hand sides of both (5.3) and (5.4), which allows us to shrink the multisets in those two equalities and obtain

$$\begin{aligned}&\{t_i - \varepsilon_i : i \neq k\} \cup \{u_1, \ldots, u_q\} \cup \{u_1, \ldots, u_q\}\\ =&\{t'_j - \varepsilon'_j : j \neq l\} \cup \{u'_1, \ldots, u'_{q'}\} \cup \{u'_1, \ldots, u'_{q'}\},\end{aligned}$$

and

$$\begin{aligned}&\{t_i - \varepsilon_i : i \neq k\} \cup \{u_1, \ldots, u_q\} \cup \{u_1 - 1, \ldots, u_q - 1\}\\ =&\{t'_j - \varepsilon'_j : j \neq l\} \cup \{u'_1, \ldots, u'_{q'}\} \cup \{u'_1 - 1, \ldots, u'_{q'} - 1\}.\end{aligned}$$

If the latter is the case, namely if $t_k - \varepsilon_k = u'_l$, then by (5.4), $u'_l - 1 = t_k - \varepsilon_k - 1$ is a minimal element in both sides of (5.4). But since $t_k - \varepsilon_k$ is minimal in the left hand side of (5.3), $t_k - \varepsilon_k - 1$, which is strictly smaller than $t_k - \varepsilon_k$, has to be among $\{u_1 - 1, \ldots, u_q - 1\}$. Namely, $t_k - \varepsilon_k - 1 = u_m - 1$ for some m, which implies

$$u_m = u'_l.$$

Hence by eliminating u_m and u'_l, we can shrink both (5.3) and (5.4) and obtain

$$\begin{aligned}&\{t_1 - \varepsilon_1, \ldots, t_p - \varepsilon_p\} \cup \{u_i : i \neq m\} \cup \{u_i : i \neq m\}\\ =&\{t'_1 - \varepsilon'_1, \ldots, t'_{p'} - \varepsilon'_{p'}\} \cup \{u'_j : j \neq l\} \cup \{u'_j : j \neq l\},\end{aligned}$$

and

$$\begin{aligned}&\{t_1 - \varepsilon_1, \ldots, t_p - \varepsilon_p\} \cup \{u_i : i \neq m\} \cup \{u_i - 1 : i \neq m\}\\ =&\{t'_1 - \varepsilon'_1, \ldots, t'_{p'} - \varepsilon'_{p'}\} \cup \{u'_j : j \neq l\} \cup \{u'_j - 1 : j \neq l\}.\end{aligned}$$

Next assume u_k is a minimal element in the left hand side of (5.3). Then either $u_k = u'_l$ or $u_k = t'_l - \varepsilon'_l$ for some l. If the former is the case, one can shrink both sides of (5.3) and (5.4) as above. If the latter is the case, by arguing in the same way one can obtain $u_k - 1 = u'_m - 1$ for some m, which gives $u_k = u'_m$ and allows us to shrink the multisets in (5.3) and (5.4) as before.

To summarize, at each step one can obtain either $t_i - \varepsilon_i = t'_j - \varepsilon'_j$ or $u_i = u'_j$ for some i, j, and by eliminating this element from (5.3) and (5.4), one can shrink the multisets from those two equalities. But the resulting equalities are of the same form, and hence we can apply the same argument to those two shrunken equalities. By repeating the process, one can obtain (5.5) and (5.6), which also imply $p = p'$ and $q = q'$.

5.2. **Step 2.** We will show the 1-dimensional constituents of φ and φ' are equal. By twisting by $\lambda_{0,s}$, we have

$$\begin{aligned}\varphi \otimes \lambda_{0,s} &= \left(\lambda_{\varepsilon_1, t_1+s} \oplus \cdots \oplus \lambda_{\varepsilon_p, t_p+s}\right) \oplus \left(\varphi_{-N_1, u_1+s} \oplus \cdots \oplus \varphi_{-N_q, u_q+s}\right)\\ \varphi' \otimes \lambda_{0,s} &= \left(\lambda_{\varepsilon'_1, t'_1+s} \oplus \cdots \oplus \lambda_{\varepsilon'_p, t'_p+s}\right) \oplus \left(\varphi_{-N'_1, u'_1+s} \oplus \cdots \oplus \varphi_{-N'_q, u'_q+s}\right),\end{aligned}$$

