Math 143 – Practice Midterm Exam 1

1. Find the following integrals:

   a) $\int (\ln x)^2 dx$. Solution: This was the trickiest one. Set $x = e^y$, then $dx = e^y dy$. Then simplify and use integration by parts.

   b) $\int x^2 \sin(\pi x)dx$. Solution: Use integration by parts twice. $u = x^2, dv = sin(\pi x)dx$ first, then ....

   c) $\int \sin^4(x)\cos^3(x)dx$. Solution: Use $\cos^2 x = 1 - \sin^2 x$, then u substitution where $u = \sin x$.

   d) $\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}dx$. Solution: there was a small typo. dx should be dt. Now use trig substitution. Set $t = sec(x)$. Since x goes from $\sqrt{2}$ to 2, t goes from $\pi/4$ to $\pi/3$. Also $dt = sec(x)tan(x)dx$, so the integral is $\int_{\pi/4}^{\pi/3} \frac{sec(x)tan(x)}{sec^3(x)tan(x)}dx = \int_{\pi/4}^{\pi/3} \frac{1}{sec^2(x)}dx = \int_{\pi/4}^{\pi/3} cos^2(x)dx = \int_{\pi/4}^{\pi/3} \frac{1}{2}(1 + cos(2x))dx = \frac{1}{2}(\pi/12 + \frac{\sqrt{3}-2}{4})$

   e) $\int \frac{2x+3}{(x-1)(x^2+1)^2}$. Solution: Standard partial fractions question.

   f) $\int_0^1 \frac{x}{x^2+4x+13}dx$. Solution: Need to complete the square, then do partial fractions. In particular, $x^2 + 4x + 13 = x^2 + 4x + 4 - 4 + 13 = (x+2)^2 + 9$. So $\int_0^1 \frac{x}{x^2+4x+13}dx = \int_0^1 \frac{x}{(x+2)^2+9}dx$. Set $u = x+2$, so get $\int_2^3 \frac{u-2}{u^2+9}du$, which equals $\int_2^3 \frac{u}{u^2+9}du - \int_2^3 \frac{2}{u^2+9}du$. The first integral is a substituion $v = u^2 + 9$, and the second integral is $2\frac{1}{3}tan^{-1}(u/3)]_2^3$.

2. Calculate the following limits:

   a) $\lim_{x\to 0} \frac{x3^x}{3^x - 1}$. Solution: L'Hopital gives $\lim_{x\to 0} \frac{xln(3)3^x + 3^x}{ln(3)3^x}$. Divide top and bottom by 3^x and get $\lim_{x\to 0} \frac{1 + xln(3)}{ln(3)} = 1/ln(3)$.

   b) $\lim_{x\to\infty}(x - \ln x)$. Solution: Raise e to the limit. So consider $e^{\lim_{x\to\infty}(x-\ln x)} = \lim_{x\to\infty} e^{(x-\ln x)} = \lim_{x\to\infty} e^x e^{-\ln(x)} = \lim_{x\to\infty}\frac{e^x}{x}$. L'Hopital says this last limit is ∞. Therefore, the original limit is also ∞ (since if e to something is ∞, then that original something is ∞).

   c) $\lim_{x\to 0^+}(\tan 2x)^x$. Solution: Standard, did something like this in class. Write $\lim_{x\to 0^+}(\tan 2x)^x = \lim_{x\to 0^+} e^{ln((\tan 2x)^x)} = e^{\lim_{x\to 0^+} ln((\tan 2x)^x)} = ...$

   d) $\lim_{x\to 0} \cot 2x \sin 6x$. Solution: Turn into sines and cosines: $\lim_{x\to 0}\cot 2x \sin 6x = \lim_{x\to 0}\frac{cos(2x)sin(6x)}{sin(2x)}$, then use L'Hopital.

   e) $\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)}$. Solution: L'Hopital.

3. (a) Find the approximations T_6, M_6, S_6 for $\int_0^\pi \sin x dx$ and the corresponding errors E_T, E_M, E_S. Solution: Let's do S_6: The formula is $S_6 = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6))$. $\Delta x = \pi/6$, and we have $x_0 = 0, x_1 = \pi/6, x_2 = 2\pi/6, x_3 = 3\pi/6, x_4 = 4\pi/6, x_5 = 5\pi/6, x_6 = 6\pi/6 = \pi$. Then $S_6 = \frac{\pi}{18}(sin(0)+4sin(\pi/6)+2sin(2\pi/6)+4sin(3\pi/6)+2sin(4\pi/6) + 4sin(5\pi/6) + sin(\pi)) = \frac{\pi}{18}(2\sqrt{3} + 8)$.

   (b) How large do we have to choose n so that the approximations T_n, M_n, S_n to the integral in part (a) are accurate to within 0.00001? Solution: Let's do the problem for M_n: $f''(x) = -sin(x)$. Then $|f''(x)| = |-sin(x)| \leq 1$ for every $0 \leq x \leq \pi$. So we can take $K = 1$. Recall the formula $|E_M| \leq \frac{K(b-a)^3}{24n^2}$. We want $|E_M| \leq .00001$. In other words, we want $\frac{K(b-a)^3}{24n^2} \leq .00001$. But $\frac{K(b-a)^3}{24n^2} = \frac{1(\pi-0)^3}{24n^2} = \frac{\pi^3}{24n^2}$. Thus, we want $\frac{\pi^3}{24n^2} \leq .00001$. In other words, we want $n^2 \geq \frac{\pi^3}{24(.00001)} = \frac{100{,}000\pi^3}{24}$, so that we want $n \geq \sqrt{\frac{100{,}000\pi^3}{24}}$.

4. Determine whether each improper integral is convergent or divergent. Evaluate those that are convergent.

   (a) $\displaystyle\int_{-\infty}^0 xe^{2x}dx$. Solution: Type 1 improper integral, then use integration by parts.

   (b) $\displaystyle\int_6^8 \frac{4}{(x-6)^3}dx$. Solution: Type 2 improper integral, then use u substitution.

   (c) $\displaystyle\int_0^9 \frac{1}{\sqrt[3]{x-1}}$. Solution: Type 2 improper integral, noting that we must write $\displaystyle\int_0^1 \frac{1}{\sqrt[3]{x-1}} = \int_1^9 \frac{1}{\sqrt[3]{x-1}} + \int_0^9 \frac{1}{\sqrt[3]{x-1}}$, and then use limits. Also, it is a simple u substitution later.

   (d) $\displaystyle\int_1^\infty \frac{1}{x^2+x}dx$. Solution: Type 1 improper integral. To solve the integral from 1 to t, use partial fractions.

5. Use the Comparison Theorem to determine whether the integral is convergent or divergent.

   (a) $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}dx$. Solution: $\frac{2+e^{-x}}{x} \geq \frac{2}{x} \geq 0$ for $x \geq 1$ since $2 + e^{-x} \geq 2$ for $x \geq 1$. So we can use comparison theorem. Show that $\displaystyle\int_1^\infty \frac{2}{x}dx$ diverges, so $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}dx$ diverges as well.

   (b) $\displaystyle\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}dx$. Solution: since $0 < cos(x) \leq 1$ for $0 \leq x \leq 1$, we have that $0 < cos^2(x) \leq 1$, so that $0 \leq 1 \leq sec^2(x)$ for $0 \leq x \leq 1$. So we can use comparison theorem. Show that $\displaystyle\int_0^1 \frac{1}{x\sqrt{x}}dx$ diverges, so $\displaystyle\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}dx$ diverges as well.