The character of a simple supercuspidal representation of $SL(2, F)$

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Abstract

Let F be a non-Archimedean local field of characteristic zero. In this paper, we give a new interpretation of the character of a simple supercuspidal representation of $SL(2, F)$, on the split torus, when p is arbitrary.

1 Introduction

Let $\mathbf{G}$ be a connected reductive group defined over a non-Archimedean local field F of characteristic zero, and let $G = \mathbf{G}(F)$ be its F-points. If π is an irreducible admissible representation of G, we denote by θ_π its distribution character, which is a linear functional on $C_c^\infty(G)$, the locally constant, compactly supported functions on G. Harish-Chandra showed that θ_π can be represented by a locally constant function on the regular semisimple set of G, which we will also denote θ_π.

Suppose now that π is a supercuspidal representation. Much is known about θ_π. The first supercuspidal characters were computed by Sally and Shalika in [10], where they investigated the supercuspidal representations of $SL(2, F)$ when $p \neq 2$. Since then, there has been much work on understanding and computing supercuspidal characters, which has recently culminated in the work of Adler and Spice [1]. The computation of these characters has yielded important consequences, including applications to the study of stability of supercuspidal L-packets constructed by Debacker and Reeder [4], Reeder [8], and Kaletha [6].

In this paper, we give a new interpretation of the character values of a simple supercuspidal representation of $SL(2, F)$, on the split torus, when p is arbitrary (see Theorem 5.3). For $p \neq 2$, this was carried out in [10]. Our computation gives a formula that is inherently geometric in nature: a polynomial in q, whose powers are lengths of affine Weyl group elements that bound an explicit interval in the standard apartment of $SL(2, F)$ (see Remark 5.4).

It is our hope that even 46 years after these characters were computed in the $p \neq 2$ case, we could still say something new and interesting. We note that even in the theory of real groups, character values on split tori are generally difficult to compute, and are not known in general.

We now briefly present an outline of the paper. In section 2, we set up the notation that we will use throughout. In section 3, we recall the definition of simple supercuspidal representation as well as the Frobenius formula for the character of an induced representation. In section 4, we present a reduction formula for the character of a simple supercuspidal representation of a split simply connected group, on the split maximal torus. In section 5, we compute the character of a simple supercuspidal representation of $SL(2, F)$ on the split torus. In section 6, we compare our result to that of Sally/Shalika in [10].

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2 Notation

Let F be a non-Archimdean local field of characteristic zero. Let $\mathfrak{o}$ be the ring of integers of F, and $\mathfrak{p}$ its maximal ideal. Fix a uniformizer ϖ in F. If $p \neq 2$, we fix a non-square unit ϵ in $\mathfrak{o}^\times$. Let $|\cdot|$ denote the standard normalized absolute value on F, and *val* the associated valuation. We fix a Haar measure on F such that $\mathfrak{o}$ has volume 1, and we use the abbreviation *vol* to denote volume. A character of $\mathfrak{o}$ is said to be of *level* 1 if it is trivial on $\mathfrak{p}$, but nontrivial on $\mathfrak{o}$. If ψ is a character of F, we may define the *twist* of ψ by $a \in F$ by $\psi_a(x) := \psi(ax)$.

3 Background

In this section, we recall the definition of a simple supercuspidal representation following Gross and Reeder [5], as well as the Frobenius formula for the character of a supercuspidal representation following [9].

Let $\mathbf{G}$ be a split, simply connected, almost simple, connected reductive group, and $\mathbf{T}$ a maximal F-split torus in $\mathbf{G}$. Associated to $\mathbf{T}$ we have the set of roots Φ of $\mathbf{T}$ in $\mathbf{G}$, a set of affine roots Ψ, and an apartment $\mathcal{A}$. Let $\mathbf{Z}$ be the center of $\mathbf{G}$, and set $G = \mathbf{G}(F)$, $T = \mathbf{T}(F)$, and $Z = \mathbf{Z}(F)$. Let $X^*(T)$ denote the character lattice of T, let T_0 be the maximal compact subgroup of T, and set

$$T_1 = \langle t \in T_0 : \lambda(t) \in 1 + \mathfrak{p} \ \forall \lambda \in X^*(T) \rangle.$$

Denoting the normalizer of T in G by N, we have the affine Weyl group $W^a := N/T_0$. For $w \in W^a$, we let $\ell(w)$ denote the length of w.

