A REMARK ON THE KOTTWITZ HOMOMORPHISM

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Abstract. We prove that for any split, almost simple, connected reductive group G over a p-adic field F, the Kottwitz homomorphism $\kappa : G(F) \to \Omega$ exhibits a homomorphic section $\Omega \hookrightarrow G(F)$. We then extend this result to certain additional split connected reductive groups.

1. Introduction

Let G be a connected reductive group over a p-adic field F. In [Kot97], Kottwitz defined a canonical homomorphism

$$\kappa : G(F) \twoheadrightarrow X^*(Z(\widehat{G})^I)^{\mathrm{Fr}}.$$

This homomorphism is surjective and, in the case that G is split, simplifies to a homomorphism

$$\kappa : G(F) \twoheadrightarrow X^*(Z(\widehat{G})) \cong X_*(T)/Q^\vee.$$

In this note, we show that the map κ has a homomorphic section in the case that G is split and almost simple, as well as for certain additional split groups. More specifically, fix a fundamental alcove in the building of G corresponding to a maximal split torus T, and let Ω be the subgroup of the extended affine Weyl group W that stabilizes C. We show that there is a homomorphic section of the canonical projection $N_G(T) \twoheadrightarrow \Omega$, where $N_G(T)$ is the normalizer of a maximal torus T in G. If G is almost-simple, then this section can be described as follows: it is known (see Proposition 3.1) that Ω may be identified with a collection of elements $\{1, \epsilon_i \rtimes w_i\} \subset W = X_*(T) \rtimes W_\circ$, where ϵ_i are certain fundamental coweights and $W_\circ$ is the finite Weyl group. By [Spr98, §9.3.3], there is a canonical map $\mathcal{N}_\circ : W_\circ \to N_G(T)$ (denoted ϕ in *loc. cit.*) that is compatible with the projection $N_G(T) \to W_\circ$. We may then consider the map

$$\iota : \Omega \to N_G(T)$$

$$\epsilon_i w_i \mapsto \epsilon_i(\varpi^{-1})\mathcal{N}_\circ(w_i),$$

where ϖ is a uniformizer in F. The map ι is a section of the projection $N_G(T) \twoheadrightarrow \Omega$, and it turns out that ι is a homomorphism in all cases except the adjoint group of type D_l where l is odd, and some cases in type A_l (see Theorem 3.5 and Remark 3.6). Nonetheless, we can still use ι to construct a homomorphic section for all almost-simple p-adic groups (see Theorem 3.5). We then show that for certain split connected groups with connected center, the Kottwitz homomorphism exhibits a homomorphic section (see Proposition 4.3).

We would like to remark that if G is any split connected reductive group with simply connected derived group, then κ has a homomorphic section. This follows from the fact that Ω is a free abelian group of finite rank, isomorphic to a free quotient of $X_*(T)$. Then one constructs a section by taking a homomorphic section of $X_*(T) \to \Omega$ and then composing that section with the map $X_*(T) \to T$, $\lambda \mapsto \lambda(\varpi^{-1})$. In particular, the image of this section lies in T, not just $N_G(T)$. The situation where Ω is *finite* is much more subtle, which is what this paper is about.

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2. Preliminaries

Let G be a split connected reductive group over a p-adic field F. Fix a pinning $(B, T, \{X_\alpha\})$ for G. This gives rise to a set of non-zero roots Φ of G with respect to T, a set of positive roots Π in Φ, and a basis $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_l\}$ of the set of positive roots, so that l is the rank of G. We recall that for each $\alpha \in \Phi$, there exists an isomorphism u_α of F onto a unique closed subgroup U_α of G such that $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$, for $t \in T, x \in F$ [Spr98, §8.1.1].

Let $X^*(T)$, $X_*(T)$ be the character, cocharacter lattices of T, respectively. Let Q be the lattice generated by Φ, and $P^\vee$ the coweight lattice. Namely, $P^\vee$ is the $\mathbb{Z}$-dual of Q relative to the standard pairing $(\cdot,\cdot) : X^*(T) \times X_*(T) \to \mathbb{Z}$. We let $\Phi^\vee$ be the system of coroots, $Q^\vee$ the lattice generated by $\Phi^\vee$, and P the weight lattice. Then $P^\vee$ is spanned by the l fundamental coweights, which are denoted $\epsilon_1, \epsilon_2, ..., \epsilon_l$. We recall that the ϵ_i are defined by the relation $(\epsilon_i, \alpha_j) = \delta_{ij}$. If α is a root, we denote its associated coroot by $\alpha^\vee$.

We now let $W_\circ = N_G(T)/T$ be the Weyl group of G relative to T. For each $\alpha \in \Phi$, we let $s_\alpha \in W_\circ$ be the simple reflection associated to α. Then the u_α may be chosen such that for all $\alpha \in R$, $n_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$ lies in $N_G(T)$ and has image s_α in $W_\circ$ (see [Spr98, §8.1.4]). Relative to the pinning that we have chosen, there is a canonical, well-defined map $\mathcal{N}_\circ : W_\circ \to N_G(T)$ [Spr98, §9.3.3] (the map is denoted ϕ in loc.cit.), defined by $\mathcal{N}_\circ(w) = n_{\beta_1} n_{\beta_2} \cdots n_{\beta_m}$ for a reduced expression $w = s_{\beta_1} s_{\beta_2} \cdots s_{\beta_m}$.

2.1. The map $\mathcal{N}_\circ$.

In this section, we recall a result about the map $\mathcal{N}_\circ$ from [Ros16].

Definition 2.1. For $u, v \in W_\circ$, we define

$$\mathcal{F}(u, v) = \{\alpha \in \Pi \mid v(\alpha) \in -\Pi, u(v(\alpha)) \in \Pi\}.$$

The following proposition describes the failure of $\mathcal{N}_\circ$ to be a homomorphism.

Proposition 2.2. [Ros16, Proposition 3.1.2] *For $u, v \in W_\circ$,*

$$\mathcal{N}_\circ(u) \cdot \mathcal{N}_\circ(v) = \mathcal{N}_\circ(u \cdot v) \cdot \prod_{\alpha \in \mathcal{F}(u,v)} \alpha^\vee(-1).$$

Definition 2.3. For $w \in W_\circ, i \in \mathbb{N}$, we define

$$\mathcal{F}_w(i) = \{\alpha \in \Pi \mid w^i(\alpha) \in -\Pi, w^{i+1}(\alpha) \in \Pi\}.$$

Corollary 2.4. *If $w \in W_\circ$ and $n \in \mathbb{N}$, then*

$$\mathcal{N}_\circ(w)^n = \mathcal{N}_\circ(w^n) \cdot \prod_{m=1}^{n-1} \prod_{\alpha \in \mathcal{F}_w(m)} \alpha^\vee(-1).$$

Proof. By Proposition 2.2, $\mathcal{N}_\circ(w)^2 = \mathcal{N}_\circ(w^2) \cdot \prod_{\alpha \in \mathcal{F}_w(1)} \alpha^\vee(-1)$. Multiplying by $\mathcal{N}_\circ(w)$ on the left and using Proposition 2.2 again, we get $\mathcal{N}_\circ(w)^3 = \mathcal{N}_\circ(w^3) \cdot \prod_{\alpha \in \mathcal{F}_w(2)} \alpha^\vee(-1) \cdot \prod_{\alpha \in \mathcal{F}_w(1)} \alpha^\vee(-1)$. Continuing in this way, the claim follows. □

3. Embedding Ω into G

Let G be a split, almost-simple p-adic group. We set $W = N_G(T)/T_\circ$, where $T_\circ$ is the maximal bounded subgroup of T. The group W is the extended affine Weyl group, and we note that we have a semidirect product decomposition $W = X_*(T) \rtimes W_\circ$. We also set $\Omega = W/W^\circ$, where $W^\circ = Q^\vee \rtimes W_\circ$ is the affine Weyl group. We therefore have a canonical projection $N_G(T) \to \Omega$. This projection is exactly the restriction of κ to $N_G(T)$.

