**Research Article**

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# A local converse theorem for Archimedean GL($n$)

https://doi.org/10.1515/forum-2022-0230
Received August 6, 2022; revised December 28, 2022

**Abstract:** We prove a local converse theorem for $\mathrm{GL}_n$ over the Archimedean local fields which characterizes an infinitesimal equivalence class of irreducible admissible generic representations of $\mathrm{GL}_n(\mathbb{R})$ or $\mathrm{GL}_n(\mathbb{C})$ in terms of twisted local gamma factors.

**Keywords:** Representations of reductive groups, Langlands program

**MSC 2020:** 20G05

**Communicated by:** Freydoon Shahidi

## 1 Introduction

Let $F$ be a local field of characteristic 0, and let $\mathrm{Irr}_n$ be the set of (infinitesimal) equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(F)$. A so-called local converse theorem for $\mathrm{GL}_n(F)$ characterizes the set $\mathrm{Irr}_n$ in terms of local factors with some suitable twists. If $F$ is non-Archimedean, the first major result is the one by Henniart [5] in which he shows that if two *generic* representations $\pi, \pi' \in \mathrm{Irr}_n$ satisfy

$$\gamma(s, \pi \times \tau, \psi) = \gamma(s, \pi' \times \tau, \psi)$$

for all generic $\tau \in \mathrm{Irr}_t$ for all $t = 1, \dots, n-1$, where the $\gamma$-factor is the one defined by Jacquet, Piatetski-Shapiro and Shalika, then $\pi = \pi'$. Later, Chen [4] improved this result by requiring $t$ be only up to $n-2$ with the extra assumption that $\pi$ and $\pi'$ have the same central character. It had been conjectured by Jacquet for some time that one only needs $t \leq [\frac{n}{2}]$. Recently, this conjecture has been proven by Chai [3], and Jacquet and Liu [9] (see also [1, 10]). Let us also mention that Nien [13] has shown an analogous result when $F$ is a finite field.

In this paper, we prove the Archimedean analogue of the local converse theorem as follows.

**Theorem.** *Let $F = \mathbb{C}$ or $\mathbb{R}$. If $\pi, \pi' \in \mathrm{Irr}_n$ are generic representations of $\mathrm{GL}_n(F)$ that satisfy*

$$\gamma(s, \pi \times \chi, \psi) = \gamma(s, \pi' \times \chi, \psi)$$

*for all unitary characters $\chi$ on $F^\times$, then $\pi = \pi'$.*

Here, the gamma factors are defined on the "Galois side" via the local Langlands correspondence (LLC); namely they are the gamma factors of Artin type. The basic idea of our proof is that we pass to the Galois side via the LLC so that the gamma factors, which are essentially products of gamma functions, can be explicitly computed in terms of the data for the corresponding representations of the Weil group. Then we will compare poles of the gamma functions. In this sense, what we actually prove is the following assertion: given two local Langlands parameters $\varphi, \varphi' : W_F \to \mathrm{GL}_n(\mathbb{C})$ of generic type, if $\gamma(s, \varphi \otimes \chi, \psi) = \gamma(s, \varphi' \otimes \chi, \psi)$ for all 1-dimensional characters $\chi$, then $\varphi = \varphi'$.

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We note that it can be shown that the LLC for Archimedean GL($n$) is uniquely characterized by the local $L$-factors. This uniqueness result was originally announced by Henniart in [6, Section 1.10, p. 592], although his proof has never appeared to the best of our knowledge. Yet in [2], we have proven a refined version of the theorem announced by Henniart. This result will appear elsewhere.

It should also be noted that these gamma factors of Artin type are known to coincide with the local coefficients of Shahidi [14]. Moreover, in [7], Jacquet has shown that these gamma factors appear as constants of the functional equations satisfied by Rankin–Selberg integrals. The authors do not know if it is possible to prove the local converse theorem as above by using purely representation theoretic methods using this theory of Jacquet without passing to the Galois side, and this is certainly an interesting question to be answered.

**Notation 1.1.** Throughout, $F$ is either $\mathbb{R}$ or $\mathbb{C}$. We let $\mathrm{Irr}_n$ be the set of infinitesimal equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(F)$. For $z \in F$, we let $|z| = \sqrt{z\bar{z}}$, so that if $F = \mathbb{R}$, it is the absolute value of $z$, and if $F = \mathbb{C}$, it is the usual modulus of $z$. We also let $\|z\| = z\bar{z} = |z|^2$. By a character we always mean a quasi-character, and $\mathrm{Irr}_1$ is the set of characters of $F^\times$. We let $\psi_F$ be the standard choice of additive character on $F$; namely if $F = \mathbb{R}$, then $\psi_\mathbb{R}(r) = e^{2\pi i r}$, and if $F = \mathbb{C}$, then

$$\psi_\mathbb{C}(z) = \psi_\mathbb{R} \circ \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}(z) = e^{2\pi i(z+\bar{z})}.$$

We let $\Gamma(s)$ be the gamma function. Recall that $\Gamma(s)$ has no zeroes, and has infinitely many poles, which are precisely at $s = 0, -1, -2, \ldots$, all of which are simple.

Finally, if $w, z \in \mathbb{C}$, then we write $w \preceq z$ if $z - w \in \mathbb{Z}^{\geq 0}$. This is a partial order on $\mathbb{C}$. Also, $w \prec z$ means $w \preceq z$ and $w \neq z$. For fixed $z, w \in \mathbb{C}$, the gamma functions $\Gamma(s + z)$ and $\Gamma(s + w)$ have a common pole if and only if $z$ and $w$ are comparable under $\preceq$, namely $z - w \in \mathbb{Z}$. We use this fact repeatedly throughout the paper.

# 2 Complex case

In this section, we consider the complex case, so we set $F = \mathbb{C}$.

## 2.1 Weil group and its representations

We let $W_F$ be the Weil group of $F$, namely

$$W_\mathbb{C} = \mathbb{C}^\times.$$

Each (not necessarily unitary) character of $\mathbb{C}^\times$, which we also view as a 1-dimensional representation of $W_\mathbb{C}$, is of the form

$$\chi_{-N,t}(z) := z^{-N}\|z\|^t$$

for $z \in \mathbb{C}^\times$, where $N \in \mathbb{Z}$ and $t \in \mathbb{C}$. Let us note that if we write $z = re^{i\theta}$ with $r, \theta \in \mathbb{R}$ as usual, we have

$$\chi_{-N,t}(z) = r^{2t-N}e^{-iN\theta}.$$

But when dealing with the local factors, it seems to be more convenient to denote each character as $z^{-N}\|z\|^t$ instead of using $re^{i\theta}$, and hence we choose this convention. Let us note that

$$\overline{\chi_{-N,t}} = \chi_{N,t-N},$$

where

$$\overline{\chi_{-N,t}}(z) := \overline{\chi_{-N,t}(z)} = \chi_{-N,t}(\bar{z})$$

as usual.

Since $W_\mathbb{C}$ is abelian, $\chi_{-N,t}$ is the only irreducible semisimple representation of $W_\mathbb{C}$, and hence each $n$-dimensional semisimple representation

$$\varphi : W_\mathbb{C} \to \mathrm{GL}_n(\mathbb{C})$$

is of the form

$$\varphi = \chi_{-N_1,t_1} \oplus \cdots \oplus \chi_{-N_n,t_n}. \tag{2.1}$$

Note that the contragredient $\varphi^\vee$ is

$$\varphi^\vee = \chi_{N_1,-t_1} \oplus \cdots \oplus \chi_{N_n,-t_n},$$

because $\chi_{-N,t}^\vee = \chi_{-N,t}^{-1} = \chi_{N,-t}$.

## 2.2 Local factors

Recall that the $L$-, $\epsilon$- and $\gamma$-factors of the character $\chi_{-N,t}$ are defined as follows:

$$L(\chi_{-N,t}) = 2(2\pi)^{-(t-\frac{N}{2}+\frac{|N|}{2})}\Gamma\Big(t - \frac{N}{2} + \frac{|N|}{2}\Big), \tag{2.2}$$

$$\epsilon(\chi_{-N,t}, \psi_{\mathbb{C}}) = i^{|N|}, \tag{2.3}$$

$$\gamma(\chi_{-N,t}, \psi_{\mathbb{C}}) = \epsilon(\chi_{-N,t}, \psi_{\mathbb{C}})\frac{L(\chi_{-N,t}^\vee\|\cdot\|)}{L(\chi_{-N,t})} = i^{|N|}(2\pi)^{-1+2t-N}\frac{\Gamma(1-t+\frac{N}{2}+\frac{|N|}{2})}{\Gamma(t-\frac{N}{2}+\frac{|N|}{2})}. \tag{2.4}$$

If $\varphi : W_{\mathbb{C}} \to \mathrm{GL}_n(\mathbb{C})$ is an $n$-dimensional representation as in (2.1), we define the local factors multiplicatively as follows:

$$\begin{aligned} L(\varphi) &= L(\chi_{-N_1,t_1})\cdots L(\chi_{-N_n,t_n}),\\ \epsilon(\varphi, \psi_{\mathbb{C}}) &= \epsilon(\chi_{-N_1,t_1}, \psi_{\mathbb{C}})\cdots\epsilon(\chi_{-N_n,t_n}, \psi_{\mathbb{C}}),\\ \gamma(\varphi, \psi_{\mathbb{C}}) &= \gamma(\chi_{-N_1,t_1}, \psi_{\mathbb{C}})\cdots\gamma(\chi_{-N_n,t_n}, \psi_{\mathbb{C}}). \end{aligned}$$