Now fix a Chevalley basis in the Lie algebra of $\mathbf{G}$. To each $\psi \in \Psi$ we have an associated affine root group U_ψ in G. Fix an alcove C in the apartment with corresponding simple affine and positive affine roots $\Pi \subset \Psi^+$. Let Φ^+, Φ^- be the corresponding set of positive, negative roots, respectively. Set

$$I = \langle T_0, U_\psi : \psi \in \Psi^+ \rangle,$$
$$I_+ = \langle T_1, U_\psi : \psi \in \Psi^+ \rangle,$$
$$I_{++} = \langle T_1, U_\psi : \psi \in \Psi^+ \setminus \Pi \rangle.$$

Then I is the Iwahori subgroup corresponding to C and I_+ is its pro-unipotent radical. We set $H = ZI_+$.

The following results can be found in [5, §9].

Lemma 3.1. *The subgroup I_{++} is normal in I_+, with quotient*

$$I_+/I_{++} \cong \bigoplus_{\psi \in \Pi} U_\psi / U_{\psi+1}$$

as T_0-modules.

Definition 3.2. A character $\chi : H \to \mathbb{C}^*$ is called *affine generic* if

(i) χ is trivial on I_{++} and

(ii) χ is nontrivial on $U_\psi/U_{\psi+1}$ for every $\psi \in \Pi$ (see Lemma 3.1).

Theorem 3.3. *Let $\chi : H \to \mathbb{C}^*$ be an affine generic character. Then $\mathrm{cInd}_H^G \chi$ is an irreducible supercuspidal representation, where* cInd *denotes compact induction.*

Gross and Reeder have named the representations from Theorem 3.3 *simple supercuspidal representations.*

Now suppose that π is an irreducible smooth supercuspidal representation of G. Let K be a compact open subgroup of G, and suppose that σ is an irreducible representation of K such that

$$\pi = \mathrm{cInd}_K^G \sigma.$$

Let θ_σ denote the character of σ. The following is the Frobenius formula for the induced character θ_π, see [9, Theorem 1.9].

Theorem 3.4. *Let g be a regular element of G. Then*

$$\theta_\pi(g) = \sum_{x \in K\backslash G/K} \sum_{y \in K \backslash KxK} \dot{\theta}_\sigma(ygy^{-1})$$

where

$$\dot{\theta}_\sigma(k) = \begin{cases} \theta_\sigma(k) & \text{if } k \in K \\ 0 & \text{if } k \in G \setminus K \end{cases}$$

4 Reduction formula for the character on the split torus

In this section, we show that the formula in Theorem 3.4 simplifies if G is simply connected, π is a simple supercuspidal representation, and $g \in T$. Let us first recall the following basic fact about double coset decompositions. Suppose that K is a compact open subgroup of G, and let us choose any set of representatives $\{t_\alpha\}$ for the double cosets of $K\backslash G/K$. Then $Kt_\alpha K$ is the disjoint union of the cosets $Kt_\alpha s_1, Kt_\alpha s_2, ..., Kt_\alpha s_m$, where $s_1, s_2, ..., s_m$ is a set of representatives of $K/(K \cap t_\alpha^{-1} K t_\alpha)$. We will use this fact repeatedly in this paper.

Let $[W^a]$ denote any set of representatives for the group W^a.

Proposition 4.1. *Let $\pi = \mathrm{cInd}_H^G \chi$ be a simple supercuspidal representation of G, as in Theorem 3.3. If $g \in T$, then*

$$\theta_\pi(g) = |T_0/ZT_1| \sum_{x \in [W^a]} \sum_{y \in H\backslash HxH} \dot{\chi}(ygy^{-1}),$$

where

$$\dot{\chi}(h) = \begin{cases} \chi(h) & \text{if } h \in H \\ 0 & \text{if } h \in G \setminus H \end{cases}$$