The group Ω can be identified with the subgroup of W that stabilizes a fundamental alcove $\mathcal{C}$. Moreover, it is known that Ω acts on the set $\{1 - \alpha_0, \alpha_1, \alpha_2, ..., \alpha_l\}$, where α_0 is the highest root in Φ. The action of Ω on this set can be found in [IM65, p. 18-19]. We let Ω_{ad} be the analogous group for the adjoint group G_{ad}.

It is known that there exists in $W_\circ$ an element w_Δ such that $w_\Delta(\Delta) = -\Delta$. The element w_Δ is unique and satisfies $w_\Delta^2 = 1$. Moreover, if we denote the subset $\Delta - \{\alpha_i\}$ by Δ_i, then the subgroup W_i of $W_\circ$ generated by $s_{\alpha_1}, ..., \hat{s}_{\alpha_i}, ..., s_{\alpha_l}$ ($\hat{s}_{\alpha_i}$ means that s_{α_i} is omitted) contains an element w_{Δ_i} such that $w_{\Delta_i}(\Delta_i) = -\Delta_i$ and $w_{\Delta_i}^2 = 1$.

We recall the following result from [IM65].

Proposition 3.1. [IM65, Proposition 1.18] *The mapping from the set* $\{0\} \cup \{\epsilon_i : (\alpha_0, \epsilon_i) = 1\}$ *onto* Ω_{ad} *defined by* $0 \mapsto 1, \epsilon_i \mapsto \epsilon_i w_{\Delta_i} w_\Delta$ *is bijective.*

The notation ρ_i (and sometimes ρ) is used in [IM65] to denote the element $\epsilon_i w_{\Pi_i} w_\Delta$. We will adopt the same notation. We will also let S_{ad} denote the set $\{0\} \cup \{\epsilon_i : (\alpha_0, \epsilon_i) = 1\}$. We note that every lattice between $Q^\vee$ and $P^\vee$ arises as $\langle Q^\vee, S\rangle$, for some subset $S \subset S_{\mathrm{ad}}$. If S is such a subset, we will talk of the almost-simple p-adic group G that is determined by the lattice $\langle Q^\vee, S\rangle$. We note in particular that if Ω_G denotes the omega group for G, then one can see that $\Omega_G = W/W^\circ \cong \langle 1, \rho_i : \epsilon_i \in S\rangle$.

We now assume that G is not simply connected. For otherwise, $\Omega = 1$, so the claim that $\kappa : G \to \Omega$ has a homomorphic section is vacuous.

Let ϖ be a uniformizer of F. There is a natural map $X_*(T) \to N_G(T)$ given by $\lambda \mapsto \lambda(\varpi^{-1})$ (see [Tit79, p. 31]). We also have the map $\mathcal{N}_\circ : W_\circ \to N_G(T)$. Coupling these maps together, we obtain a natural map

$$W \to N_G(T)$$
$$(\lambda, w) \mapsto \lambda(\varpi^{-1})\mathcal{N}_\circ(w)$$

for $\lambda \in X_*(T), w \in W_\circ$. Most of the time, we will write λw instead of (λ, w). Proposition 3.1 gives us a set-theoretic embedding $\Omega \hookrightarrow W$. We can then consider the composite map $\Omega \hookrightarrow W \to N_G(T)$, which gives us a section of the canonical projection $N_G(T) \to \Omega$:

$$\iota : \Omega \to N_G(T)$$
$$\omega = \epsilon_i w_{\Delta_i} w_\Delta \mapsto \epsilon_i(\varpi^{-1})\mathcal{N}_\circ(w_{\Delta_i} w_\Delta)$$

That $\epsilon_i(\varpi^{-1})$ is well-defined follows from the fact that $\epsilon_i \in X_*(T)$ by our definition of G earlier. That ι is a section follows from Proposition 3.1. In particular, ι is injective. We will sometimes identify ϵ_i with $\epsilon_i(\varpi^{-1})$ for ease of notation.

We will show that ι is a homomorphic embedding for all types except A_l, and the specific case when G is adjoint of type D_l where l is odd. Nonetheless, we will still produce a homomorphic embedding $\Omega \hookrightarrow N_G(T)$ which is a section of $N_G(T) \to \Omega$, in these two outlier cases.

Suppose $\omega = \epsilon_i w_{\Delta_i} w_\Delta$ is a generator of Ω, whose order is r. Propositions 3.2 and 3.3 will be dedicated to showing that $\iota(\omega)$ also has order r. Let $w_i = w_{\Delta_i} w_\Delta$ for convenience of notation. We compute

$$\iota(\omega)^r = (\epsilon_i(\varpi^{-1})\mathcal{N}_\circ(w_i))^r = \epsilon_i(\varpi^{-1}) \cdot \left(\mathcal{N}_\circ(w_i)\epsilon_i(\varpi^{-1})\mathcal{N}_\circ(w_i)^{-1}\right) \cdot \left(\mathcal{N}_\circ(w_i)^2\epsilon_i(\varpi^{-1})\mathcal{N}_\circ(w_i)^{-2}\right)$$
$$\cdots \left(\mathcal{N}_\circ(w_i)^{r-1}\epsilon_i(\varpi^{-1})\mathcal{N}_\circ(w_i)^{1-r}\right)\mathcal{N}_\circ(w_i)^r = (\epsilon_i + w_i(\epsilon_i) + w_i^2(\epsilon_i) + \cdots + w_i^{r-1}(\epsilon_i))(\varpi^{-1})\mathcal{N}_\circ(w_i)^r.$$

We will now show that $\epsilon_i + w_i(\epsilon_i) + w_i^2(\epsilon_i) + \cdots + w_i^{r-1}(\epsilon_i) = 0$ and $\mathcal{N}_\circ(w_i)^r = 1$.