Note that we have

$$\gamma(\varphi, \psi_{\mathbb{C}}) = \epsilon(\varphi, \psi_{\mathbb{C}})\frac{L(\varphi^\vee\|\cdot\|)}{L(\varphi)}.$$

## 2.3 GL(1)-twist

Let $\chi_{-M,s}$ be another character on $\mathbb{C}^\times$, and let $\varphi$ be an $n$-dimensional representation of $W_{\mathbb{C}}$ as in (2.1). Then the twist $\varphi \otimes \chi_{-M,s}$ by $\chi_{-M,s}$ is given by

$$\varphi \otimes \chi_{-M,s} = \chi_{-(N_1+M),t_1+s} \oplus \cdots \oplus \chi_{-(N_n+M),t_n+s}. \tag{2.5}$$

We set

$$\begin{aligned} L(s, \varphi) &= L(\varphi \otimes \chi_{0,s}),\\ \epsilon(s, \varphi, \psi_{\mathbb{C}}) &= \epsilon(\varphi \otimes \chi_{0,s}, \psi_{\mathbb{C}}),\\ \gamma(s, \varphi, \psi_{\mathbb{C}}) &= \gamma(\varphi \otimes \chi_{0,s}, \psi_{\mathbb{C}}). \end{aligned}$$

We then have

$$\gamma(s, \varphi, \psi_{\mathbb{C}}) = \epsilon(\varphi, \psi_{\mathbb{C}})\frac{L(1-s, \varphi^\vee)}{L(s, \varphi)}.$$

## 2.4 Local Langlands correspondence for $\mathrm{GL}_n(\mathbb{C})$

By the Archimedean local Langlands correspondence, originally established by Langlands [11], there is a one-to-one correspondence between the set $\mathrm{Irr}_n$ of (infinitesimal equivalence classes) of irreducible admissible representations of $\mathrm{GL}_n(\mathbb{C})$ and the set $\Phi_n$ of (conjugacy classes of) all continuous semisimple $n$-dimensional representations of $W_{\mathbb{C}}$. This correspondence can be fairly explicitly described as follows. For each

$$\varphi = \chi_{-N_1,t_1} \oplus \cdots \oplus \chi_{-N_n,t_n} \in \Phi_n,$$

consider the (normalized) induced representation

$$I(\varphi) := \mathrm{Ind}_{B(\mathbb{C})}^{\mathrm{GL}_n(\mathbb{C})} \chi_{-N_1,t_1} \otimes \cdots \otimes \chi_{-N_n,t_n},$$

where $B(\mathbb{C})$ is the Borel subgroup of $\mathrm{GL}_n(\mathbb{C})$ and the character $\chi_{-N_1,t_1} \otimes \cdots \otimes \chi_{-N_n,t_n}$ is viewed as a character on $B(\mathbb{C})$ as usual. Let us reorder the constituents of $\varphi$ in the Langlands situation, which means

$$\mathrm{Re}(t_1) \geq \cdots \geq \mathrm{Re}(t_n).$$

By the Langlands quotient theorem, $I(\varphi)$ has a unique irreducible quotient, which we denote by $\pi_\varphi$. Then the local Langlands correspondence is obtained by the map

$$\Phi_n \to \mathrm{Irr}_n, \quad \varphi \mapsto \pi_\varphi.$$

### 2.4.1 Genericity conditions

It is well known that the (full) induced representation $I(\varphi)$ is always generic (see, for example, [16, Theorem 15.4.1, p. 381].) The following proposition characterizes when the Langlands quotient $\pi_\varphi$ is generic.

**Proposition 2.1.** *Let*

$$\varphi = \chi_{-N_1,t_1} \oplus \cdots \oplus \chi_{-N_n,t_n}$$

*be such that*

$$N_1 \leq \cdots \leq N_n.$$

*Then the following statements are all equivalent.*
(i) *The representation $\pi_\varphi$ that corresponds to $\varphi$ under the local Langlands correspondence is generic.*
(ii) *$\pi_\varphi = I(\varphi)$, namely $I(\varphi)$ is irreducible.*
(iii) *For all $i \leq j$, whenever $t_j - t_i \in \mathbb{Z}$, we have*

$$0 \leq t_j - t_i \leq N_j - N_i.$$

*In particular, if $t_j - t_i \in \mathbb{Z}$, then $t_i \preceq t_j$, where we recall from Notation 1.1 that $t_i \preceq t_j$ means $t_j - t_i \in \mathbb{Z}^{\geq 0}$ (note that if $t_j - t_i \notin \mathbb{Z}$, then there no condition.)*
(iv) *The Rankin–Selberg L-factor*

$$L(s, \varphi \otimes \varphi^\vee) := L(\varphi \otimes \varphi^\vee \otimes \chi_{0,s})$$

*is holomorphic at $s = 1$.*

*Proof.* The equivalence of (i) and (ii) is well known. The equivalence of (ii) and (iii) is a special case of [15]; though, presumably, the case of $\mathrm{GL}_n(\mathbb{C})$ had been known much before. Since the authors were not able to find an explicit reference for $\mathrm{GL}_n(\mathbb{C})$, we reproduce essential parts of the proof.

First consider the principal series

$$\mathrm{Ind}_{B(\mathbb{C})}^{\mathrm{GL}_n(\mathbb{C})} \chi_1 \otimes \cdots \otimes \chi_n,$$

where $\chi_i : \mathbb{C}^\times \to \mathbb{C}^\times$ is a character. This is reducible if and only if for some $i \neq j$ the character $\chi_i \chi_j^{-1}$ is of the form

$$\chi_i \chi_j^{-1}(z) = z^p \overline{z}^q, \quad p - q \in \mathbb{Z},$$

where either both $p$ and $q$ are in $\mathbb{Z}^{>0}$ or both $p$ and $q$ are in $\mathbb{Z}^{<0}$. (One can prove this by reducing to the $\mathrm{GL}_2(\mathbb{C})$ situation by induction in stages and applying [8, Theorem 6.2], or one may apply the general result of [15, Theorem 1.1] to $\mathrm{GL}_n(\mathbb{C})$.)

Now for each $i < j$ we have

$$(\chi_{-N_i,t_i})(\chi_{-N_j,t_j})^{-1}(z) = z^{-N_i+N_j} \|z\|^{t_i-t_j} = z^{N_j-N_i+t_i-t_j} \overline{z}^{t_i-t_j}.$$

Noting $N_i \leq N_j$, we know that $I(\varphi)$ is reducible if and only if $t_i - t_j \in \mathbb{Z}^{>0}$ or otherwise both $N_j - N_i + t_i - t_j \in \mathbb{Z}^{<0}$ and $t_i - t_j \in \mathbb{Z}^{<0}$. Hence $I(\varphi)$ is irreducible if and only if, whenever $t_i - t_j \in \mathbb{Z} \setminus \{0\}$, we have $t_i - t_j \notin \mathbb{Z}^{>0}$ and $N_j - N_i + t_i - t_j \notin \mathbb{Z}^{<0}$. One can then see that these conditions are precisely (iii).

We show the equivalence of (iii) and (iv). Since

$$\varphi \otimes \varphi^\vee \otimes \chi_{0,s} = \sum_{i,j} \chi_{-N_i,t_i} \otimes \chi_{N_j,-t_j} \otimes \chi_{0,s} = \sum_{i,j} \chi_{-(N_i-N_j),\, s+t_i-t_j},$$

we have

$$L(s,\varphi\otimes\varphi^\vee)=\prod_{i,j}L(\chi_{-(N_i-N_j),\,s+t_i-t_j})=F(s)\prod_i\Gamma(s)\prod_{i<j}\Gamma(s+t_i-t_j-N_i+N_j)\Gamma(s+t_j-t_i),$$

where $F(s)$ is a holomorphic function without zeros (here to compute the $L$-factors we used that $N_i$'s are in the increasing order). Hence $L(s,\varphi\otimes\varphi^\vee)$ is holomorphic at $s=1$ if and only if

$$t_i-t_j-N_i+N_j\notin\mathbb{Z}^{<0}\quad\text{and}\quad t_j-t_i\notin\mathbb{Z}^{<0}.$$

But this condition is equivalent to

$$0\le t_j-t_i\le N_j-N_i$$

whenever $t_j-t_i\in\mathbb{Z}$ with $i\le j$. □

## 2.5 Local converse theorem for $\mathrm{GL}_n(\mathbb{C})$

For two characters $\chi_{N,t}$ and $\chi_{M,s}$, we define

$$\chi_{N,t}\sim\chi_{M,s}\quad\text{if } t-s\in\mathbb{Z}.$$

This is certainly an equivalence relation. Then, given a Langlands parameter $\varphi$ of $\mathrm{GL}_n(\mathbb{C})$, by grouping the constituents by this equivalence relation we can write

$$\varphi=\varphi_1\oplus\cdots\oplus\varphi_k,$$

where all constituents of $\varphi_i$ are equivalent under $\sim$ and the constituents of different $\varphi_i$ and $\varphi_j$ are inequivalent under $\sim$. Then we know that in the $\gamma$-factor

$$\gamma(s,\varphi,\psi_{\mathbb{C}})=\gamma(s,\varphi_1,\psi_{\mathbb{C}})\cdots\gamma(s,\varphi_n,\psi_{\mathbb{C}})$$

the zeros and the poles coming from $\gamma(s,\varphi_i,\psi_{\mathbb{C}})$ do not interfere with those coming from $\gamma(s,\varphi_j,\psi_{\mathbb{C}})$ for $j\neq i$.