so the equality $L(\varphi \otimes \lambda_{0,s}) = L(\varphi' \otimes \lambda_{0,s})$ implies

$$F(s)\prod_{i=1}^{p}\Gamma\left(\frac{s+t_i}{2}\right)\cdot\prod_{i=1}^{q}\Gamma(s+u_i)=\prod_{j=1}^{p}\Gamma\left(\frac{s+t'_j}{2}\right)\cdot\prod_{j=1}^{q}\Gamma(s+u'_j),$$

where $F(s)$ is a function without a zero or pole. Since we know $\{u_1,...,u_q\} = \{u'_1,...,u'_q\}$ as multisets, we have

$$F(s)\prod_{i=1}^{p}\Gamma\left(\frac{s+t_i}{2}\right)=\prod_{j=1}^{p}\Gamma\left(\frac{s+t'_j}{2}\right),$$

from which we have

$$\{t_1,\dots,t_p\} = \{t'_1,\dots,t'_p\} \tag{5.7}$$

as multisets by Lemma 3.1.

By using (5.5) and (5.7), we will show that the 1-dimensional constituents of φ and φ' are equal. But of course, by induction, it suffices to show $\lambda_{\varepsilon_i,t_i} = \lambda_{\varepsilon'_j,t'_j}$ for some i and j. We will show this by examining minimal elements in those multisets.

First, let us note that since we have (5.7), we may reorder the indices so that $t_i = t'_i$ for all $i \in \{1,\dots,n\}$. Let $t_k = t'_k$ be minimal in $\{t_1,\dots,t_p\}$. Clearly if $\varepsilon_k = \varepsilon'_k$, then $\lambda_{\varepsilon_k,t_k} = \lambda_{\varepsilon'_k,t'_k}$, so we are done. So let us assume $\varepsilon_k \neq \varepsilon'_k$. By symmetry, we may assume $\varepsilon_k = 1$ and $\varepsilon'_k = 0$. Note, then, that $t_k - \varepsilon_k$ is minimal in $\{t_1-\varepsilon_1,\dots,t_p-\varepsilon_p\}$. By (5.5), we must have $t_k - \varepsilon_k = t'_l - \varepsilon'_l$ for some l. Assume $\varepsilon'_l = 0$. Then $t_k - 1 = t'_l$, which implies $t'_l = t_l \prec t_k$. This contradicts the minimality of t_k. Hence $\varepsilon'_l = 1 = \varepsilon_k$. Then $t_k - \varepsilon_k = t'_l - \varepsilon'_l$ implies $t_l = t'_k$. Namely we have $\lambda_{\varepsilon_k,t_k} = \lambda_{\varepsilon'_l,t'_l}$.

This completes the proof that φ and φ' have the same 1-dimensional constituents.

5.3. **Step 3.** Finally, we will show that the 2-dimensional constituents are equal. Since we have already shown the 1-dimensional constituents are equal, we may assume

$$\begin{aligned}\varphi &= \varphi_{-N_1,u_1}\oplus\cdots\oplus\varphi_{-N_q,u_q}\\ \varphi' &= \varphi_{-N'_1,u'_1}\oplus\cdots\oplus\varphi_{-N'_{q'},u'_q}.\end{aligned}$$

Then the proof is very similar to the complex case. First note that since we already have (5.6), we can reorder the indices in such a way that

$$u_1 = u'_1, u_2 = u'_2, \dots, u_n = u'_n. \tag{5.8}$$

Let us write

$$\begin{aligned}N_{\min} &= \min\{N_1,\dots,N'_q\}\\ N'_{\min} &= \min\{N'_1,\dots,N'_q\},\end{aligned}$$

and we will show $N'_{\min} = N_{\min}$. Assume $N'_{\min} < N_{\min}$, and consider the twist by $\varphi_{-M,s}$ with $M = N_{\min}$. Then the equality $L(\varphi \otimes \varphi_{-N_{\min},s}) = L(\varphi' \otimes \varphi_{-N_{\min},s})$ is written as

$$F(s)\prod_{i=1}^{q}\Gamma(s+u_i-N_{\min})=\prod_{j=1}^{q}\Gamma(s+u'_j-M'_j),\quad M'_j=\min\{N'_j,N_{\min}\}, \tag{5.9}$$

for some $F(s)$ without a zero or a pole, where we used (2.17). Let $J = \{j \in \{1, \dots, n\} : N'_j < N_{\min}\}$. By our assumption, J is not empty. Since we have (5.8), the equality (5.9) simplifies to

$$F(s) \prod_{i \in J} \Gamma(s + u_i - N_{\min}) = \prod_{j \in J} \Gamma(s + u'_j - N'_j). \tag{5.10}$$