Proposition 3.2. $\epsilon_i + w_i(\epsilon_i) + w_i^2(\epsilon_i) + \cdots + w_i^{r-1}(\epsilon_i) = 0.$

Proof. We compute $w_i^j(\epsilon_i)$ for $j = 1, 2, ..., r-1$. The tables on pages 18-19 of [IM65] give the values of i for each type, and the explicit action of w_i on the set $\{-\alpha_0, \alpha_1, \alpha_2, ..., \alpha_l\}$. We also note that the order of ω equals the order of w_i (see [IM65, p. 18]). We begin with type B_l and end with type A_l (since, computationally, A_l is the most intricate).

- In type B_l, we have that $r = 2$ and $i = 1$, so we wish to show that $\epsilon_1 + w_1(\epsilon_1) = 0$. Note that $w_1(\alpha_1) = -\alpha_0$ and w_1 fixes the other simple roots. To compute $w_1(\epsilon_1)$, we pair $w_1(\epsilon_1)$ with all of the simple roots. By Weyl-invariance of the inner product $(\cdot, \cdot)$ and the fact that $w_1^2 = 1$, we have

$$(w_1(\epsilon_1), \alpha_j) = \begin{cases} 0 & \text{if} \quad j \neq 1 \\ (\epsilon_1, -\alpha_0) & \text{if} \quad j = 1 \end{cases}$$

  But since $\alpha_0 = \alpha_1 + 2(\alpha_2 + ... + \alpha_l)$, we have that $(\epsilon_1, -\alpha_0) = -1$. Therefore, $w_1(\epsilon_1) = -\epsilon_1$, so that $\epsilon_1 + w_1(\epsilon_1) = 0$.
- In type C_l, we have that $i = l$, and the same argument as in type B_l holds. Indeed, $w_l(\alpha_l) = -\alpha_0$, w_l permutes the simple roots other than α_l, and $\alpha_0 = 2(\alpha_1 + ... + \alpha_{l-1}) + \alpha_l$. Thus, $(w_l(\epsilon_l), \alpha_l) = -1$, so $w_l(\epsilon_l) = -\epsilon_l$, and the result follows.

- We consider type D_l. We first consider the case that l is odd and G is adjoint. In this case $\Omega \cong \mathbb{Z}/4\mathbb{Z}$ and it is enough to consider $i = l$. The claim is that $\epsilon_l + w_l(\epsilon_l) + w_l^2(\epsilon_l) + w_l^3(\epsilon_l) = 0$. One can see from the table on [IM65, p. 19] that w_l permutes $\alpha_2, \alpha_3, ..., \alpha_{l-2}$, and also acts by $-\alpha_0 \mapsto \alpha_l \mapsto \alpha_1 \mapsto \alpha_{l-1} \mapsto -\alpha_0$. We therefore conclude that $(w_l(\epsilon_l), \alpha_j) = (\epsilon_l, w_l^3(\alpha_j)) = 0$ if $j = 2, 3, ..., l-2$. Moreover, since $w_l^3(\alpha_1) = \alpha_l, w_l^3(\alpha_{l-1}) = \alpha_1$, $w_l^3(\alpha_l) = -\alpha_0$, we conclude that $(w_l(\epsilon_l), \alpha_1) = 1, (w_l(\epsilon_l), \alpha_{l-1}) = 0$, and $(w_l(\epsilon_l), \alpha_l) = -1$. Therefore, $w_l(\epsilon_l) = \epsilon_1 - \epsilon_l$. One can compute similarly that $w_l^2(\epsilon_l) = \epsilon_{l-1} - \epsilon_1$ and $w_l^3(\epsilon_l) = -\epsilon_{l-1}$. Therefore, $\epsilon_l + w_l(\epsilon_l) + w_l^2(\epsilon_l) + w_l^3(\epsilon_l) = 0$.

  We now consider the case where l is odd and G is neither adjoint nor simply connected. We have that $\rho_l^2 = \rho_1$ generates Ω. Thus, we need to show that $\epsilon_1 + w_1(\epsilon_1) = 0$. First, we note that w_1 fixes α_j, for $j = 2, 3, ..., l-2$, it exchanges $-\alpha_0$ and α_1, and it exchanges α_{l-1} and α_l. Since w_1 has order 2, we compute that
$$(w_1(\epsilon_1), \alpha_j) = (\epsilon_1, w_1(\alpha_j)) = 0$$
  if $j = 2, ..., l$. We also have $(w_1(\epsilon_1), \alpha_1) = (\epsilon_1, -\alpha_0) = -1$. Thus, $w_1(\epsilon_1) = -\epsilon_1$, so the result follows.

  We now consider the case that l is even and G is adjoint. In this case, $\Omega \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In the notation of [IM65, p. 19], the generators of Ω are $\rho_1, \rho_{l-1}, \rho_l$. It is straightforward to compute that $w_1(\epsilon_1) = -\epsilon_1, w_l(\epsilon_l) = -\epsilon_l$, and that $w_{l-1}(\epsilon_{l-1}) = -\epsilon_{l-1}$, proving the claim for G.

  If l is even and G is neither adjoint nor simply connected, the result follows readily from the adjoint case.
- We now consider type E_6. Then $r = 3, i = 1$, and w_1 acts by $\alpha_1 \mapsto \alpha_6 \mapsto -\alpha_0$. Since w_1 has order 3, we compute that
$$(w_1(\epsilon_1), \alpha_j) = (\epsilon_1, w_1^2(\alpha_j)) = \begin{cases} 0 & \text{if} \quad j \neq 1, 6 \\ -1 & \text{if} \quad j = 1 \\ 1 & \text{if} \quad j = 6 \end{cases}$$
  which implies that $w_1(\epsilon_1) = -\epsilon_1 + \epsilon_6$. Similarly one may compute that $w_1^2(\epsilon_1) = -\epsilon_6$. Therefore, $\epsilon_1 + w_1(\epsilon_1) + w_1^2(\epsilon_1) = 0$.
- Type E_7 is analogous to types B_l and types C_l. Just note that in this case we have $i = 1$ and $w_1(\alpha_1) = -\alpha_0$, and from [IM65, p. 19] we see that the coefficient of α_1 in α_0 is 1.
- We finally consider type A_l. We may identify roots and co-roots, fundamental weights and fundamental co-weights. Recall that we may take $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_l\}$ to be $\alpha_i = \alpha_i^\vee = e_i - e_{i+1}$ for $1 \leq i \leq l$. The corresponding fundamental coweights are
$$\epsilon_i = \epsilon_i^\vee = \left\{ \tfrac{1}{l+1}[(l+1-i)(e_1 + e_2 + ... + e_i) - i(e_{i+1} + e_{i+2} + ... + e_{l+1})] \quad \text{if} \quad i \leq l \right.$$
  We recall that the isogenies of type A_l are in one to one correspondence with the subgroups of $\Omega_{\text{ad}} = \mathbb{Z}/(l+1)\mathbb{Z}$. The element ρ_1 generates Ω_{ad}. Let $a, b \in \mathbb{N}$ such that $l + 1 = ab$. Let $\omega = \rho_1^a$, so that $\omega^b = 1$. Let G be the group of type A_l that is given by the subgroup $\langle \omega \rangle$ of Ω_{ad}. In particular, its associated cocharacter lattice, which we denote by $X_*(A_l^a)$, is given by $\langle Q^\vee, \epsilon_a \rangle$. For ease of notation, let $n = l + 1$. Then $\omega = \epsilon_a w_a$, where w_a is the a-th power of the n-cycle $(1\ 2\ \cdots\ n)$. We need to show that $\epsilon_a + w_a(\epsilon_a) + ... + w_a^{b-1}(\epsilon_a) = 0$. A computation shows that this sum is
$$\frac{1}{n}[(n-a)(e_1 + e_2 + ... + e_a) - a(e_{a+1} + ... + e_n)]$$
$$+\frac{1}{n}[(n-a)(e_{a+1} + e_{a+2} + ... + e_{2a}) - a(e_{2a+1} + ... + e_n + e_1 + e_2 + ... + e_a)]$$
$$+ ... + \frac{1}{n}[(n-a)(e_{n-a+1} + e_{n-a+2} + ... + e_n) - a(e_1 + e_2 + ... + e_{n-a})],$$
  which equals zero.