Let us first prove the following proposition.

**Proposition 2.2.** *Let*

$$\varphi=\chi_{-N_1,t_1}\oplus\cdots\oplus\chi_{-N_n,t_n}\quad\textit{and}\quad\varphi'=\chi_{-N'_1,t'_1}\oplus\cdots\oplus\chi_{-N'_{n'},t'_{n'}}$$

*be generic parameters of* $\mathrm{GL}_n(\mathbb{C})$ *and* $\mathrm{GL}_{n'}(\mathbb{C})$*, respectively, such that all constituents* $\chi_{-N_i,t_i}$ *and* $\chi_{-N'_j,t'_j}$ *are equivalent under $\sim$, namely $t_i-t'_j\in\mathbb{Z}$ for all $i,j$. Assume*

$$F_\chi(s)\gamma(s,\varphi\otimes\chi,\psi_{\mathbb{C}})=\gamma(s,\varphi'\otimes\chi,\psi_{\mathbb{C}})\tag{2.6}$$

*for all characters $\chi$, where $F_\chi(s)$ is a meromorphic function (depending on $\chi$) whose poles and zeros do not interfere with those from the gamma factors. Then $\varphi=\varphi'$ (and hence $n=n'$).*

*Proof.* Since all constituents $\chi_{-N_i,t_i}$ and $\chi_{-N'_j,t'_j}$ are equivalent under $\sim$, there exists $s_0$ with $\mathrm{Re}(s_0)$ large enough such that all of the $\gamma(s,\chi_{-N_i,t_i},\psi_{\mathbb{C}})$ and $\gamma(s,\chi_{-N'_j,t'_j},\psi_{\mathbb{C}})$ have a simple pole at $s=s_0$, so that, at $s=s_0$, $\gamma(s,\varphi\otimes\chi,\psi_{\mathbb{C}})$ has a pole of order $n$ and $\gamma(s,\varphi'\otimes\chi,\psi_{\mathbb{C}})$ has a pole of order $n'$. Hence we have $n=n'$.

Without loss of generality, we may assume

$$N_1\le\cdots\le N_n\quad\text{and}\quad N'_1\le\cdots\le N'_n.$$

Since $\varphi$ is generic and $t_j-t_i\in\mathbb{Z}$, by Proposition 2.1 (iii) we have

$$0\le t_j-t_i\le N_j-N_i\tag{2.7}$$

for $i\le j$. In particular, $t_1\preceq\cdots\preceq t_n$. Similarly, this holds for the $t'_i$'s and $N'_i$'s.

Let $\chi = \chi_{-M,0}$ be such that $M + N_i > 0$ and $M + N'_i > 0$ for all $i$. Then (the reciprocal of) identity (2.6) is equivalent to

$$F(s)\prod_{i=1}^{n}\frac{\Gamma(s+t_i)}{\Gamma(1-s-t_i+N_i+M)} = \prod_{i=1}^{n}\frac{\Gamma(s+t'_i)}{\Gamma(1-s-t'_i+N'_i+M)}, \tag{2.8}$$

where $F(s)$ is a meromorphic function whose poles and zeros do not interfere.

Set $M$ to be large enough so that all of the gamma functions in the denominators in (2.8) are holomorphic at $s = -t_1, \dots, -t_n$. Then on the left-hand side, we have a pole at $s = -t_1$ coming from $\Gamma(s + t_1)$. Hence on the right-hand side we must have a pole at $s = -t_1$ from some $\Gamma(s + t'_i)$. If $t'_1$ is such that $t'_1 \succ t_1$ (strict inequality), then, since the $t'_i$'s are in increasing order (with respect to $\preceq$), we never have a pole at $s = -t_1$ for any of the $\Gamma(s + t'_i)$'s. Hence $t'_1 \preceq t_1$. By switching the roles of $t_1$ and $t'_1$, we have $t'_1 \succeq t_1$. Hence we have $t_1 = t'_1$. Thus, $\Gamma(s + t_1)$ and $\Gamma(s + t'_1)$ can be removed from (2.8). Arguing inductively, we have

$$t_i = t'_i$$

for all $i = 1, \dots, n$.

Thus we can reduce (2.8) to

$$F(s)\prod_{i=1}^{n}\Gamma(1-s-t'_i+N'_i+M) = \prod_{i=1}^{n}\Gamma(1-s-t_i+N_i+M), \tag{2.9}$$

where $M$ is a fixed integer. Let $k, \ell \in \{1, \dots, n\}$ be such that

$$t_k - N_k \succeq t_i - N_i \quad \text{and} \quad t'_\ell - N'_\ell \succeq t'_i - N'_i$$

for all $i$. Then at $s = -t_k + N_k + M$, the right-hand side (and hence the left-hand side) is holomorphic, which implies $t_k - N_k \preceq t'_\ell - N'_\ell$. By switching the roles, we obtain $t_k - N_k \succeq t'_\ell - N'_\ell$, and hence $t_k - N_k = t'_\ell - N'_\ell$. By arguing inductively, we obtain

$$\{t_1 - N_1, \dots, t_q - N_q\} = \{t'_1 - N'_1, \dots, t'_q - N'_q\}$$

as multisets.

Now, we will show $N_i = N'_i$ for all $i = 1, \dots, n$. By the above identity of the multisets, we must have $t_1 - N_1 = t'_i - N'_i$ for some $i$. Since we already know $t_i = t'_i$, we have

$$N'_i - N_1 = t'_i - t_1 = t'_i - t'_1 \le N'_i - N'_1,$$

where the last inequality is by the genericity condition (2.7). Hence we have $N'_1 \le N_1$. Also, we must have $t'_1 - N'_1 = t_j - N_j$ for some $j$. By applying the same argument, we must have $N_1 \le N'_1$. Thus we must have $N_1 = N'_1$. By arguing inductively, we have

$$N_i = N'_i$$

for all $i = 1, \dots, n$. □

Now, we are ready to prove the local converse theorem.

**Theorem 2.3.** *Let $\pi$ and $\pi'$ be generic irreducible admissible representations of* $\mathrm{GL}_n(\mathbb{C})$. *Assume that*

$$\gamma(s, \pi \times \chi, \psi_{\mathbb{C}}) = \gamma(s, \pi' \times \chi, \psi_{\mathbb{C}})$$

*for all unitary characters $\chi$. Then $\pi = \pi'$.*

*Proof.* Let us first note that if $\chi$ is not unitary, then $\gamma(s, \pi \times \chi, \psi_{\mathbb{C}}) = \gamma(s + t, \pi \times \chi', \psi_{\mathbb{C}})$ for some $t \in \mathbb{C}$ and some unitary character $\chi'$. Hence we may assume that the identity of the gamma factors holds for all (not necessarily unitary) characters $\chi$.

Let $\varphi$ and $\varphi'$ be the Langlands parameters of $\mathrm{GL}_n(\mathbb{C})$ corresponding to $\pi$ and $\pi'$, respectively. Let us write

$$\varphi = \varphi_1 \oplus \cdots \oplus \varphi_k \quad \text{and} \quad \varphi' = \varphi'_1 \oplus \cdots \oplus \varphi'_{k'},$$

where all constituents of each $\varphi_j$ are equivalent under $\sim$ and the constituents of $\varphi_i$ and $\varphi_j$ are inequivalent under $\sim$ for $i \neq j$, and similarly for $\varphi'$.

We then have

$$\prod_{i=1}^{k} \gamma(s, \varphi_i \otimes \chi, \psi_{\mathbb{C}}) = \prod_{i=1}^{k'} \gamma(s, \varphi'_i \otimes \chi, \psi_{\mathbb{C}}).$$

Note that, for $i \neq j$, the gamma factors $\gamma(s, \varphi_i \otimes \chi, \psi_{\mathbb{C}})$ and $\gamma(s, \varphi_j \otimes \chi, \psi_{\mathbb{C}})$ do not share a zero or a pole, and similarly for $\varphi'$.

Now, assume that $\varphi$ and $\varphi'$ do not share any constituents equivalent under $\sim$. Then $\gamma(s, \varphi \otimes \chi, \psi_{\mathbb{C}})$ and $\gamma(s, \varphi' \otimes \chi, \psi_{\mathbb{C}})$ do not share a zero or pole. So there are at least some $\varphi_i$ and $\varphi'_j$ having constituents equivalent under $\sim$. By reordering the indices, we may assume $i = j = 1$. Then the equality of the gamma factors is written as

$$F_\chi(s)\gamma(s, \varphi_1 \otimes \chi, \psi_{\mathbb{C}}) = \gamma(s, \varphi'_1 \otimes \chi, \psi_{\mathbb{C}}),$$

where $F_\chi(s)$ is a meromorphic function whose poles and zeros do not interfere with those of the above two gamma factors. Hence by the above proposition, we have $\varphi_1 = \varphi'_1$. Arguing inductively, we conclude $\varphi = \varphi'$. □

# 3 Real case

In this section, we consider the real case, so we set $F = \mathbb{R}$.