Proof. Our starting point is the well-known affine Bruhat decomposition $I\backslash G/I \longleftrightarrow W^a$. Since $I/I_+ \cong T_0/T_1$ and since G is simply connected, the affine Bruhat decomposition descends to a decomposition

$$H\backslash G/H \longleftrightarrow N/ZT_1.$$

Therefore,

$$\theta_\pi(g) = \sum_{x \in N/ZT_1} \sum_{y \in H\backslash HxH} \dot{\chi}(ygy^{-1}). \tag{1}$$

We now analyze the inner sum. Write

$$\tau(x) := \sum_{y \in H\backslash HxH} \dot{\chi}(ygy^{-1}),$$

for $x \in N$. We claim that $\tau(x) = \tau(xt)\ \forall t \in T_0$. To see this, write

$$HxH = \bigcup_{i=1}^{m} Hxx_i,$$

where x_i are representatives of $H/(H \cap x^{-1}Hx)$. Then the map $Hxy \mapsto Hxty$ gives a bijection $H\backslash HxH \to H\backslash HxtH$, for $t \in T_0$. Therefore, in particular,

$$HxtH = \bigcup_{i=1}^{m} Hxtx_i.$$

We conclude that

$$\tau(xt) = \sum_{y \in H\backslash HxtH} \dot{\chi}(ygy^{-1}) = \sum_{i=1}^{m} \dot{\chi}(xtx_igx_i^{-1}t^{-1}x^{-1}) =$$

$$\sum_{i=1}^{m} \dot{\chi}(xtx_it^{-1}tgt^{-1}tx_i^{-1}t^{-1}x^{-1}) = \sum_{i=1}^{m} \dot{\chi}(xtx_it^{-1}gtx_i^{-1}t^{-1}x^{-1}),$$

the last equality coming from the fact that $g \in T$. Since $H/(H \cap x^{-1}Hx)$ is by certain spaces of the form $U_\gamma/U_{\gamma+n}$, and since conjugation by an element $t \in T_0$ preserves $U_\gamma/U_{\gamma+n}$, we get that

$$\sum_{i=1}^{m} \dot{\chi}(xtx_it^{-1}gtx_i^{-1}t^{-1}x^{-1}) = \sum_{i=1}^{m} \dot{\chi}(xx_igx_i^{-1}x^{-1}) = \tau(x).$$

We have therefore concluded that $\tau(xt) = \tau(x)\ \forall t \in T_0$.

Finally, note that we have the following short exact sequence

$$1 \to T_0/ZT_1 \to N/ZT_1 \to W^a = N/T_0 \to 1.$$

Since we just showed that τ is constant along fibers of the above exact sequence, together with (1), we now have that

$$\theta_\pi(g) = |T_0/ZT_1| \sum_{x \in [W^a]} \sum_{y \in H\backslash HxH} \dot{\chi}(ygy^{-1}).$$

□

5 The character formula on the split torus

In this section we prove Theorem 5.3. In particular, we will compute the inner sum $\tau(x)$ in Proposition 4.1 for any set of representatives of W^a by decomposing HxH into a union of left cosets. Afterwards, we sum over all x to obtain θ_π.

Henceforth we will consider the standard Iwahori $I = \begin{pmatrix} \mathfrak{o}^\times & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o}^\times \end{pmatrix}$. By definition of affine generic, we have that $\chi|_{I_+}$ is given by

$$\chi|_{I_+} : I_+ \to \mathbb{C}^*$$

$$\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \mapsto \chi_1(d_{12})\chi_2(d_{21})$$

where χ_1 is some level 1 character of $\mathfrak{o}$ and where $\chi_2(d_{21}) = \chi_2'(\frac{1}{\varpi}d_{21})$, where χ_2' is some level 1 character of $\mathfrak{o}$. Extend χ_1 to a function on F by setting $\dot\chi_1(z) := \chi_1(z)\ \forall z \in \mathfrak{o}$ and $\dot\chi_1(z) = 0\ \forall z \in F \setminus \mathfrak{o}$. Moreover, extend χ_2' to a function on F by setting $\dot\chi_2'(z) := \chi_2'(z)\ \forall z \in \mathfrak{o}$ and $\dot\chi_2(z) = 0\ \forall z \in F \setminus \mathfrak{o}$.