$\square$

Proposition 3.3. $\mathcal{N}_\circ(w_i)^r = 1$.

Proof. We again assume that G is not simply connected. We proceed on a type by type basis, beginning with B_l and ending again with type A_l. To compute $\mathcal{N}_\circ(w_i)^r$, we use Corollary 2.4. We remind the reader again that the order of $\epsilon_i w_i$ equals the order of w_i (see [IM65, p. 18]).

- Suppose that G is of type B_l and adjoint. We recall that the roots may be identified with the functionals $\pm e_i (1 \leq i \leq l)$ and $\pm e_i \pm e_j (1 \leq i < j \leq l)$. The corresponding coroots may be identified (in the obvious way) with the functionals $\pm 2e_i, \pm e_i \pm e_j$. The fundamental coweights corresponding to the standard choice of simple roots are given by $\epsilon_i = e_1 + ... + e_i$, for $1 \leq i \leq l$.

  We note that the cocharacter lattice of type B_l adjoint is $\langle Q^\vee, \epsilon_1 \rangle$. The action of w_1 exchanges $-\alpha_0$ and α_1. Therefore, $\mathcal{F}_{w_1}(1)$ is the set of positive roots that contain α_1. In other words, $\mathcal{F}_{w_1}(1) = \{e_1 + e_j : j = 2, 3, ..., l\} \cup \{e_1 - e_j : j = 2, 3, ..., l\} \cup \{e_1\}$. One may therefore compute that
$$\sum_{\alpha \in \mathcal{F}_{w_1}(1)} \alpha^\vee = \sum_{j>1} (e_1 + e_j)^\vee + \sum_{j>1} (e_1 - e_j)^\vee + e_1^\vee = \sum_{j>1} (e_1 + e_j) + \sum_{j>1} (e_1 - e_j) + 2e_1,$$
  where we have identified $e_1^\vee$ with $2e_1$ in the usual way. Writing $e_1 + e_j$ and $e_1 - e_j$ as sums of simple coroots, one may compute that
$$\sum_{j>1} (e_1 + e_j) + \sum_{j>1} (e_1 - e_j) + 2e_1 = 2l\alpha_1^\vee + 2l\alpha_2^\vee + ... + 2l\alpha_{l-1}^\vee + l\alpha_l^\vee.$$
  Noting that $\epsilon_1 = \alpha_1^\vee + \alpha_2^\vee + ... + \alpha_{l-1}^\vee + \frac{1}{2}\alpha_l^\vee$, we have that $\sum_{\alpha \in \mathcal{F}_{w_1}(1)} \alpha^\vee = 2l\epsilon_1$. Therefore, $\mathcal{N}_\circ(w_1)^2 = (\epsilon_1)(-1)^{2l} = 1$.
- We now turn to type C_l adjoint. We recall that the roots may be identified with the functionals $\pm 2e_i (1 \leq i \leq l)$ and $\pm e_i \pm e_j (1 \leq i < j \leq l)$. The corresponding coroots may be identified (in the obvious way) with the functionals $\pm e_i, \pm e_i \pm e_j$. The fundamental coweights corresponding to the standard choice of simple roots are $\epsilon_i = e_1 + ... + e_i$, for $1 \leq i < l$, and $\epsilon_l = \frac{1}{2}(e_1 + e_2 + ... + e_l)$.

  We note that the cocharacter lattice of type C_l adjoint is $\langle Q^\vee, \epsilon_l \rangle$. The action of w_l exchanges $-\alpha_0$ and α_l. Therefore, $\mathcal{F}_{w_l}(1)$ is the set of all positive roots that contain α_l, so that $\mathcal{F}_{w_l}(1) = \{e_i + e_j : 1 \leq i < j \leq l\} \cup \{2e_i : 1 \leq i \leq l\}$. Therefore, one may compute that
$$\sum_{\alpha \in \mathcal{F}_{w_l}(1)} \alpha^\vee = \sum_{i<j} (e_i + e_j)^\vee + \sum_i (2e_i)^\vee = \sum_{i<j} (e_i + e_j) + \sum_i e_i.$$
  Writing $e_i + e_j$ and e_i as sums of simple coroots, we may compute that
$$\sum_{i<j} (e_i + e_j) + \sum_i e_i = l\alpha_1^\vee + 2l\alpha_2^\vee + 3l\alpha_3^\vee + ... + l^2\alpha_l^\vee.$$
  Recalling that $\epsilon_l = \frac{1}{2}(\alpha_1^\vee + 2\alpha_2^\vee + 3\alpha_3^\vee + ... + l\alpha_l^\vee)$, we have that $\sum_{\alpha \in \mathcal{F}_{w_l}(1)} \alpha^\vee = 2l\epsilon_l$. Therefore, $\mathcal{N}_\circ(w)^2 = (\epsilon_l)(-1)^{2l} = 1$.
- We now consider type D_l. We recall that the root system of type D_l is realized as the set of all $\pm e_i \pm e_j$, with $i < j$. Since all roots α satisfy $||\alpha||^2 = 2$, we may identify roots and co-roots, fundamental weights and fundamental co-weights. Recall that we may take $\Delta = \{\alpha_1, \alpha_2, ..., \alpha_l\}$ to be
$$\alpha_i = \alpha_i^\vee = \begin{cases} e_i - e_{i+1} & \text{if} \quad i \leq l-1 \\ e_{l-1} + e_l & \text{if} \quad i = l \end{cases}$$
  The corresponding fundamental weights are
$$\epsilon_i = \epsilon_i^\vee = \begin{cases} e_1 + ... + e_i & \text{if} \quad i < l-1 \\ \frac{1}{2}(e_1 + ... + e_{l-1} - e_l) & \text{if} \quad i = l-1 \\ \frac{1}{2}(e_1 + ... + e_{l-1} + e_l) & \text{if} \quad i = l \end{cases}$$
  First we consider the case where l is even. By [IM65, p. 19], Ω_{ad} is generated by the elements $\rho_1, \rho_{l-1}, \rho_l$, and the actions of their corresponding Weyl elements w_1, w_{l-1}, w_l on the set