## 3.1 Weil group and its representations

Recall that the Weil group $W_{\mathbb{R}}$ of $\mathbb{R}$ is defined by

$$W_{\mathbb{R}} = \mathbb{C}^\times \cup j\mathbb{C}^\times, \quad j^2 = -1,\ jzj^{-1} = \bar{z},$$

where $z \in \mathbb{C}^\times$. We naturally view $W_{\mathbb{C}} = \mathbb{C}^\times$ as a subgroup of $W_{\mathbb{R}}$. Note that $\mathbb{R}^\times \cong W_{\mathbb{R}}^{ab}$ because we have a surjective map

$$W_{\mathbb{R}} \to \mathbb{R}^\times, \quad z \mapsto z\bar{z},\ j \mapsto -1, \tag{3.1}$$

whose kernel is the commutator group $[W_{\mathbb{R}}, W_{\mathbb{R}}]$, which is of the form $\{z \in \mathbb{C}^\times : |z| = 1\}$.

An irreducible representation of $W_{\mathbb{R}}$ is 1- or 2-dimensional. If it is 1-dimensional, it factors through $W_{\mathbb{R}}^{ab} \cong \mathbb{R}^\times$, and hence is identified with a character, which is of the form

$$\lambda_{\varepsilon,t}(r) := r^{-\varepsilon}|r|^t = \operatorname{sign}(r)^\varepsilon |r|^{t-\varepsilon}, \quad r \in \mathbb{R}^\times,$$

where $\varepsilon \in \{0, 1\}$, $t \in \mathbb{C}$ and sign is the sign character. Also, we often write $\lambda_{0,t} = |\cdot|^t$. If it is 2-dimensional, it is of the form

$$\varphi_{-N,t} := \operatorname{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \chi_{-N,t},$$

where $\chi_{-N,t}$ is the character on $\mathbb{C}^\times$ as before, namely

$$\chi_{-N,t}(z) = z^{-N}\|z\|^t$$

for $z \in \mathbb{C}^\times$.

If $N = 0$, then the representation $\varphi_{-N,t}$ is not irreducible, but we have

$$\varphi_{0,t} = \lambda_{0,t} \oplus \lambda_{1,t+1}.$$

But otherwise it is irreducible. Furthermore, since

$$\operatorname{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \chi_{-N,t} = \operatorname{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \overline{\chi_{-N,t}},$$

we have

$$\varphi_{-N,t} = \varphi_{N,t-N}.$$

Hence we may and do assume that $N \geq 0$. Also, we consider $\lambda_{0,t} \oplus \lambda_{1,t+1}$ as the induced representation $\varphi_{0,t}$. In general, an $n$-dimensional representation $\varphi : W_{\mathbb{R}} \to \mathrm{GL}_n(\mathbb{C})$ is of the form

$$\varphi = (\lambda_{\varepsilon_1,t_1} \oplus \cdots \oplus \lambda_{\varepsilon_p,t_p}) \oplus (\varphi_{-N_1,u_1} \oplus \cdots \oplus \varphi_{-N_q,u_q}), \tag{3.2}$$

where we may assume that $N_i \geq 0$ for all $i$ and a representation of the form $\lambda_{0,t} \oplus \lambda_{1,t+1}$ is treated as $\varphi_{0,t}$.

Note that

$$\lambda_{\varepsilon,t}{}^{\vee} = \lambda_{\varepsilon,t}{}^{-1} = \lambda_{\varepsilon,-t+2\varepsilon}$$

and

$$\varphi_{-N,t}{}^{\vee} = \mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \overline{\chi_{-N,t}}^{-1} = \varphi_{-N,N-t}.$$

## 3.2 *L*-, *ϵ*- and *γ*-factors

For the 1-dimensional $\lambda_{\varepsilon,t}$, the $L$-, $\epsilon$- and $\gamma$-factors are defined as follows:

$$\begin{aligned} L(\lambda_{\varepsilon,t}) &= \pi^{-\frac{t}{2}}\Gamma\Big(\frac{t}{2}\Big),\\ \epsilon(\lambda_{\varepsilon,t}, \psi_{\mathbb{R}}) &= (-i)^{\varepsilon},\\ \gamma(\lambda_{\varepsilon,t}, \psi_{\mathbb{R}}) &= \epsilon(\lambda_{\varepsilon,t}, \psi_{\mathbb{R}})\frac{L(\lambda_{\varepsilon,t}{}^{\vee}|\cdot|)}{L(\lambda_{\varepsilon,t})} = (-i)^{\varepsilon}\pi^{t-\varepsilon-\frac{1}{2}}\frac{\Gamma(\frac{1-t+2\varepsilon}{2})}{\Gamma(\frac{t}{2})}. \end{aligned}$$

For the 2-dimensional representation $\varphi_{-N,t}$ with $N \geq 0$, the local factors are defined as follows:

$$\begin{aligned} L(\varphi_{-N,t}) &= L(\chi_{-N,t}) = 2(2\pi)^{-t}\Gamma(t),\\ \epsilon(\varphi_{-N,t}, \psi_{\mathbb{R}}) &= -i \cdot \epsilon(\chi_{-N,t}, \psi_{\mathbb{C}}) = -i^{|N|+1},\\ \gamma(\varphi_{-N,t}, \psi_{\mathbb{R}}) &= \epsilon(\varphi_{-N,t}, \psi_{\mathbb{R}}) \cdot \frac{L(\varphi_{-N,t}{}^{\vee}|\cdot|)}{L(\varphi_{-N,t})} = -i^{|N|+1}(2\pi)^{2t-N-1} \cdot \frac{\Gamma(1-t+N)}{\Gamma(t)}. \end{aligned}$$

In general, if $\varphi : W_{\mathbb{R}} \to \mathrm{GL}_n(\mathbb{C})$ is an $n$-dimensional representation as in (3.2), we again define the local factors multiplicatively by

$$\begin{aligned} L(\varphi) &= L(\lambda_{\varepsilon_1,t_1}) \cdots L(\lambda_{\varepsilon_p,t_p}) \cdot L(\varphi_{-N_1,u_1}) \cdots L(\chi_{-N_q,u_q}),\\ \epsilon(\varphi, \psi_{\mathbb{R}}) &= \epsilon(\lambda_{\varepsilon_1,t_1}, \psi_{\mathbb{R}}) \cdots \epsilon(\lambda_{\varepsilon_p,t_p}, \psi_{\mathbb{R}}) \cdot \epsilon(\varphi_{-N_1,u_1}, \psi_{\mathbb{R}}) \cdots \epsilon(\chi_{-N_q,u_q}, \psi_{\mathbb{R}}),\\ \gamma(\varphi, \psi_{\mathbb{R}}) &= \gamma(\lambda_{\varepsilon_1,t_1}, \psi_{\mathbb{R}}) \cdots \gamma(\lambda_{\varepsilon_p,t_p}, \psi_{\mathbb{R}}) \cdot \gamma(\varphi_{-N_1,u_1}, \psi_{\mathbb{R}}) \cdots \gamma(\chi_{-N_q,u_q}, \psi_{\mathbb{R}}). \end{aligned}$$

Let us note that for the parameter $\varphi_{0,t}$ one can check

$$\begin{aligned} L(\varphi_{0,t}) &= L(\lambda_{0,t})L(\lambda_{1,t+1}),\\ \epsilon(\varphi_{0,t}, \psi_{\mathbb{R}}) &= \epsilon(\lambda_{0,t}, \psi_{\mathbb{R}})\epsilon(\lambda_{1,t+1}, \psi_{\mathbb{R}}),\\ \gamma(\varphi_{0,t}, \psi_{\mathbb{R}}) &= \gamma(\lambda_{0,t}, \psi_{\mathbb{R}})\gamma(\lambda_{1,t+1}, \psi_{\mathbb{R}}) \end{aligned}$$

by using the duplication formula

$$\Gamma\Big(\frac{t}{2}\Big)\Gamma\Big(\frac{t+1}{2}\Big) = 2^{1-t}\sqrt{\pi}\Gamma(t).$$

## 3.3 GL(1)-twist

Let $\lambda_{\varepsilon,t}$ and $\lambda_{\delta,s}$ be characters on $\mathbb{R}^{\times}$. We set

$$\eta = \begin{cases} 2 & \text{if } \varepsilon = \delta = 1,\\ 0 & \text{otherwise,} \end{cases}$$

so that

$$\varepsilon + \delta - \eta \in \{0, 1\} \quad \text{and} \quad \varepsilon + \delta - \eta = \varepsilon + \delta \ (\mathrm{mod}\ 2).$$

We then have

$$\lambda_{\varepsilon,t} \otimes \lambda_{\delta,s} = \lambda_{\varepsilon+\delta-\eta,\, s+t-\eta}, \tag{3.3}$$

and hence

$$\begin{aligned} L(\lambda_{\varepsilon,t} \otimes \lambda_{\delta,s}) &= \pi^{-\frac{s+t-\eta}{2}}\Gamma\Big(\frac{s+t-\eta}{2}\Big), \\ \epsilon(\lambda_{\varepsilon,t} \otimes \lambda_{\delta,s}, \psi_{\mathbb{R}}) &= (-i)^{\varepsilon+\delta-\eta}, \\ \gamma(\lambda_{\varepsilon,t} \otimes \lambda_{\delta,s}, \psi_{\mathbb{R}}) &= (-i)^{\varepsilon+\delta-\eta}\pi^{s+t-\varepsilon-\delta+\eta-\frac{1}{2}}\frac{\Gamma(\frac{1-s-t+2(\varepsilon+\delta)-\eta}{2})}{\Gamma(\frac{s+t-\eta}{2})}. \end{aligned} \tag{3.4}$$