By Lemma 3.1, we have

$$\{u_i - N_{\min} : i \in J\} = \{u'_j - N'_j : j \in J\}$$

as multisets. Now suppose $u_k - N_{\min}$ is minimal in the left hand side, which is the same as supposing u_k is minimal in $\{u_i : i \in J\}$. There exists $l \in J$ such that $u_k - N_{\min} = u'_l - N'_l$, which gives $u_k = u'_l + (N_{\min} - N'_l)$. Since $N_{\min} - N'_l > 0$, this implies $u_l = u'_l \prec u_k$, which contradicts the minimality of u_k. Hence $N'_{\min} \geq N_{\min}$. By symmetry, we have $N'_{\min} \leq N_{\min}$, namely $N'_{\min} = N_{\min}$.

Then we can proceed analogously to Step 3 of the complex case. Namely, first let us reorder the indices so that in addition to (5.8) we have

$$0 < N_1 = \cdots = N_d < N_{d+1} \leq \cdots \leq N_q. \tag{5.11}$$

By twisting by $\varphi_{-M,s}$ with $M = N_1 + 1$, we have

$$F(s) \prod_{i=1}^{d} \Gamma(s + u_i - N_1) \prod_{i=d+1}^{q} \Gamma(s + u_i - N_1 - 1) = \prod_{j=1}^{q} \Gamma(s + u_i - N_1 - 1 + \varepsilon_j), \tag{5.12}$$

where

$$\varepsilon_j = \begin{cases} 1, & \text{if } N'_j = N_1 = N'_{\min} \\ 0, & \text{if } N'_j > N_1. \end{cases}$$

By shifting $s \mapsto s + N_1 + 1$, the equality (5.12) becomes

$$F(s + N_1 + 1) \prod_{i=1}^{d} \Gamma(s + u_i + 1) \cdot \prod_{i=d+1}^{q} \Gamma(s + u_i) = \prod_{j=1}^{q} \Gamma(s + u_i + \varepsilon_j). \tag{5.13}$$

This is exactly the same as (4.8) of the complex case. Hence by arguing in the same way, we can show that there exists a pair of indices i and j such that $u_i = u'_j$ and $N_i = N'_j$. Hence for such indices, we have $\varphi_{-N_i, u_i} = \varphi_{-N'_j, u'_j}$, namely φ and φ' have a common constituent. By arguing inductively, we have $\varphi = \varphi'$.

The proof is complete.

6. Low rank cases

In this section, we will consider low rank cases for $F = \mathbb{R}$. In particular, we will show that twisting by GL(2) is sharp in that one cannot characterize the set Irr_n only by GL(1) twists even for $n = 2$. Yet, we will show that for $n = 2, 3$, we can characterize it if we assume the central characters are equal. But for $n \geq 4$, even with the central character assumption, the GL(1)-twist is not enough.

6.1. **Case for** $\mathrm{GL}_2(\mathbb{R})$ **and** $\mathrm{GL}_3(\mathbb{R})$**.** We will prove the following which characterizes the sets Irr_2 and Irr_3 only by GL(1)-twists and the central character assumption.

Proposition 6.1. *Let $n = 2$ or 3. Assume $\pi, \pi' \in \mathrm{Irr}_n$ are such that*

$$L(s, \pi \times \lambda) = L(s, \pi' \times \lambda)$$

for all characters λ on $\mathbb{R}^\times$. Further assume that the central characters of π and π' are equal. Then $\pi = \pi'$.

Proof. We will give a proof only for $n = 3$. (The cae for $n = 2$ is simpler and left to the reader.) Assume φ and φ' are the local Langlands parameters for π and π'.