$\{-\alpha_0, \alpha_1, \alpha_2, ..., \alpha_l\}$ are given by

$$w_1(-\alpha_0) = \alpha_1, \quad w_1(\alpha_1) = -\alpha_0, \quad w_1(\alpha_i) = \alpha_i \qquad (2 \leq i \leq l-2)$$

$$w_1(\alpha_{l-1}) = \alpha_l, \quad w_1(\alpha_l) = \alpha_{l-1}.$$

$$w_l(-\alpha_0) = \alpha_l, \quad w_l(\alpha_l) = -\alpha_0, \quad w_l(\alpha_i) = \alpha_{l-i} \quad (1 \leq i \leq l-1). \qquad w_{l-1} = w_l w_1.$$

Consider the isogeny given by the subgroup $\langle \rho_l \rangle$ in Ω_{ad}. Its associated cocharacter lattice, which we denote by $X_*(D_l^l)$, is given by $X_*(D_l^l) = \langle Q^\vee, \epsilon_l \rangle$. We must compute $\mathcal{F}_{w_l}(1)$. As w_l exchanges $-\alpha_0$ and α_l, $\mathcal{F}_{w_l}(1)$ is the set of positive roots that contain α_l. By [Bou02, Plate IV], the sum of all of the (co)roots in $\mathcal{F}_{w_l}(1)$ is then equal to

$$(l-1)(\alpha_1 + 2\alpha_2 + 3\alpha_3 + ... + (l-2)\alpha_{l-2} + \frac{1}{2}(l-2)\alpha_{l-1} + \frac{1}{2}l\alpha_l).$$

But notice that $\epsilon_l = \frac{1}{2}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + ... + (l-2)\alpha_{l-2} + \frac{1}{2}(l-2)\alpha_{l-1} + \frac{1}{2}l\alpha_l)$. Therefore, $\mathcal{N}_\circ(w_l)^2 = (\epsilon_l)(-1)^{2(l-1)} = 1$.

We now consider the isogeny given by the subgroup $\langle \rho_1 \rangle$ of Ω_{ad}. Its associated cocharacter lattice, which we denote by $X_*(D_l^1)$, is given by$X_*(D_l^1) = \langle Q^\vee, \epsilon_1 \rangle$. Since w_1 exchanges $-\alpha_0$ and α_1, $\mathcal{F}_{w_1}(1)$ is the set of all positive roots containing α_1. By [Bou02, Plate IV], the sum of all of the (co)roots in $\mathcal{F}_{w_1}(1)$ is

$$(l-1)(2\alpha_1 + 2\alpha_2 + 2\alpha_3 + ... + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l).$$

But notice that $\epsilon_1 = \alpha_1 + \alpha_2 + \alpha_3 + ... + \alpha_{l-2} + \frac{1}{2}\alpha_{l-1} + \frac{1}{2}\alpha_l$, so $\mathcal{N}_\circ(w_1)^2 = (\epsilon_1)(-1)^{2(l-1)} = 1$.

We now consider the isogeny given by the subgroup $\langle \rho_{l-1} \rangle$ of Ω_{ad}, whose associated cocharacter lattice we denote by $X_*(D_l^{l-1})$. To describe $\mathcal{F}_{w_{l-1}}(1)$, we must describe all positive roots that contain α_{l-1}. By [Bou02, Plate IV], the sum of all of the (co)roots in $\mathcal{F}_{w_{l-1}}(1)$ is

$$(l-1)(\alpha_1 + 2\alpha_2 + 3\alpha_3 + ... + (l-2)\alpha_{l-2} + \frac{l}{2}\alpha_{l-1} + \frac{l-2}{2}\alpha_l).$$

But $X_*(D_l^{l-1}) = \langle Q^\vee, \epsilon_{l-1} \rangle$, and $\epsilon_{l-1} = \frac{1}{2}(\alpha_1 + 2\alpha_2 + 3\alpha_3 + ... + (l-2)\alpha_{l-2} + \frac{l}{2}\alpha_{l-1} + \frac{l-2}{2}\alpha_l)$. Thus, $\mathcal{N}_\circ(w_{l-1})^2 = (\epsilon_{l-1})(-1)^{2(l-1)} = 1$.

We now turn to D_l with l odd. First we consider the adjoint case, denoting the associated cocharacter lattice by $X_*(D_l^{ad})$. To show that $\mathcal{N}_\circ(w_l)^4 = 1$, we need to compute the sum

$$\gamma := \sum_{\alpha \in \mathcal{F}_{w_l}(1)} \alpha + \sum_{\beta \in \mathcal{F}_{w_l}(2)} \beta + \sum_{\delta \in \mathcal{F}_{w_l}(3)} \delta.$$

Noting that

$$\mathcal{F}_{w_l}(1) = \{\alpha \in \Pi : \alpha \text{ contains } \alpha_{l-1} \text{ but doesn't contain } \alpha_1\}$$

$$\mathcal{F}_{w_l}(2) = \{\alpha \in \Pi : \alpha \text{ contains } \alpha_1 \text{ but doesn't contain } \alpha_l\}$$

$$\mathcal{F}_{w_l}(3) = \{\alpha \in \Pi : \alpha \text{ contains } \alpha_l\},$$

one computes that

$$\gamma = 2(l-2)(\alpha_1 + 2\alpha_2 + ... + (l-2)\alpha_{l-2}) + 2(\alpha_1 + \alpha_2 + ... + \alpha_{l-2})$$
$$+((l-2)(l-1)+1)\alpha_{l-1} + \left(\frac{(l-3)(l-2)}{2} + \frac{(l-1)l}{2}\right)\alpha_l.$$

Modulo $2X_*(D_l^{ad})$, γ is equivalent to $\alpha_{l-1}+\alpha_l$. But $\alpha_{l-1}+\alpha_l \equiv 2\epsilon_1 \pmod{2X_*(D_l^{ad})}$, so $\mathcal{N}_\circ(w_l)^4 = 1$ as needed.

We now consider the group G, of type D_l, with l odd, that is neither simply connected nor adjoint. We need to show that $\mathcal{N}_\circ(w_l^2)^2 = 1$. First, we recall that w_l^2 fixes α_i, for $i = 2, 3, ..., l-2$ and it exchanges $-\alpha_0$ and α_1, and exchanges α_{l-1} and α_l. We must therefore count the positive roots that contain α_1. But this has already been computed in the D_l cases with l even, and our results there imply that $\mathcal{N}_\circ(w_l^2)^2 = 1$, noting that the cocharacter lattice in the current case is given by $\langle Q^\vee, \epsilon_1 \rangle$.