Fix an element $g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in T_1$, set $r := val(a - a^{-1})$, and again define

$$\tau(x) := \sum_{y \in H \backslash HxH} \dot\chi(ygy^{-1}),$$

for $x \in N$.

Proposition 5.1. *Let* $x = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$.

(a) If $val(b) = n \geq 0$ *and* $2n < r$*, then* $\tau(x) = q^{2n}$.

(b) If $val(b) = n < 0$ *and* $-2n < r$*, then* $\tau(x) = q^{-2n}$.

(c) Otherwise, $\tau(x) = 0$.

Proof. It will be convenient for the proof to introduce the following notation. For $r_1, r_2 \in \mathbb{N}$, we write

$$\begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p}^{r_1} \\ \mathfrak{p}^{r_2} & 1+\mathfrak{p} \end{pmatrix} := \left\{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in SL(2,F) : x_1, x_4 \in 1+\mathfrak{p}, x_2 \in \mathfrak{p}^{r_1}, x_3 \in \mathfrak{p}^{r_2} \right\}.$$

We prove (a). Let $val(b) = n \geq 0$. We first write the double coset HxH as a finite union of single right cosets. Since

$$H \cap x^{-1}Hx = Z \cdot \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{o} \\ \mathfrak{p}^{2n+1} & 1+\mathfrak{p} \end{pmatrix},$$

we obtain an explicit disjoint union decomposition

$$HxH = \coprod_{z \in \mathfrak{p}/\mathfrak{p}^{2n+1}} Hx \cdot \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}.$$

Suppose that $y \in HxH$. We need to check when $ygy^{-1} \in H$ since $\dot{\chi}$ vanishes outside H. Using our above double coset decomposition, write $y = hxu$, where u is of the form $\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}$ for some $z \in \mathfrak{p}/\mathfrak{p}^{2n+1}$ and for some $h \in H$. Then $ygy^{-1} \in H \Leftrightarrow xugu^{-1}x^{-1} \in H$. Moreover,

$$xugu^{-1}x^{-1} = \begin{pmatrix} a & 0 \\ b^{-2}z(a - a^{-1}) & a^{-1} \end{pmatrix}.$$

We now write $a - a^{-1} = \varpi^r \nu$ for some unit $\nu \in \mathfrak{o}^\times$. Then $b^{-2}z(a - a^{-1}) = \varpi^{-2n}\varpi^r z'$ for some $z' \in \mathfrak{p}/\mathfrak{p}^{2n+1}$.

Therefore,

$$\sum_{y \in H \backslash HxH} \dot{\chi}(ygy^{-1}) = \sum_{z' \in \mathfrak{p}/\mathfrak{p}^{2n+1}} \dot{\chi}_2'(\varpi^{-2n+r-1}z')$$

Making a change of variables, we get

$$\sum_{z' \in \mathfrak{p}/\mathfrak{p}^{2n+1}} \dot{\chi}_2'(\varpi^{-2n+r-1}z') = \sum_{v \in \mathfrak{p}^{-2n+r}/\mathfrak{p}^r} \dot{\chi}_2'(v) = vol(\mathfrak{p}^r)^{-1} \int_{\mathfrak{p}^{-2n+r} \cap \mathfrak{o}} \dot{\chi}_2'(v) dv$$

since $\dot{\chi}$ vanishes outside H. If $\mathfrak{p}^{-2n+r} \supseteq \mathfrak{o}$, then this integral vanishes since the integral of a nontrivial character over a group vanishes. However, if $\mathfrak{p}^{-2n+r} \subsetneq \mathfrak{o}$, which is precisely the condition that $2n < r$, then

$$vol(\mathfrak{p}^r)^{-1} \int_{\mathfrak{p}^{-2n+r} \cap \mathfrak{o}} \dot{\chi}_2'(v)' dv = vol(\mathfrak{p}^r)^{-1} \int_{\mathfrak{p}^{-2n+r}} dv = vol(\mathfrak{p}^r)^{-1} vol(\mathfrak{p}^{-2n+r})$$

since $\dot{\chi}_2'$ is trivial on $\mathfrak{p}$. By our choice of measure, we have that $vol(\mathfrak{p}^d) = q^{-d}$ for $d > 0$, thus proving part (a).