First assume

$$\varphi = \lambda_{\varepsilon_1,t_1} \oplus \lambda_{\varepsilon_2,t_2} \oplus \lambda_{\varepsilon_3,t_3} \tag{6.2}$$

$$\varphi' = \varphi_{-N',u'} \oplus \lambda_{\varepsilon',t'} \tag{6.3}$$

with $N' > 0$, and we will derive a contradiction. First note that the assumption on the central characters implies

$$-(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + t_1 + t_2 + t_3 = 2u' - N' - \varepsilon' + t', \tag{6.4}$$

where we used (2.3) to obtain the right hand side. Next note that for each $\lambda_{\delta,s}$ we have

$$\begin{aligned}\varphi \otimes \lambda_{\delta,s} &= \lambda_{\delta_1,t_1+s-\gamma_1} \oplus \lambda_{\delta_2,t_2+s-\gamma_2} \oplus \lambda_{\delta_3,t_3+s-\gamma_3}\\ \varphi' \otimes \lambda_{\delta,s} &= \varphi_{-N',u'+s-\delta} \oplus \lambda_{\delta',t'+s-\gamma'}\end{aligned}$$

where $\delta_i = \varepsilon_i + \delta \pmod 2$ and $\gamma_i = 2$ (resp. 0) if $\varepsilon_i = \delta = 1$ (resp. otherwise), and similarly for δ' and γ'. Now the equality of the L-functions implies

$$F(s)\Gamma\left(\frac{s+t_1-\gamma_1}{2}\right)\cdot\Gamma\left(\frac{s+t_2-\gamma_2}{2}\right)\cdot\Gamma\left(\frac{s+t_3-\gamma_3}{2}\right) = \Gamma(s+u'-\delta)\cdot\Gamma\left(\frac{s+t'-\gamma'}{2}\right)$$

for some $F(s)$ without a zero or pole, which is, by the duplication formula, written as

$$\begin{aligned}&F(s)\Gamma\left(\frac{s+t_1-\gamma_1}{2}\right)\cdot\Gamma\left(\frac{s+t_2-\gamma_2}{2}\right)\cdot\Gamma\left(\frac{s+t_3-\gamma_3}{2}\right)\\ =&\Gamma\left(\frac{s+u'-\delta}{2}\right)\cdot\Gamma\left(\frac{s+u'-\delta+1}{2}\right)\cdot\Gamma\left(\frac{s+t'-\gamma'}{2}\right)\end{aligned}$$

for some other $F(s)$. By Lemma 3.1, we have

$$\{t_1-\gamma_1, t_2-\gamma_2, t_3-\gamma_3\} = \{u'-\delta, u'-\delta+1, t'-\gamma'\} \tag{6.5}$$

as multisets. First choose $\delta = 0$, which forces $\gamma_1 = \gamma_2 = \gamma_3 = \gamma' = 0$, and hence we have

$$\{t_1, t_2, t_3\} = \{u', u'+1, t'\}$$

as multisets. Up to reordering of the indices, we may assume

$$t_1 = u', t_2 = u'+1, t_3 = t',$$

which makes (6.5) into

$$\{t_1-\gamma_1, t_1+1-\gamma_2, t_3-\gamma_3\} = \{t_1-\delta, t_1+1-\delta, t_3-\gamma'\}. \tag{6.6}$$

Also note that the central character assumption (6.4) implies

$$-(\varepsilon_1+\varepsilon_2+\varepsilon_3)+u'+u'+1+t' = 2u'-N'-\varepsilon'+t',$$

which gives

$$N' = (\varepsilon_1+\varepsilon_2+\varepsilon_3)-\varepsilon'-1.$$

Since $N'>0$, we must always have

$$\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon' > 1. \tag{6.7}$$

By letting $\delta=1$ in (6.6), we have

$$\{t_1-\gamma_1, t_1+1-\gamma_2, t_3-\gamma_3\} = \{t_1-1, t_1, t_3-\gamma'\}. \tag{6.8}$$

Assume $t_1-\gamma_1$ is a minimal element in the left hand side . Then we must have $\gamma_1=2$ (so $\varepsilon_1=1$), because t_1-1 is in the left hand side and so we must have $t_1-\gamma_1 \preceq t_1-1$, and hence we have

$$\{t_1-2, t_1+1-\gamma_2, t_3-\gamma_3\} = \{t_1-1, t_1, t_3-\gamma'\}.$$

Then we must have $t_1-2=t_3-\gamma'$, which implies

$$\{t_1+1-\gamma_2, t_1-2-\gamma_3+\gamma'\} = \{t_1-1, t_1\}.$$

Considering $1-\gamma_2$ is always odd, we conclude $t_1+1-\gamma_2=t_1-1$, which implies $\gamma_2=2$ (so $\varepsilon_2=1$), and $t_1-2-\gamma_3+\gamma'=t_1$, which implies $\gamma_3=0$ (so $\varepsilon_3=0$) and $\gamma'=2$ (so $\varepsilon'=1$). But this contradicts the condition (6.7) coming from the central character assumption. Hence $t_1-\gamma_1$ cannot be minimal in the left hand side of (6.8).