- We now turn to the group G of type E_6 and adjoint. We follow here [Bou02, Plate V], which has different conventions than [IM65]. The Weyl element w in question acts by $\alpha_1 \mapsto \alpha_6 \mapsto -\alpha_0$. Therefore, we need to compute the sum of all roots α that contain α_1, together with all roots that contain α_6 that also do not contain α_1 . One computes that this sum is
$$\sum_{\alpha\in\mathcal{F}_w(1)} \alpha^\vee + \sum_{\beta\in\mathcal{F}_w(2)} \beta^\vee = 16\alpha_1 + 16\alpha_2 + 24\alpha_3 + 32\alpha_4 + 24\alpha_5 + 16\alpha_6 \in 2Q^\vee.$$
Therefore, $\mathcal{N}_\circ(w)^3 = 1$.
- We now turn to E_7 adjoint. We need to show that $\mathcal{N}_\circ(w)^2 = 1$, where w is the Weyl element in question. We follow here [Bou02, Plate VI], which has different conventions than [IM65]. Using the fact that w exchanges α_7 and $-\alpha_0$, one counts that the sum of all of the positive roots that contain α_7 is
$$\sum_{\alpha\in\mathcal{F}_w(1)} \alpha^\vee = 18\alpha_1 + 27\alpha_2 + 36\alpha_3 + 54\alpha_4 + 45\alpha_5 + 36\alpha_6 + 27\alpha_7.$$
But this sum is exactly equal to $18\epsilon_7$, so $\mathcal{N}_\circ(w)^2 = (\epsilon_7)(-1)^{18} = 1$.
- We finally consider type A_l. We re-adopt our conventions and notation from the proof of Proposition 3.2 in the case of type A_l. That is, we let $a, b \in \mathbb{N}$ such that $l+1 = ab$. Let $\omega = \rho_1^a$, so that $\omega^b = 1$, and for ease of notation, let $n = l+1$. Then $\omega = \epsilon_a w_a$, where w_a is the a-th power of the n-cycle $(1\ 2\ \cdots\ n)$.

  A computation then shows that
$$\sum_{m=1}^{b-1} \sum_{\alpha\in\mathcal{F}_{w_a}(m)} \alpha^\vee = (n-a)[e_1 + e_2 + ... + e_a] + (n-3a)[e_{a+1} + e_{a+2} + ... + e_{2a}]$$
$$+(n-5a)[e_{2a+1} + ... + e_{3a}] + ... + (a-n)[e_{n-a+1} + e_{n-a+2} + ... + e_{n-1} + e_n].$$
We denote this sum by γ. We recall that the cocharacter lattice of this isogeny is given by $X_*(A_l^a) = \langle Q^\vee, \epsilon_a\rangle$, where
$$\epsilon_a = \frac{1}{n}[(n-a)(e_1 + e_2 + ... + e_a) - a(e_{a+1} + ... + e_n)].$$
Suppose first that n is odd, so that a is also odd. Therefore, $n-a, n-3a, n-5a, ..., a-n$ are all even, so one can see that $\gamma \in 2Q^\vee$, which implies that $\mathcal{N}_\circ(w_a)^b = 1$. Suppose now that n is even. Then
$$\gamma - n\epsilon_a = (n-2a)[e_{a+1} + e_{a+2} + ... + e_{2a}] + (n-4a)[e_{2a+1} + ... + e_{3a}]$$
$$+ ... + (2a-n)[e_{n-a+1} + e_{n-a+2} + ... + e_{n-1} + e_n].$$
One can see that $\gamma - n\epsilon_a =: \eta \in 2Q^\vee$. Therefore, $\gamma = \eta + n\epsilon_a$ is twice a cocharacter, so $\mathcal{N}_\circ(w_a)^b = 1$.
□

Remark 3.4. The previous argument in the case of type A_l depends on the group not being simply connected. Otherwise, it may not be that $\mathcal{N}_\circ(w_a)^b = 1$. Indeed, if n is even, we relied on the fundamental coweight ϵ_a being contained in the cocharacter lattice in order to conclude that $\mathcal{N}_\circ(w_a)^b = 1$. If n is odd, however, it was automatic that $\mathcal{N}_\circ(w_a)^b = 1$. Indeed, this does not conflict with a basic known example; if $G = SL(n)$ and w is the long Weyl element, then
$$\mathcal{N}_\circ(w)^n = \begin{cases} -1 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

Theorem 3.5. *For G a split, almost simple, p-adic group, there exists an embedding $\Omega \hookrightarrow N_G(T)$ that is also a section of the canonical map $N_G(T) \to \Omega$.*

Proof. Suppose that Ω is cyclic of order n. We have shown that if $\omega = \epsilon_i w_{\Delta_i} w_\Delta \in \Omega$ is a generator, then $\iota(\omega)^n = 1$. But in fact $\iota(\omega)$ has order n. To see this, note that if $m < n$, then $\iota(\omega)^m = (\epsilon_i + w_i(\epsilon_i) + ... + w_i^{m-1}(\epsilon_i))(\varpi^{-1}) \cdot \mathcal{N}_\circ(w_i)^m$. But $\mathcal{N}_\circ(w_i)^m$ has a nontrivial projection to $W_\circ$ since w_i has order n. Therefore, $\iota(\omega)^m$ has a nontrivial projection to $W_\circ$ as well, so in particular must be nontrivial. Since $\iota(\omega)$ has order n, we may define a homomorphism $\Omega \to N_G(T)$ by sending ω^j to $\iota(\omega)^j$, and one may check that this map is in fact a section of the map $N_G(T) \to \Omega$.

It remains to consider the case where G is adjoint of type D_l with l even, since its fundamental group is not cyclic. We denote the associated cocharacter lattice by $X_*(D_l^{ad})$. We show that ι is a homomorphism in this case. Recall that in Proposition 3.3, we showed that $\iota(\omega)^2 = 1$ for each $\omega \in \Omega$. We need to show that $\iota(\rho_1\rho_l) = \iota(\rho_1)\iota(\rho_l), \iota(\rho_1\rho_{l-1}) = \iota(\rho_1)\iota(\rho_{l-1})$, and $\iota(\rho_{l-1}\rho_l) = \iota(\rho_{l-1})\iota(\rho_l)$. We will carry out the case $\iota(\rho_1\rho_l) = \iota(\rho_1)\iota(\rho_l)$, noting that the other cases are similar. First note that $\iota(\rho_1\rho_l) = \iota(\rho_{l-1}) = \epsilon_{l-1}\mathcal{N}_\circ(w_{l-1})$ and $\iota(\rho_1)\iota(\rho_l) = \epsilon_1\mathcal{N}_\circ(w_1)\epsilon_l\mathcal{N}_\circ(w_l) = \epsilon_1\mathcal{N}_\circ(w_1)\epsilon_l\mathcal{N}_\circ(w_1)^{-1}\mathcal{N}_\circ(w_1)\mathcal{N}_\circ(w_l)$. One can compute that $\epsilon_1\mathcal{N}_\circ(w_1)\epsilon_l\mathcal{N}_\circ(w_1)^{-1} = \epsilon_{l-1}$, so it suffices to show that $\mathcal{N}_\circ(w_1)\mathcal{N}_\circ(w_l) = \mathcal{N}_\circ(w_{l-1})$. By Corollary 2.2, we need to show that $\prod_{\alpha \in \mathcal{F}(w_1,w_l)} \alpha^\vee(-1) = 1$. One computes that

$$\mathcal{F}(w_1,w_l) = \{\alpha \in \Pi : w_l(\alpha) \in -\Pi, w_1w_l(\alpha) \in \Pi\} = \{\alpha \in \Pi : \alpha \text{ contains } \alpha_l \text{ and } \alpha \text{ does not contain } \alpha_{l-1}\}.$$