For the 2-dimensional parameter $\varphi_{-N,t} = \mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \chi_{-N,t}$, the twisted parameter $\varphi_{-N,t} \otimes \lambda_{\delta,s}$ is computed as

$$\begin{aligned} \varphi_{-N,t} \otimes \lambda_{\delta,s} &= \mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}}(\chi_{-N,t} \otimes (\lambda_{\delta,s} \circ N_{\mathbb{C}/\mathbb{R}})) \\ &= \mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}}(\chi_{-N,t} \otimes \chi_{0,s-\delta}) \\ &= \mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \chi_{-N,t+s-\delta} \end{aligned}$$

and

$$\varphi_{-N,t} \otimes \lambda_{\delta,s} = \varphi_{-N,t+s-\delta}.$$

Accordingly, we have

$$\begin{aligned} L(\varphi_{-N,t} \otimes \lambda_{\delta,s}) &= 2(2\pi)^{-(s+t-\delta)}\Gamma(s+t-\delta), \\ \epsilon(\varphi_{-N,t} \otimes \lambda_{\delta,s}, \psi_{\mathbb{R}}) &= -i^{|N|+1}, \\ \gamma(\chi_{-N,t} \otimes \lambda_{\delta,s}, \psi_{\mathbb{R}}) &= -i^{|N|+1}(2\pi)^{2(s+t-\delta)-N} \cdot \frac{\Gamma(1-s-t+\delta+N)}{\Gamma(s+t)}. \end{aligned}$$

If $\varphi : W_{\mathbb{R}} \to \mathrm{GL}_n(\mathbb{C})$ is an $n$-dimensional representation as in (3.2), we have

$$\varphi \otimes \lambda_{\delta,s} = (\lambda_{\varepsilon_1+\delta-\eta_1, s+t_1-\eta_1} \oplus \cdots \oplus \lambda_{\varepsilon_p+\delta-\eta_p, s+t_p-\eta_p}) \oplus (\varphi_{-N_1, s+u_1-\delta} \oplus \cdots \oplus \varphi_{-N_q, s+u_q-\delta}),$$

where $\eta_i$ is defined as before, namely $\eta_i = 2$ if $\varepsilon_i = \delta_i = 1$, and $\eta_i = 0$ otherwise. Accordingly, we have

$$\gamma(\varphi \otimes \lambda_{\delta,s}, \psi_{\mathbb{R}}) = F(s)\prod_{i=1}^{p}\frac{\Gamma(\frac{1-s-u_i+2(\varepsilon_i+\delta)-\eta_i}{2})}{\Gamma(\frac{s+u_i-\eta_i}{2})}\prod_{i=1}^{q}\frac{\Gamma(1-s-t_i+\delta_i+N_i)}{\Gamma(s+t_i-\delta_i)},$$

where $F(s)$ is a holomorphic function without a zero.

We set

$$\begin{aligned} L(s, \varphi) &= L(\varphi \otimes \chi_{0,s}), \\ \epsilon(s, \varphi, \psi_{\mathbb{R}}) &= \epsilon(\varphi \otimes \chi_{0,s}, \psi_{\mathbb{R}}), \\ \gamma(s, \varphi, \psi_{\mathbb{R}}) &= \gamma(\varphi \otimes \chi_{0,s}, \psi_{\mathbb{R}}). \end{aligned}$$

We then have

$$\gamma(s, \varphi, \psi_{\mathbb{R}}) = \epsilon(\varphi, \psi_{\mathbb{R}})\frac{L(1-s, \varphi^\vee)}{L(s, \varphi)}.$$

## 3.4 GL(2)-twist

For 2-dimensional representations $\varphi_{-N,t}$ and $\varphi_{-M,s}$ of $W_{\mathbb{R}}$, we have

$$\begin{aligned} \varphi_{-N,t} \otimes \varphi_{-M,s} &= (\mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \chi_{-N,t}) \otimes (\mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \chi_{-M,s}) \\ &= (\mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \chi_{-N,t} \cdot \chi_{-M,s}) \oplus (\mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \chi_{-N,t} \cdot \overline{\chi_{-M,s}}) \\ &= (\mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \chi_{-(N+M),t+s}) \oplus (\mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \chi_{-N,t} \cdot \chi_{M,s-M}) \\ &= \varphi_{-(N+M),t+s} \oplus (\mathrm{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} \chi_{-(N-M),t+s-M}) \\ &= \varphi_{-(N+M),t+s} \oplus \varphi_{-(N-M),t+s-M}. \end{aligned}$$

## 3.5 Local Langlands correspondence for $\mathrm{GL}_n(\mathbb{R})$

By the Archimedean local Langlands correspondence, originally established by Langlands [11], there is a one-to-one correspondence between the set $\mathrm{Irr}_n$ of (infinitesimal equivalence classes) of irreducible admissible representations of $\mathrm{GL}_n(\mathbb{R})$ and the set $\Phi_n$ of (conjugacy classes of) all continuous semisimple $n$-dimensional representations of $W_{\mathbb{R}}$. This correspondence is explicitly described as follows.

The 1-dimensional representation $\lambda_{\varepsilon,t}$ corresponds to the character on $\mathrm{GL}_1(\mathbb{R})$ in the obvious way. The 2-dimensional representation $\varphi_{-N,t}$ corresponds to the representation of $\mathrm{GL}_2(\mathbb{R})$ of the form

$$D_N \otimes |\det|^{t-\frac{N}{2}},$$

where $D_N$ is the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ if $N \geq 1$, and the limit of discrete series if $N = 0$.

In general, let

$$\varphi = \varphi_1 \oplus \cdots \oplus \varphi_k \in \Phi_n,$$

where each $\varphi_i$ is either $\lambda_{\varepsilon_i,t_i}$ or $\varphi_{-N_i,t_i}$ with $N_i \geq 0$, with the proviso that $\lambda_{0,t} \oplus \lambda_{1,t+1}$ is considered as $\varphi_{0,t}$. For each $i$, we let $\pi_i$ be the representation of $\mathrm{GL}_{n_i}(\mathbb{R})$ corresponding to $\varphi_i$ as above, so that $\pi_i$ is a character with $n_i = 1$ or a (limit of) discrete series with $n_i = 2$. Note that $n_1 + \cdots + n_k = n$. Let $P(\mathbb{R})$ be the $(n_1, \dots, n_k)$-parabolic of $\mathrm{GL}_n(\mathbb{R})$, so that the Levi part is $\mathrm{GL}_{n_1}(\mathbb{R}) \times \cdots \times \mathrm{GL}_{n_k}(\mathbb{R})$, where $n_i = 1, 2$. Consider the (normalized) induced representation

$$I(\varphi) := \mathrm{Ind}_{P(\mathbb{R})}^{\mathrm{GL}_n(\mathbb{R})} \pi_1 \otimes \cdots \otimes \pi_k.$$

Let us reorder the constituents of $\varphi$ in the Langlands situation, which means

$$\mathrm{Re}(t_1) \geq \cdots \geq \mathrm{Re}(t_k).$$

By the Langlands quotient theorem, the induced representation $I(\varphi)$ has a unique irreducible quotient (the Langlands quotient), which we denote by $\pi_\varphi$. Then the local Langlands correspondence is obtained by the map

$$\Phi_n \to \mathrm{Irr}_n, \quad \varphi \mapsto \pi_\varphi.$$

## 3.6 Genericity conditions

It is well known that the induced representation $I(\varphi)$ is generic (see, for example, [16, Theorem 15.4.1, p. 381]). The following proposition characterizes when the Langlands quotient $\pi_\varphi$ is generic.

**Proposition 3.1.** *Let*

$$\varphi = (\lambda_{\varepsilon_1,t_1} \oplus \cdots \oplus \lambda_{\varepsilon_p,t_p}) \oplus (\varphi_{-N_1,u_1} \oplus \cdots \oplus \varphi_{-N_q,u_q})$$

*be a Langlands parameter, where $\lambda_{0,t} \oplus \lambda_{1,t+1}$ (if there is any) is considered as $\varphi_{0,t}$. Assume*

$$\mathrm{Re}(t_1) \leq \cdots \leq \mathrm{Re}(t_p) \quad \textit{and} \quad N_1 \leq \cdots \leq N_q.$$

*Then the following statements are all equivalent:*

(i) *The representation $\pi_\varphi$ that corresponds to $\varphi$ under the local Langlands correspondence is generic.*
(ii) *$\pi_\varphi = I(\varphi)$, namely $I(\varphi)$ is irreducible.*
(iii) *The following three assertions hold:*
  (a) *If $t_i - t_j \in \mathbb{Z}$, then $t_i - t_j \in 2\mathbb{Z}$.*
  (b) *If $u_i - t_j \in \mathbb{Z}$, then $-\varepsilon_j \leq u_i - t_j \leq N_i - \varepsilon_j$.*
  (c) *If $u_i - u_j \in \mathbb{Z}$, then $0 \leq u_j - u_i \leq N_j - N_i$ for $i \leq j$. In particular, if $u_i - u_j \in \mathbb{Z}$, then $u_i \leq u_j$ for $i \leq j$.*
  *(Note that if $t_i - t_j \notin \mathbb{Z}$, $u_i - t_j \notin \mathbb{Z}$ or $u_i - u_j \notin \mathbb{Z}$, then there is no condition for the corresponding case.)*
(iv) *The Rankin–Selberg L-factor*

$$L(s, \varphi \otimes \varphi^\vee) := L(\varphi \otimes \varphi^\vee \otimes \chi_{0,s})$$

*is holomorphic at $s = 1$.*

*Proof.* The equivalence of (i) and (ii) is well known. The equivalence of (ii) and (iii) was obtained by Speh in her Ph.D thesis, and the results are nicely summarized in [12, Theorem 10b, p. 164]. (But one has to translate [12] to our situation. For doing that, reorder the constituents of $\varphi$ in the Langlands situation, and use that her $p_i$ is our $N_i$, her $s_i$ with $n_i = 1$ is our $t_i$, and her $s_i$ with $n_i = 2$ is our $u_i - \frac{N_i}{2}$. The details are left to the reader.)