Suppose now that $t_1+1-\gamma_2$ is minimal. Then this implies that $\gamma_1=0$ and $\gamma_2=2$, in which case $\varepsilon_1=0$ and $\varepsilon_2=1$. Then (6.8) implies $t_3-\gamma_3=t_3-\gamma'$, which implies $\gamma_3=\gamma'$, which in turn implies $\varepsilon_3=\varepsilon'$. Then the left hand side of (6.7) will be 1, which is a contradiction. Hence neither $t_1-\gamma_1$ nor $t_2+1-\gamma_2$ can be minimal.

So $t_3-\gamma_3$ can be the only minimal element, which means

$$t_3-\gamma_3 \prec t_1-1 \prec t_1$$

or

$$t_3-\gamma_3 \prec t_1-2 \prec t_1+1-\gamma_2.$$

The first case comes from the situation where $\gamma_1=0, \gamma_2=2$, which we argued earlier cannot happen. We consider the second case. Notice that all the elements are comparable and the inequalities are all strict. Hence the same has to happen in the right hand side. But one can easily see that this is impossible because the right hand side already contains t_1-1 and t_1 while the second largest element in the left hand side is t_1-2.

Hence we conclude that we cannot have both (6.2) and (6.3) at the same time; namely, if φ is a sum of three irreducible representations, so is φ', and if φ is a sum of two irreducible representations, so is φ'.

First, assume

$$\varphi = \lambda_{\varepsilon_1,t_1}\oplus\lambda_{\varepsilon_2,t_2}\oplus\lambda_{\varepsilon_3,t_3}$$
$$\varphi' = \lambda_{\varepsilon_1',t_1'}\oplus\lambda_{\varepsilon_2',t_2'}\oplus\lambda_{\varepsilon_3',t_3'}.$$

Then the argument is similar to Step 2 of the real case. Namely by twisting by $\lambda_{0,s}$ and equating the L-factors, one can get

$$\{t_1, t_2, t_3\} = \{t'_1, t'_2, t'_3\}$$

as multisets. Also by twisting by $\lambda_{1,s}$ and equating the L-factors, one can get

$$\{t_1 - \gamma_1, t_2 - \gamma_2, t_3 - \gamma_3\} = \{t'_1 - \gamma'_1, t'_2 - \gamma'_2, t'_3 - \gamma'_3\}$$

as multisets, where $\gamma_i = 2$ if $\varepsilon_i = 1$, and $\gamma_i = 0$ if $\varepsilon_i = 0$, and similarly for γ'_i. Then the argument of Step 2 of the real case goes through with the ε's replaced by γ's. Hence we get $\varphi = \varphi'$.

Next, assume

$$\varphi = \varphi_{-N,t} \oplus \lambda_{\varepsilon,u}$$
$$\varphi' = \varphi_{-N',t'} \oplus \lambda_{\varepsilon',u'}.$$

By twisting $\lambda_{\delta,s}$, equating the L-factors and using the duplication formula, one obtains

$$\{t - \delta, t + 1 - \delta, u - \gamma\} = \{t' - \delta, t' + 1 - \delta, u' - \gamma'\}$$

where $\gamma = 2$ if $\varepsilon = \delta = 1$ and 0 otherwise, and similarly for γ'. By examining minimal elements, one can see that this implies $t = t'$, $u = u'$ and $\gamma = \gamma'$ (so $\varepsilon = \varepsilon'$). (The detail is left to the reader.) Finally the central character assumption implies $N = N'$. The proposition follows. □

Let us remark that even when $n = 2$, the assumption on the central character is necessary. Consider the following example.

Example 1. Let

$$\varphi = \varphi_{-N,t}$$
$$\varphi' = \varphi_{-N',t}$$

with $N > 0, N' > 0$ but $N \neq N'$. Then for any $\lambda_{\delta,s}$, we have

$$L(\varphi \otimes \lambda_{\delta,s}) = L(\varphi' \otimes \lambda_{\delta,s}) = 2(2\pi)^{-(t+s-\delta)}\Gamma(t + s - \delta).$$

Of course, there are infinitely many such choices for N and N'. If we further assume $N = N'$ (mod 4), not only the L-factors but the ϵ-factors coincide. Indeed, we can compute

$$\epsilon(\varphi \otimes \lambda_{\delta,s}, \psi_{\mathbb{R}}) = \epsilon(\varphi' \otimes \lambda_{\delta,s}, \psi_{\mathbb{R}}) = -i^{N+1} = -i^{N'+1},$$

and again there are infinitely many such choices.