This last set, by [Bou02, Plate IV], is the set $\{e_i + e_l : 1 \leq i < l\}$. Adding these roots together gives $\gamma := \alpha_1 + 2\alpha_2 + 3\alpha_3 + ... + (l-2)\alpha_{l-2} + (l-1)\alpha_l$. But $X_*(D_l^{ad})$ contains $\epsilon_{l-1}, \epsilon_l$, and we see that $\gamma = (2-l)\epsilon_{l-1} + l\epsilon_l$, which lives in $2X_*(D_l^{ad})$ since l is even. The result follows. □

Remark 3.6.

(1) It is not difficult to show that ι is a homomorphism in the case that G is adjoint of type E_6. Since we cannot claim this for all types, we do not include the computation.
(2) In the case that G is adjoint of type D_l where l is odd, one can show that ι is not a homomorphism. In fact, one can show that $\iota(\rho_l)^2 = \iota(\rho_l^2)$, but it turns out that $\iota(\rho_l)^3 \neq \iota(\rho_l^3)$. This boils down to computing that the sum of all (co)roots in $\mathcal{F}_{w_l}(2)$ equals $le_1 - (e_1 + e_2 + ... + e_l)$, which when evaluated at -1 is nontrivial.
(3) In the case that G is type A_l, it turns out that ι is sometimes a homomorphism and sometimes not. For example, if $a = 1$ (in the notation of Proposition 3.3), then the group in consideration if PGL_n (recall that in our notation, $n = l + 1 = ab$), and one can show that $\iota(\rho_1)^2 \neq \iota(\rho_1^2)$. On the other hand, if both n and a are even, then ι is a homomorphism.

4. Beyond split almost-simple groups

One may ask about generalizing Theorem 3.5 to more general connected reductive groups. The biggest obstacle to generalizing the result, using the methods in this paper, revolves around the fact that if $W_\circ(\Omega)$ denotes the projection of Ω onto the finite Weyl group, then $\mathcal{N}_\circ|_{W_\circ(\Omega)} : W_\circ(\Omega) \to N_G(T)$ is not necessarily a homomorphism. This problem occurred in some A_l types, as well as adjoint D_l with l odd. But in these cases, we were able to skirt this issue by adjusting ι as in Theorem 3.5, using the fact that Ω is cyclic.

On the other hand, we are able to extend our result to certain additional split connected reductive groups. Note first that since G_{ad} is a product of split, almost-simple groups, Theorem 3.5 gives a section $s_{G_{\text{ad}}}$ of $\kappa_{G_{\text{ad}}} : G_{\text{ad}}(F) \to \Omega_{G_{\text{ad}}}$.

Definition 4.1. Call a homomorphic section s_G of κ_G *good* if it is compatible with the one constructed for G_{ad}. In other words, the following diagram commutes:

$$\begin{array}{ccc} G(F) & \longrightarrow & G_{\text{ad}}(F) \\ \kappa_G \downarrow \uparrow s_G & & \kappa_{G_{\text{ad}}} \downarrow \uparrow s_{G_{\text{ad}}} \\ \Omega_G & \longrightarrow & \Omega_{\text{ad}} \end{array}$$

Remark 4.2. Recall that when $G_{\text{der}} = G_{\text{sc}}$, there is an easy way to produce a homomorphic section with values in $T(F)$. However, this will not generally make the diagram commute, so it is not good.

Proposition 4.3. *Let G be a split connected reductive group over F. Let C be an alcove in the apartment corresponding to a split maximal torus T, with associated extended affine Weyl group $W = X_*(T) \rtimes W_\circ$. Then:*

(1) *If $Z = Z(G)$ is connected, then the induced map $G(F)/Z(\mathcal{O}_F) \to \Omega_G$ has a good homomorphic section (the analogue of the diagram above commutes).*

(2) *If Z is connected and $\Omega_G \cong \mathbb{Z}$ (e.g. $G = GSp(2n)$), then κ_G has a good homomorphic section.*
(3) *If Z is connected, $\Omega_G \cong \mathbb{Z}^n$, with $n > 1$, and $(|\Omega_{G_{\mathrm{ad}}}|, q(q-1)) = 1$, where q is the cardinality of the residue field, then κ_G has a good homomorphic section.*

Proof. We start with (1). It follows from Theorem 3.5 that $\kappa_{G_{\mathrm{ad}}}$ has a homomorphic section $s_{G_{\mathrm{ad}}}$, since G_{ad} is known to be a product of almost-simple groups. Moreover, if κ_Z denotes the Kottwitz homomorphism for $Z(F)$, then κ_Z also has a homomorphic section, which we denote s_Z. As $H^1(F, Z) = 1$, we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z(F) & \longrightarrow & G(F) & \xrightarrow{\mathrm{pr}} & G_{\mathrm{ad}}(F) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \kappa_Z} & & \downarrow{\scriptstyle \kappa_G} & & \downarrow{\scriptstyle \kappa_{G_{\mathrm{ad}}}} & & \\ 1 & \longrightarrow & \Omega_Z & \longrightarrow & \Omega_G & \xrightarrow{\mathrm{pr}} & \Omega_{G_{\mathrm{ad}}} & \longrightarrow & 1 \end{array}$$

We naturally have $Z(F)/Z(\mathcal{O}_F) \cong X_*(Z)$, therefore obtaining another diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z(F)/Z(\mathcal{O}_F) & \longrightarrow & G(F)/Z(\mathcal{O}_F) & \xrightarrow{\overline{\mathrm{pr}}} & G_{\mathrm{ad}}(F) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \overline{\kappa_Z}} & & \downarrow{\scriptstyle \overline{\kappa_G}} & & \downarrow{\scriptstyle \kappa_{G_{\mathrm{ad}}}} & & \\ 1 & \longrightarrow & \Omega_Z & \longrightarrow & \Omega_G & \xrightarrow{\mathrm{pr}} & \Omega_{G_{\mathrm{ad}}} & \longrightarrow & 1 \end{array}$$

where $\overline{\kappa_Z}, \overline{\kappa_G}$ are the induced maps. Let $\overline{\kappa_{G_{\mathrm{ad}}}}$ denote the map induced from $\kappa_{G_{\mathrm{ad}}}$ on $s_{\mathrm{ad}}(\Omega_{\mathrm{ad}})$. Then we have a commutative diagram of groups:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z(F)/Z(\mathcal{O}_F) & \longrightarrow & \overline{\mathrm{pr}}^{-1}(s_{\mathrm{ad}}(\Omega_{\mathrm{ad}})) & \xrightarrow{\mathrm{pr}} & s_{\mathrm{ad}}(\Omega_{\mathrm{ad}}) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \overline{\kappa_Z}} & & \downarrow{\scriptstyle \overline{\kappa_G}} & & \downarrow{\scriptstyle \overline{\kappa_{G_{\mathrm{ad}}}}} & & \\ 1 & \longrightarrow & \Omega_Z & \longrightarrow & \Omega_G & \xrightarrow{\mathrm{pr}} & \Omega_{G_{\mathrm{ad}}} & \longrightarrow & 1 \end{array}$$