To show the equivalence of (iii) and (iv), note that, since

$$\varphi^\vee = (\lambda_{\varepsilon_1, -t_1+2\varepsilon_1} \oplus \cdots \oplus \lambda_{\varepsilon_p, -t_p+2\varepsilon_p}) \oplus (\varphi_{-N_1, N_1-u_1} \oplus \cdots \oplus \varphi_{-N_q, N_q-u_q}),$$

one can compute

$$\begin{aligned}\varphi \otimes \varphi^\vee \otimes \lambda_{0,s} = {} & \bigoplus_{i,j} \lambda_{\varepsilon_i+\varepsilon_j,\, s+t_i-t_j+2\varepsilon_j-\gamma_{ij}} \bigoplus_{i,j} \varphi_{-N_i,\, s+u_i-t_j+\varepsilon_j} \\ & \times \bigoplus_{i,j} \varphi_{-N_i,\, s+N_i-u_i+t_j-\varepsilon_j} \bigoplus_{i,j} \varphi_{-(N_i+N_j),\, s+u_i+N_j-u_j} \oplus \varphi_{-(N_i-N_j),\, s+u_i-u_j},\end{aligned}$$

where $\varepsilon_i + \varepsilon_j$ is viewed modulo 2 as before, and $\gamma_{ij} = 2$ if $\varepsilon_i = \varepsilon_j = 2$, and 0 otherwise. Hence,

$$\begin{aligned} L(\varphi \otimes \varphi^\vee \otimes \lambda_{0,s}) &= \prod_{i,j} L(\lambda_{\varepsilon_i+\varepsilon_j,\, s+t_i-t_j+2\varepsilon_j-\gamma_{ij}}) \prod_{i,j} L(\varphi_{-N_i,\, s+u_i-t_j+\varepsilon_j}) \\ &\quad \times \prod_{i,j} L(\varphi_{-N_i,\, s+N_i-u_i+t_j-\varepsilon_j}) \prod_{i,j} L(\varphi_{-(N_i+N_j),\, s+u_i+N_j-u_j}) L(\varphi_{-(N_i-N_j),\, s+u_i-u_j}) \\ &= F(s) \prod_{i,j} \Gamma\Big(\frac{s+t_i-t_j+2\varepsilon_j-\gamma_{ij}}{2}\Big) \prod_{i,j} \Gamma(s+u_i-t_j+\varepsilon_j) \\ &\quad \times \prod_{i,j} \Gamma(s+N_i-u_i+t_j-\varepsilon_j) \prod_{i,j} \Gamma(s+u_i+N_j-u_j) \\ &\quad \times \prod_{i\geq j} \Gamma(s+u_i-u_j) \prod_{i<j} \Gamma(s+u_i-u_j-(N_i-N_j)) \\ &= F(s) \prod_{i,j} \Gamma\Big(\frac{s+t_i-t_j+2\varepsilon_j-\gamma_{ij}}{2}\Big) \prod_{i,j} \Gamma(s+u_i-t_j+\varepsilon_j) \\ &\quad \times \prod_{i,j} \Gamma(s+N_i-u_i+t_j-\varepsilon_j) \prod_{i,j} \Gamma(s+u_i+N_j-u_j) \\ &\quad \times \prod_{i} \Gamma(s) \prod_{i<j} \Gamma(s+u_j-u_i)\Gamma(s+u_i-u_j-(N_i-N_j)), \end{aligned}$$

where $F(s)$ is a holomorphic function without a zero. We want this to be holomorphic at $s = 1$.

To derive (a), assume that

$$\Gamma\Big(\frac{s+t_i-t_j+2\varepsilon_j-\gamma_{ij}}{2}\Big)$$

is holomorphic at $s = 1$. If $t_i - t_j \notin \mathbb{Z}$, this is automatic. Assume $t_i - t_j \in \mathbb{Z}$. Then we must have either

$$t_i - t_j + 2\varepsilon_j - \gamma_{ij} \geq 0 \quad \text{or} \quad t_i - t_j + 2\varepsilon_j - \gamma_{ij} \in 2\mathbb{Z}.$$

The second condition is equivalent to $t_i - t_j \in 2\mathbb{Z}$. For the first condition, by switching the roles of $i$ and $j$, we also have $t_j - t_i + 2\varepsilon_i - \gamma_{ij} \geq 0$. By combining the two, we obtain

$$-2\varepsilon_j + \gamma_{ij} \leq t_i - t_j \leq 2\varepsilon_i - \gamma_{ij}.$$

If $\varepsilon_i = \varepsilon_j$, then $0 \leq t_i - t_j \leq 0$, which implies $t_i - t_j = 0 \in 2\mathbb{Z}$. If $\varepsilon_i = 0$ and $\varepsilon_j = 1$, then we have $-2 \leq t_i - t_j \leq 0$. Hence either

$$t_i - t_j \in \{-2, 0\} \subseteq 2\mathbb{Z} \quad \text{or} \quad t_j - t_i = 1.$$

But the latter would give us a constituent of the form $\lambda_{0,t_i} \oplus \lambda_{1,t_i+1}$, which is considered as $\varphi_{0,t_i}$.

To derive (b), assume $\Gamma(s + u_i - t_j + \varepsilon_j)$ is holomorphic at $s = 1$. Then we must have $u_i - t_j + \varepsilon_j \notin \mathbb{Z}^{<0}$. If $u_i - t_j \notin \mathbb{Z}$, this is automatic. If $u_i - t_j \in \mathbb{Z}$, then we must have $u_i - t_j + \varepsilon_j \geq 0$, which implies $-\varepsilon_j \leq u_i - t_j$. The other inequality of (iii) follows in the same way from $\Gamma(s + N_i - u_i + t_j - \varepsilon_j)$.

To derive (c), we argue in the same way by looking at $\Gamma(s+u_j-u_i)$ and $\Gamma(s+u_i-u_j-(N_i-N_j))$ for $i \leq j$.

As for the gamma function $\Gamma(s+u_i+N_j-u_j)$, if this is holomorphic at $s=1$ and $u_i-u_j \in \mathbb{Z}$, then we must have $u_i+N_j-u_j \geq 0$. By switching the roles of $i$ and $j$, we also have $u_j+N_i-u_i \geq 0$. By combining the two, we obtain $-N_j \leq u_i-u_j \leq N_i$. But this is subsumed under (iii).

Hence we have proven that if $\pi$ is generic, then conditions (a), (b) and (c) are satisfied. The converse is clear by looking at $s=1$ in the above gamma functions. □

Let us note that condition (iii) (c) in the above lemma is essentially the same as the complex case.

### 3.7 Local converse theorem for $\mathrm{GL}_n(\mathbb{R})$

As we did in the complex case, we define

$$\lambda_{\varepsilon,t} \sim \lambda_{\varepsilon',t'} \quad \text{if } t-t' \in \mathbb{Z},$$

and

$$\varphi_{-N,u} \sim \varphi_{-N',u'} \quad \text{if } u-u' \in \mathbb{Z}.$$

Further, we define

$$\lambda_{\varepsilon,t} \sim \varphi_{-N,u} \quad \text{if } t-u \in \mathbb{Z}.$$

The relation $\sim$ is certainly an equivalence relation.

Let us first prove the following lemma.

**Lemma 3.2.** *Let*

$$\begin{aligned}\varphi &= (\lambda_{\varepsilon_1,t_1} \oplus \cdots \oplus \lambda_{\varepsilon_p,t_p}) \oplus (\varphi_{-N_1,u_1} \oplus \cdots \oplus \varphi_{-N_q,u_q}),\\ \varphi' &= (\lambda_{\varepsilon_1',t_1'} \oplus \cdots \oplus \lambda_{\varepsilon_{p'}',t_{p'}'}) \oplus (\varphi_{-N_1',u_1'} \oplus \cdots \oplus \varphi_{-N_{q'}',u_{q'}'})\end{aligned}$$

*be generic parameters such that all constituents are equivalent under $\sim$. Further, we assume*

$$t_1 \preceq \cdots \preceq t_p \quad \textit{and} \quad 0 \leq N_1 \leq \cdots \leq N_q,$$

*so the genericity condition* (iii) (c) *implies*

$$u_1 \preceq \cdots \preceq u_p,$$

*and similarly for the $t_i'$'s, $u_i'$'s and $N_i'$'s.*

*Assume*

$$F(s)\gamma(s,\varphi,\psi_{\mathbb{R}}) = \gamma(s,\varphi',\psi_{\mathbb{R}}),$$

*where $F(s)$ is a meromorphic function whose zeros and poles do not interfere with those of $\gamma(s,\varphi,\psi_{\mathbb{R}})$ and $\gamma(s,\varphi',\psi_{\mathbb{R}})$. Then $p=p'$ and $q=q'$, and $t_i-t_j' \in 2\mathbb{Z}$ for all $i,j$.*

*Proof.* By computing (the reciprocals of) the gamma factors, we have

$$F(s)\prod_{i=1}^{p}\frac{\Gamma(\frac{s+t_i}{2})}{\Gamma(\frac{1-s-t_i+2\varepsilon_i}{2})}\cdot\prod_{i=1}^{q}\frac{\Gamma(s+u_i)}{\Gamma(1-s-u_i+N_i)} = \prod_{i=1}^{p'}\frac{\Gamma(\frac{s+t_i'}{2})}{\Gamma(\frac{1-s-t_i'+2\varepsilon_i'}{2})}\cdot\prod_{i=1}^{q'}\frac{\Gamma(s+u_i')}{\Gamma(1-s-u_i'+N_i')},$$

where $F(s)$ is a meromorphic function (possibly different from the one in the lemma) whose zeros and poles do not interfere with those of the gamma functions appearing here.