6.2. **Case for** $\mathrm{GL}_4(\mathbb{R})$**.** Finally, the following example for $\mathrm{GL}_4(\mathbb{R})$ shows that, even with the central character assumption and the equality of the ϵ-factors, one will still need GL(2)-twists.

Example 2. Let

$$\varphi = \varphi_{-N_1,t_1} \oplus \varphi_{-N_2,t_2}$$
$$\varphi' = \varphi_{-N'_1,t'_1} \oplus \varphi_{-N'_2,t'_2}.$$

Let ω and ω' be the central characters of the representations corresponding to φ and φ', respectively. Then by (2.3) we know

$$\omega(r) = r^{-(N_1+N_2)+2(t_1+t_2)}$$
$$\omega'(r) = r^{-(N_1'+N_2')+2(t_1'+t_2')}$$

for $r \in \mathbb{R}^\times$. For each $\lambda_{\delta,s}$, we have

$$L(\varphi \otimes \lambda_{\delta,s}) = 4(2\pi)^{-(2s+t_1+t_2-2\delta)}\Gamma(s+t_1-\delta)\Gamma(s+t_2-\delta)$$
$$L(\varphi' \otimes \lambda_{\delta,s}) = 4(2\pi)^{-(2s+t_1'+t_2'-2\delta)}\Gamma(s+t_1'-\delta)\Gamma(s+t_2'-\delta)$$

and

$$\epsilon(\varphi \otimes \lambda_{\delta,s}, \psi_{\mathbb{R}}) = i^{N_1+N_2+2}$$
$$\epsilon(\varphi' \otimes \lambda_{\delta,s}, \psi_{\mathbb{R}}) = i^{N_1'+N_2'+2}.$$

Hence if we have

$$N_1 + N_2 = N_1' + N_2' \quad \text{and} \quad \{t_1, t_2\} = \{t_1', t_2'\},$$

then the L-factors, ϵ-factors and central characters will all coincide. Of course there are infinitely many such choices.

References

[ALSX] M. Adrian, B. Liu, S. Stevens, P. Xu, *On the Jacquet conjecture on the local converse problem for p-adic* GL_N, Represent. Theory 20 (2016), 1–13.

[Chai] J. Chai, *Bessel functions and local converse conjecture of Jacquet* , to appear in the Journal of the European Mathematical Society (JEMS).

[Ch] J. Chen, *The* $n\times(n-2)$ *local converse theorem for* $\mathrm{GL}(n)$ *over a p-adic field*, Journal of Number Theory 120 (2006) 193–205.

[H1] G. Henniart, *Caracterisation de la correspondance de Langlands locale par les facteurs* ε *de paires*, Invent. Math. 113 (1993) 339–350.

[H2] G. Henniart, *Une caractrisation de la correspondance de Langlands locale pour* $\mathrm{GL}(n)$, Bull. Soc. Math. France 130 (2002), 587–602.

[JNS] D. Jiang, C. Nien and S. Stevens, *Towards the Jacquet conjecture on the Local Converse Problem for p-adic* GL_n, J. Eur. Math. Soc. 17 (2015), 991–1007.

[JL] H. Jacquet and B. Liu, *On the local converse theorem for p-adic* GL_n, American Journal of Math (to appear).

[L] R. Langlands, *On the classification of irreducible representations of real algebraic groups*, Representation theory and harmonic analysis on semisimple Lie groups, 101–170, Math. Surveys Monogr., 31, Amer. Math. Soc., Providence, RI, 1989.

[N] C. Nien, *A proof of the finite field analogue of Jacquets conjecture*, Amer. J. Math. 136 (2014), 653–674.

[T] J. Tate, *Number theoretic background*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, pp. 3–26, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979.

Department of Mathematics Queens College, CUNY 65-30 Kissena Blvd., Queens, NY 11367-15971

E-mail address: moshe.adrian@qc.cuny.edu

Mathematics Department, University of Missouri-Columbia, 202 Math Sciences Building, Columbia, MO, 65211

E-mail address: takedas@missouri.edu