We have that $\overline{\kappa_Z}, \overline{\kappa_{G_{\mathrm{ad}}}}$ are isomorphisms, so by the five lemma, $\overline{\kappa_G}$ is an isomorphism, and thus the map $\overline{\kappa_G} : G(F)/Z(\mathcal{O}_F) \to \Omega_G$ has a homomorphic section.

We now prove (2). Make an initial choice of a homomorphic section s_Z^0 of κ_Z. Given $\sigma \in \Omega_G$, let $s^0(\sigma)$ be any lift in $G(F)$ of $s_{G_{\mathrm{ad}}}(\mathrm{pr}(\sigma)) \in N_{G_{\mathrm{ad}}}(T_{\mathrm{ad}})(F)$; it automatically lies in $N_G(T)(F)$. It might happen that s^0 is not a section of κ_G. However, for all $\sigma \in \Omega_G$, we have $\mathrm{pr}(\kappa_G(s^0(\sigma))) = \kappa_{G_{\mathrm{ad}}}(\mathrm{pr}(s^0(\sigma))) = \kappa_{G_{\mathrm{ad}}}(s_{G_{\mathrm{ad}}}(\mathrm{pr}(\sigma))) = \mathrm{pr}(\sigma)$. Thus, the difference between σ and $\kappa_G(s^0(\sigma))$ belongs to Ω_Z. Since κ_Z is surjective, we may alter each $s^0(\sigma)$ by an element $z_\sigma^0 \in Z(F)$ in such a way that $\sigma \mapsto s^0(\sigma) z_\sigma^0$ is a section of κ_G.

So we may assume s^0 is a set-theoretic section of κ_G, taking values in $N_G(T)(F)$. Because $s_{G_{\mathrm{ad}}}$ is homomorphic, the map

$$(\sigma_1, \sigma_2) \mapsto s^0(\sigma_1) s^0(\sigma_2) s^0(\sigma_1 \sigma_2)^{-1}$$

is a 2-cocycle of Ω_G with values in $Z(F)$, with Ω_G acting trivially on $Z(F)$. Therefore, we get an element of $H^2(\Omega_G, Z(F))$. This group parameterizes isomorphism classes of extensions of Ω_G by $Z(F)$ where the induced action of Ω_G on the normal subgroup $Z(F)$ is trivial (i.e. $Z(F)$ is central in the extension group). We claim that the extension corresponding to the 2-cocycle is the direct product $Z(F) \times \Omega_G$. This follows because $\Omega_G = \mathbb{Z}$ and $H^2(\mathbb{Z}, A) = 1$ for any abelian group A with trivial $\mathbb{Z}$-action.

The fact that the extension is trivial means that the 2-cocycle defining it is a 2-coboundary. This means that we may alter our initial choice of set-theoretic section s^0 to give a homomorphism $s : \Omega_G \to G(F)$, taking values again in $N_G(T)(F)$.

The problem now is that s might not be a section of κ_G, which we take care of as before. By construction, $\sigma^{-1}\kappa_G(s(\sigma)) \in \Omega_Z$ for every $\sigma \in \Omega_G$. So we may define $z_\sigma := s_Z^0(\sigma(\kappa_G(s(\sigma)))^{-1}) \in Z(F)$, for $\sigma \in \Omega_G$. Note that $\sigma \mapsto z_\sigma$ is a homomorphism $\Omega_G \to Z(F)$. Now define

$$s_G(\sigma) := z_\sigma s(\sigma).$$

Then s_G is the desired homomorphic section of κ_G in case (2).

In case (3), the same argument works, as long as we can prove that the 2-cocycle defined by s^0 is still a 2-coboundary. But when $n > 1$ it is no longer true that $H^2(\mathbb{Z}, A)$ always vanishes for abelian groups A with trivial $\mathbb{Z}^n$-action. Nevertheless, we will show that the extension corresponding to the given 2-cocycle is still trivial. Write $\dot{e}_i = s^0(e_i)$, where e_i corresponds to a standard basis vector in $\Omega_G \cong \mathbb{Z}^n$. Then the extension is the exact sequence

$$1 \to Z(F) \to Z(F)\langle \dot{e}_1, \cdots, \dot{e}_n \rangle \xrightarrow{\kappa_G} \Omega_G \to 1.$$

Write $N := |\Omega_{G_{\mathrm{ad}}}|$. As $\mathrm{pr}(\dot{e}_j) \in \mathrm{im}(s_{G_{\mathrm{ad}}}) \cong \Omega_{G_{\mathrm{ad}}}$, we have $\mathrm{pr}(\dot{e}_j)^N = 1$ and hence $\dot{e}_j^N \in Z(F)$. Moreover, $\dot{e}_i \dot{e}_j \dot{e}_i^{-1} \dot{e}_j^{-1} \in Z(F)$. We may write

$$a\dot{e}_j = \dot{e}_i \dot{e}_j \dot{e}_i^{-1},$$

for some $a \in Z(F)$. Raising to the N-th power, we get

$$a^N \dot{e}_j^N = \dot{e}_j^N,$$

and hence $a^N = 1$. Therefore, $a \in Z(\mathcal{O}_F)$. Moreover, since N is coprime to the pro-order of the profinite group $Z(\mathcal{O}_F)$, we conclude that $a = 1$, and therefore the elements $\dot{e}_i$ pairwise commute. Therefore, the extension is an abelian group. But then the extension is trivial, since $\Omega_G \cong \mathbb{Z}^n$.

This concludes the proof of the proposition. But we make one additional comment. By construction, the map $s_G|_{\Omega_Z}$ has image in $Z(F)$ and so gives a homomorphic section s_Z of κ_Z. This section might be different from the initial choice s_Z^0. But now we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z(F) & \longrightarrow & G(F) & \longrightarrow & G_{\mathrm{ad}}(F) & \longrightarrow & 1 \\ & & \kappa_Z \downarrow \uparrow s_Z & & \kappa_G \downarrow \uparrow s_G & & \kappa_{G_{\mathrm{ad}}} \downarrow \uparrow s_{G_{\mathrm{ad}}} & & \\ 1 & \longrightarrow & \Omega_Z & \longrightarrow & \Omega_G & \longrightarrow & \Omega_{\mathrm{ad}} & \longrightarrow & 1 \end{array}$$

□

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