We now prove (b). So suppose that $val(b) = n < 0$. Then similarly as in (a), we get

$$H \cap x^{-1}Hx = Z \cdot \begin{pmatrix} 1 + \mathfrak{p} & \mathfrak{p}^{-2n} \\ \mathfrak{p} & 1 + \mathfrak{p} \end{pmatrix},$$

$$HxH = \coprod_{z \in \mathfrak{o}/\mathfrak{p}^{-2n}} Hx \cdot \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}.$$

Suppose that $y \in HxH$. By the above double coset decomposition, we may write $y = hxu$, where u is of the form $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ for some $z \in \mathfrak{o}/\mathfrak{p}^{-2n}$, and for some $h \in H$. A computation yields $xugu^{-1}x^{-1} = \begin{pmatrix} a & b^2 z(a^{-1} - a) \\ 0 & a^{-1} \end{pmatrix}$. Therefore,

$$\sum_{y \in H \backslash HxH} \dot{\chi}(ygy^{-1}) = \sum_{z \in \mathfrak{o}/\mathfrak{p}^{-2n}} \dot{\chi}_1(b^2 z(a^{-1} - a)).$$

Rewriting $b^2 z(a^{-1} - a) = \varpi^{2n}\varpi^r z'$ where $z' \in \mathfrak{o}/\mathfrak{p}^{-2n}$, then after a change of variables, we obtain

$$\sum_{z \in \mathfrak{o}/\mathfrak{p}^{-2n}} \dot{\chi}_1(b^2 z(a^{-1} - a)) = \sum_{z' \in \mathfrak{o}/\mathfrak{p}^{-2n}} \dot{\chi}_1(\varpi^{2n}\varpi^r z') =$$

$$\sum_{v\in\mathfrak{p}^{2n+r}/\mathfrak{p}^r}\dot\chi_1(v)=vol(\mathfrak{p}^r)^{-1}\int_{\mathfrak{p}^{2n+r}\cap\mathfrak{o}}\dot\chi_1(v)dv.$$

If $\mathfrak{p}^{2n+r}\supseteq\mathfrak{o}$, then again this integral vanishes. However, if $\mathfrak{p}^{2n+r}\subsetneq\mathfrak{o}$, which is precisely the condition that $-2n<r$ then

$$vol(\mathfrak{p}^r)^{-1}\int_{\mathfrak{p}^{2n+r}\cap\mathfrak{o}}\dot\chi_1(v)dv=vol(\mathfrak{p}^r)^{-1}\int_{\mathfrak{p}^{2n+r}}dv=vol(\mathfrak{p}^r)^{-1}vol(\mathfrak{p}^{2n+r}),$$

since $\dot\chi_1$ is trivial on $\mathfrak{p}$.

The proofs of parts (a) and (b) immediately imply part (c). □

Proposition 5.2. *Let* $x=\begin{pmatrix}0&c\\-c^{-1}&0\end{pmatrix}$.

(a) If $val(c)=n\geq 0$ *and* $2n+1<r$, *then* $\tau(x)=q^{2n+1}$.

(b) If $val(c)=n<0$ *and* $-2n-1<r$, *then* $\tau(x)=q^{-2n-1}$.

(c) Otherwise, $\tau(x)=0$.

Proof. The proof is completely analogous to that of Proposition 5.1. □

Theorem 5.3.

(a) Let π *be a simple supercuspidal representation of* $SL(2,F)$. *Let* $g=\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}\in T_1$, *and set* $r=val(a-a^{-1})$. *Let* $W^a(r):=\{w\in W^a:\ell(w)<r\}$. *Then*

$$\theta_\pi(g)=c_q\sum_{w\in W^a(r)}q^{\ell(w)}$$

where

$$c_q:=\begin{cases}\frac{q-1}{2} & \text{if } p\neq 2\\ q-1 & \text{if } p=2\end{cases}$$

(b) If $g\in T\setminus ZT_1$, *then* $\theta_\pi(g)=0$.