Since all constituents are equivalent under $\sim$, we know $t_i-t_j \in \mathbb{Z}$, $u_i-u_j \in \mathbb{Z}$ and $t_i-u_j \in \mathbb{Z}$, and similarly for the $t_i'$'s and $u_i'$'s. By the genericity condition (iii) (a), we know that $t_i-t_j \in 2\mathbb{Z}$ and $t_i'-t_j' \in 2\mathbb{Z}$. Hence either $t_i-t_j' \in 2\mathbb{Z}$ for all $i,j$, or $t_i-t_j' \in 2\mathbb{Z}+1$ for all $i,j$. Assume $t_i-t_j' \notin 2\mathbb{Z}$. Then

$$\prod_{i=1}^{p}\Gamma\Big(\frac{s+t_i}{2}\Big) \quad \text{and} \quad \prod_{i=1}^{p}\Gamma\Big(\frac{s+t_i'}{2}\Big)$$

do not share a pole. Hence by choosing $M \in \mathbb{C}$ comparable to the $t_i$'s and "large enough" with respect to $\preceq$, we get a pole of order $p+q$ at $s=-M$ on the left-hand side, and a pole of order $q'$ on the right-hand side (note that, by taking $M$ large enough, the denominators never have a pole at $s=-M$). Hence we must have $p+q=q'$. By switching the roles of $\varphi$ and $\varphi'$, we have $p'+q'=q$. These two imply $p+p'=0$ (namely $p=p'=0$) and $q=q'$. Apparently, in this case the assertion $t_i-t'_j \in 2\mathbb{Z}$ is vacuously true and $p+q=p'+q'$. If $p+p' \neq 0$, we must have $t_i-t'_j \in 2\mathbb{Z}$ for all $i,j$.

Next consider $M \in \mathbb{C}$ such that $-M+t_i+1 \in 2\mathbb{Z}$ for some (and hence all) $i$. Then, at $s=-M$, none of $\Gamma(\frac{s+t_i}{2})$ and $\Gamma(\frac{s+t'_i}{2})$ has a pole. Further, by making $M$ "large enough", we know that, at $s=-M$, all denominators do not have a pole and the left-hand side has a pole of order $q$, namely the poles coming from $\Gamma(s+u_i)$. Similarly, we have a pole of order $q'$ on the right-hand side, which implies $q=q'$, and hence $p=p'$. □

Next we prove the following proposition.

**Proposition 3.3.** *Let*

$$\varphi=(\lambda_{\varepsilon_1,t_1}\oplus\cdots\oplus\lambda_{\varepsilon_p,t_p})\oplus(\varphi_{-N_1,u_1}\oplus\cdots\oplus\varphi_{-N_q,u_q}),$$
$$\varphi'=(\lambda_{\varepsilon'_1,t'_1}\oplus\cdots\oplus\lambda_{\varepsilon'_{p'},t'_{p'}})\oplus(\varphi_{-N'_1,u'_1}\oplus\cdots\oplus\varphi_{-N'_{q'},u'_{q'}})$$

*be generic parameters such that all constituents are equivalent under $\sim$, where the $t_i$'s, $N_i$'s, $t'_i$'s and $N'_i$'s are ordered as in the above lemma. Assume*

$$F_\chi(s)\gamma(s,\varphi\otimes\chi,\psi_{\mathbb{R}})=\gamma(s,\varphi'\otimes\chi,\psi_{\mathbb{R}}) \tag{3.5}$$

*for all characters $\chi$, where $F_\chi(s)$ is a meromorphic function (depending on $\chi$) whose zeros and poles do not interfere with those of $\gamma(s,\varphi\otimes\chi,\psi_{\mathbb{R}})$ and $\gamma(s,\varphi'\otimes\chi,\psi_{\mathbb{R}})$. Then $\varphi=\varphi'$.*

*Proof.* From the above lemma, we already know that $p=p'$, $q=q'$ and $t_i-t'_j \in 2\mathbb{Z}$.

By choosing $\chi$ to be trivial, (the reciprocal of) identity (3.5) is written as

$$F(s)\prod_{i=1}^{p}\frac{\Gamma(\frac{s+t_i}{2})}{\Gamma(\frac{1-s-t_i+2\varepsilon_i}{2})}\cdot\prod_{i=1}^{q}\frac{\Gamma(s+u_i)}{\Gamma(1-s-u_i+N_i)}=\prod_{i=1}^{p}\frac{\Gamma(\frac{s+t'_i}{2})}{\Gamma(\frac{1-s-t'_i+2\varepsilon'_i}{2})}\cdot\prod_{i=1}^{q}\frac{\Gamma(s+u'_i)}{\Gamma(1-s-u'_i+N'_i)}, \tag{3.6}$$

where $F(s)$ is a meromorphic function whose zeros and poles do not interfere with those of the gamma functions appearing here.

We will show $t_i=t'_i$ for $i=1,\dots,q$ by looking at poles of $\Gamma(\frac{s+t_i}{2})$ and $\Gamma(\frac{s+t'_i}{2})$. By the genericity condition (iii) (a), we know that $t_i-t_j \in 2\mathbb{Z}$ for all $i,j$. Recall that the $t_i$'s and $t'_i$'s are in increasing order, so that $t_1 \preceq t_i$ and $t'_1 \preceq t'_i$ for all $i$. Assume $t_1 \prec t'_1$ (strict inequality). Let us consider the poles at $s=-t_1$. Since $\Gamma(\frac{s+t_1}{2})$ has a pole at $s=-t_1$, the numerator of the left-hand side has a pole. Certainly, the denominator $\Gamma(\frac{1-s-t_i+2\varepsilon_i}{2})$ does not have a pole at $s=-t_1$ because $1+t_1-t_i+2\varepsilon_i$ is odd. Also, $\Gamma(1-s-u_i+N_i)$ does not have a pole because by the genericity condition (iii) (b) we have

$$1-(-t_1)-u_i+N_i \geq 1+\varepsilon_1 \geq 1.$$

Hence the left-hand side of (3.6) has a pole. Now since we already know $t_1-t'_1 \in 2\mathbb{Z}$, our assumption $t_1 \prec t'_1$ actually implies $t_1 \prec t'_1-1$. Apparently, on the right-hand side, $\Gamma(\frac{s+t'_i}{2})$ cannot have a pole at $s=-t_1$ for all $i$. Hence some $\Gamma(s+u'_i)$ must have a pole at $s=-t_1$. But the genericity condition implies

$$-t_1+u'_i > -t'_1+1+u'_i \geq 1-\varepsilon'_1 \geq 0.$$

Hence $\Gamma(s+u'_i)$ cannot have a pole at $s=-t_1$, which is a contradiction. Thus we must have $t_1 \succeq t'_1$. By switching the roles of $t_1$ and $t'_1$, we have $t_1 \preceq t'_1$. Hence $t_1=t'_1$. Then we can cancel the gamma functions containing $t_1$ and $t'_1$. By repeating the same argument, we obtain

$$t_i=t'_i$$

for all $i=1,\dots,p$.

Next we show that the $\varepsilon_i$'s agree with the $\varepsilon'_j$'s. For this purpose, we consider the twist by $\chi = \lambda_{1,s}$. By using (3.4), (the reciprocal of) equality (3.5) is written as

$$F(s)\prod_{i=1}^{p}\frac{\Gamma(\frac{s+t_i-2\varepsilon_i}{2})}{\Gamma(\frac{3-s-t_i}{2})}\cdot\prod_{i=1}^{q}\frac{\Gamma(s+u_i-1)}{\Gamma(1-s-u_i+1+N_i)}=\prod_{i=1}^{p}\frac{\Gamma(\frac{s+t'_i-2\varepsilon'_i}{2})}{\Gamma(\frac{3-s-t'_i}{2})}\cdot\prod_{i=1}^{q}\frac{\Gamma(s+u'_i-1)}{\Gamma(1-s-u'_i+1+N'_i)},$$

where $F(s)$ is a meromorphic function whose zeros and poles do not interfere with those of the gamma functions appearing here. Now let $k \in \{1,\dots,p\}$ be such that

$$t_k - 2\varepsilon_k \preceq t_i - 2\varepsilon_i$$

for all $i$, namely $t_k - 2\varepsilon_k$ is minimal with respect to $\preceq$. Similarly, let $\ell$ be such that

$$t'_\ell - 2\varepsilon'_\ell \preceq t'_i - 2\varepsilon'_i$$

for all $i$. One can then apply the same argument as above with $s = -(t_k - 2\varepsilon_k)$ and conclude that

$$t_k - 2\varepsilon_k = t'_\ell - 2\varepsilon'_\ell.$$

By arguing inductively, we have

$$\{t_1 - 2\varepsilon_1,\cdots,t_p - 2\varepsilon_p\} = \{t_1 - 2\varepsilon'_1,\cdots,t_p - 2\varepsilon'_p\}$$

as multisets.

From this identity of multisets, we will derive the identity

$$\{(\varepsilon_1,t_1),\cdots,(\varepsilon_p,t_p)\} = \{(\varepsilon'_1,t_1),\cdots,(\varepsilon'_p,t_p)\}$$

of multisets. For this, it suffices to show $(\varepsilon_i,t_i) = (\varepsilon'_j,t_j)$ for some $i$ and $j$, because then we can argue inductively on the size of the multisets. Now, we know $t_1 - 2\varepsilon_1 = t_i - 2\varepsilon'_i$ for some $i$. But then we must have $t_i - t_1 = 2(\varepsilon'_i - \varepsilon_1) \geq 0$ because of our ordering of the $t_i$'s. So we must have $\varepsilon'_i \geq \varepsilon_1$. Suppose $\varepsilon_1 = 1$. Then we have $\varepsilon'_i = 1$. Hence the equality $t_1 - 2\varepsilon_1 = t_i - 2\varepsilon'_i$ implies $t_1 = t_i$, and so $(\varepsilon_1,t_1) = (\varepsilon'_i,t_i)$. Next suppose $\varepsilon_1 = 0$. If $\varepsilon'_1 = 0$, then we have $(\varepsilon_1,t_1) = (\varepsilon'_1,t_1)$. If $\varepsilon'_1 = 1$, then by switching the roles of $\varepsilon_1$ and $\varepsilon'_1$ we have $(\varepsilon'_1,t_1) = (\varepsilon_j,t_j)$ for some $j$. Thus in any case, we know that $(\varepsilon_i,t_i) = (\varepsilon'_j,t_j)$ for some $i$ and $j$.

Now, we can cancel from (3.6) all factors containing $t_i$, $\varepsilon_i$, $t'_i$ and $\varepsilon'_i$ and obtain

$$F(s)\prod_{i=1}^{q}\frac{\Gamma(s+u_i)}{\Gamma(1-s-u_i+N_i)}=\prod_{i=1}^{q}\frac{\Gamma(s+u'_i)}{\Gamma(1-s-u'_i+N'_i)}. \tag{3.7}$$

Recall that the $u_i$'s and $u'_i$'s are in increasing order with respect to $\preceq$. We will show $u_i = u'_i$ by induction on $i$. Assume $u_1 \prec u'_1$ (strict inequality). Then $\Gamma(s+u_1)$ has a pole at $s = -u_1$ on the left-hand side, and the denominator $\Gamma(1-s-u_i+N_i)$ does not have a pole at $s = -u_1$ because by the genericity condition (iii) (c) we have $1 + u_1 - u_i + N_i \geq 1 + N_1$ for all $i$. But since $u_1 \prec u'_1$, the right-hand side cannot have a pole at $s = -u_1$. Hence we must have $u_1 \succeq u'_1$. By switching the roles of $u_1$ and $u'_1$, we have $u_1 \preceq u'_1$, from which we have $u_1 = u'_1$. Now assume we have shown $u_i = u'_i$ for $i = 1,\dots,j$ for some $j$. Then the above identity (3.7) is reduced to

$$F(s)\prod_{i=1}^{j}\frac{1}{\Gamma(1-s-u_i+N_i)}\prod_{i=j+1}^{q}\frac{\Gamma(s+u_i)}{\Gamma(1-s-u_i+N_i)}=\prod_{i=1}^{j}\frac{1}{\Gamma(1-s-u'_i+N'_i)}\prod_{i=j+1}^{q}\frac{\Gamma(s+u'_i)}{\Gamma(1-s-u'_i+N'_i)}. \tag{3.8}$$

Assume $u_{j+1} \prec u'_{j+1}$. Then by the same reasoning as above, the product

$$\prod_{i=j+1}^{q}\frac{\Gamma(s+u_i)}{\Gamma(1-s-u_i+N_i)}$$

has a pole at $s = -u_{j+1}$. Also, $\Gamma(1-s-u_i+N_i)$ does not have a pole at $s = -u_{j+1}$ for all $i = 1,\dots,j$, because $1 + u_{j+1} - u_i + N_i \geq 1 + N_i \geq 1$ by the genericity condition (iii) (c), since $j+1 \geq i$. Hence the left-hand side of (3.8) has a pole at $s = -u_{j+1}$. But the right-hand side does not have a pole at $s = u_{j+1}$ because $u_{j+1} \prec u'_{j+1}$. Thus we must have $u_{j+1} \succeq u'_{j+1}$. By switching the roles of $u_{j+1}$ and $u'_{j+1}$, we get $u_{j+1} \preceq u'_{j+1}$, from which we have $u_{j+1} = u'_{j+1}$. Hence we have

$$u_i = u'_i$$

for all $i = 1, \dots, q$.

By cancelling the numerators from (3.7), we obtain

$$\prod_{i=1}^{q} \Gamma(1 - s - u_i + N_i) = F(s) \prod_{i=1}^{q} \Gamma(1 - s - u'_i + N'_i).$$

Let $k, \ell \in \{1, \dots, q\}$ be such that

$$u_k - N_k \succeq u_i - N_i \quad \text{and} \quad u'_\ell - N'_\ell \succeq u'_i - N'_i$$

for all $i$. Then at $s = -u_k + N_k$, the left-hand side (and hence the right-hand side) is holomorphic, which implies $u_k - N_k \preceq u'_\ell - N'_\ell$. By switching the roles, we obtain $u_k - N_k \succeq u'_\ell - N'_\ell$, and hence $u_k - N_k = u'_\ell - N'_\ell$. By arguing inductively, we obtain

$$\{u_1 - N_1, \dots, u_q - N_q\} = \{u'_1 - N'_1, \dots, u'_q - N'_q\}$$

as multisets.

Then we can show $N_i = N'_i$ for all $i = 1, \dots, n$ by exactly the same argument as in the complex case as follows. By the above identity of the multisets, we must have $u_1 - N_1 = u'_i - N'_i$ for some $i$. Since we already know $u_i = u'_i$, we have

$$N'_i - N_1 = u'_i - u_1 = u'_i - u'_1 \leq N'_i - N'_1,$$

where the last inequality is by the genericity condition (iii). Hence we have $N'_1 \leq N_1$. Also, we must have $u'_1 - N'_1 = u_j - N_j$ for some $j$. By applying the same argument, we must have $N_1 \leq N'_1$. Thus we must have $N_1 = N'_1$. By arguing inductively, we have

$$N_i = N'_i$$

for all $i = 1, \dots, n$. □

Now, we are ready to prove the local converse theorem.

**Theorem 3.4.** *Let $\pi$ and $\pi'$ be generic irreducible admissible representations of* $\mathrm{GL}_n(\mathbb{R})$. *Assume that*

$$\gamma(s, \pi \times \chi, \psi_{\mathbb{R}}) = \gamma(s, \pi' \times \chi, \psi_{\mathbb{R}})$$

*for all unitary characters $\chi$. Then $\pi = \pi'$.*

*Proof.* As in the complex case, we may assume that the identity of the gamma factors holds for all (not necessarily unitary) characters $\chi$.

Let $\varphi$ and $\varphi'$ be the Langlands parameters of $\mathrm{GL}_n(\mathbb{R})$ corresponding to $\pi$ and $\pi'$, respectively. Let us write

$$\varphi = \varphi_1 \oplus \dots \oplus \varphi_k \quad \text{and} \quad \varphi' = \varphi'_1 \oplus \dots \oplus \varphi'_{k'},$$

where all constituents of each $\varphi_j$ are equivalent under $\sim$, and the constituents of $\varphi_i$ and $\varphi_j$ are inequivalent under $\sim$ for $i \neq j$, and similarly for $\varphi'$. We then have

$$\prod_{i=1}^{k} \gamma(s, \varphi_i \otimes \chi, \psi_{\mathbb{R}}) = \prod_{i=1}^{k'} \gamma(s, \varphi'_i \otimes \chi, \psi_{\mathbb{R}}).$$

Note that, for $i \neq j$, the gamma factors $\gamma(s, \varphi_i \otimes \chi, \psi_{\mathbb{R}})$ and $\gamma(s, \varphi_j \otimes \chi, \psi_{\mathbb{R}})$ do not share a zero or a pole, and similarly for $\varphi'$.

Now, assume that $\varphi$ and $\varphi'$ do not share any constituents equivalent under $\sim$. Then $\gamma(s, \varphi \otimes \chi, \psi_{\mathbb{R}})$ and $\gamma(s, \varphi' \otimes \chi, \psi_{\mathbb{R}})$ do not share a zero or pole. So there are at least some $\varphi_i$ and $\varphi'_j$ having constituents equivalent under $\sim$. By reordering indices, we may assume $i = j = 1$. Then the equality of the gamma factors is written as

$$F_\chi(s)\gamma(s, \varphi_1 \otimes \chi, \psi_{\mathbb{R}}) = \gamma(s, \varphi'_1 \otimes \chi, \psi_{\mathbb{R}}),$$

where $F_\chi(s)$ is a meromorphic function whose poles and zeros do not interfere with those of the above two gamma factors. Hence by the above proposition, we have $\varphi_1 = \varphi'_1$. Arguing inductively, we conclude $\varphi = \varphi'$. □

**Acknowledgment:** We thank Hervé Jacquet for suggesting this problem. We also thank the anonymous referee for his/her valuable comments.

**Funding:** The first-named author was partially supported by the Simons Foundation Collaboration Grant 422638 and by a PSC-CUNY award, jointly funded by the Professional Staff Congress and The City University of New York. The second-named author was partially supported by the Simons Foundation Collaboration Grant 584704.

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