Proof. We first prove (a). Let $\langle -1+\mathfrak{p}\rangle$ denote the multiplicative group generated by $\{-1+z:z\in\mathfrak{p}\}$. Note that

$$|T_0/ZT_1|=|\mathfrak{o}^\times/\langle -1+\mathfrak{p}\rangle|=\begin{cases}\frac{q-1}{2} & \text{if } p\neq 2\\ q-1 & \text{if } p=2\end{cases}$$

Therefore, by Propositions 5.1 and 5.2, and Proposition 4.1, we have

$$\theta_\pi(g)=c_q(1+2q+2q^2+...+2q^{r-1}).$$

It is a straightforward calculation to show that if $x=\begin{pmatrix}b&0\\0&b^{-1}\end{pmatrix}$ and $val(b)=k$, then $\ell(x)=|2k|$. Moreover, if $x=\begin{pmatrix}0&c\\-c^{-1}&0\end{pmatrix}$ and $val(c)=k$, then $\ell(x)=|2k+1|$. This concludes the proof of part (a).

We now prove (b). In the proof of Proposition 5.1 (a), we showed that

$$ygy^{-1} \in H \Longleftrightarrow \begin{pmatrix} a & 0 \\ b^{-2}z(a-a^{-1}) & a^{-1} \end{pmatrix} \in H.$$

This trivially implies that $a \in \langle -1+\mathfrak{p} \rangle$. Moreover, the condition $a \in \langle -1+\mathfrak{p} \rangle$ is implied, for the same reason, when you compute the terms ygy^{-1} that appear in $\theta_\pi(g)$ for any other representative $x \in [W^a]$, as simple computations will show. Therefore, θ_π vanishes on $T \setminus ZT_1$. □

Remark 5.4.

1. Up to central character, we have computed θ_π on all of the split torus. Since the central character is given by the data forming the simple supercuspidal representation, we have computed θ_π on the entire split torus.

2. The set $W^a(r)$ has geometric meaning. If C denotes the fundamental alcove in the standard apartment for $SL(2,F)$, then $W^a(r) \cdot C$ is the symmetric interval about C containing $2r-1$ alcoves.

3. We note that the term $a - a^{-1}$ in Theorem 5.3 is, up to a sign, a canonical square root of the Weyl denominator. In particular, if D denotes the Weyl denominator, then $-D(g) = (a-a^{-1})^2$.

6 Comparison with Sally and Shalika on the split torus

In [10], Sally and Shalika have computed character values on the split torus of $SL(2,F)$ for an arbitrary supercuspidal representation of $SL(2,F)$ when $p \neq 2$. We briefly recall their result.

For any quadratic extension $V = F(\sqrt{\theta})$ of F, let C_θ denote the kernel of the norm $N_{V/F}$ from V to F, and $\mathfrak{p}_\theta$ the prime ideal in V. If V is ramified, set $C_\theta^{(h)} = (1+\mathfrak{p}_\theta^{2h+1}) \cap C_\theta, h \geq 0$. If $\psi \in \widehat{C_\theta}$, denote the conductor of ψ by cond ψ (this is the largest subgroup in the filtration $\{C_\theta^{(h)}\}$ on which ψ is trivial). The ramified discrete series are indexed by a nontrivial additive character η of F and a nontrivial character $\psi \in \widehat{C_\theta}$, where V is ramified. The corresponding representation is denoted $\Pi(\eta,\psi,V)$. If $g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in SL(2,F)$, $a \in 1+\mathfrak{p}$, and $\Pi(\eta,\psi,V)$ is a ramified discrete series, then Sally and Shalika proved that

$$\theta_{\Pi(\eta,\psi,V)}(g) = \frac{1}{|a-a^{-1}|} - \frac{1}{2}q^h\left(\frac{q+1}{q}\right).$$

If π is a simple supercuspidal representation of $SL(2,F)$, then $h=1$, and so the above formula simplifies to

$$\theta_\pi(g) = \frac{1}{|a-a^{-1}|} - \frac{1}{2}q\left(\frac{q+1}{q}\right), \tag{2}$$

which is easily seen to be equal to

$$\frac{q-1}{2}\left(1+2q+2q^2+2q^3+\ldots+2q^{r-1}\right).$$

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