THE LANGLANDS PARAMETER OF A SIMPLE SUPERCUSPIDAL REPRESENTATION: ODD ORTHOGONAL GROUPS

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Abstract. In this work, we explicitly compute a certain family of twisted gamma factors of a simple supercuspidal representation π of a p-adic odd orthogonal group. These computations, together with analogous computations for general linear groups carried out in previous work with Liu [AL14], allow us to give a prediction for the Langlands parameter of π. We then prove that our prediction is correct if p is sufficiently large.

Contents

1. Introduction

Let $\mathbf{G}$ be a connected reductive group defined over a p-adic field F. Recently, Gross, Reeder, and Yu [GR10, RY14] have constructed a class of supercuspidal representations of $G = \mathbf{G}(F)$, called *simple supercuspidal representations*. These are the supercuspidal representations of G of minimal positive depth.

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Of interest is the Langlands correspondence for simple supercuspidal representations of G. Recall that the Langlands correspondence is a certain conjectural finite-to-one map

$$\Pi(G) \to \Phi(G)$$

from equivalence classes of irreducible admissible representations of G to equivalence classes of Langlands parameters of G. In the past several years there has been much activity on the Langlands correspondence. DeBacker and Reeder [DR09] have constructed a correspondence for depth zero supercuspidal representations of unramified p-adic groups, and Reeder [R08] extended this construction to certain positive depth supercuspidal representations of unramified groups. Recently, under a mild assumption on the residual characteristic, Kaletha [K13] has constructed a correspondence for simple supercuspidal representations of p-adic groups, and has subsequently in [K15] extended these results to epipelagic supercuspidal representations of p-adic groups. Finally, we would like to remark that Gross, Reeder, and Yu have also studied the Langlands correspondence for epipelagic supercuspidal representations (see [GR10], [RY14]).

Our present work is the first in a series of papers dedicated to explicitly determining the Langlands parameter φ_π of a simple supercuspidal representation π of a classical group G, using the theory of gamma factors. Let π and τ be a pair of irreducible generic representations of G and GL_n, where G is either $SO_{2\ell}$, $SO_{2\ell+1}$ or $Sp_{2\ell}$. Here, we assume that $SO_{2\ell+1}$ is split and $SO_{2\ell}$ is quasi-split. Fix a nontrivial character ψ of F. The Rankin-Selberg integral for $G \times GL_n$ was constructed, in different settings, in a series of works including [GPSR87, G90, GPSR97, GRS98, Sou93]. If G is orthogonal, we denote this Rankin-Selberg integral by $\Phi(W, f_s)$, and $\Phi(W, \phi, f_s)$ otherwise. Here, W is a Whittaker function for π, f_s is a certain holomorphic section of a principal series (induced from τ) depending on the complex parameter s, and ϕ is a Schwartz function. Applying a standard normalized intertwining operator $M^*(\tau, s)$ to f_s, one obtains a similar integral $\Phi^*(W, f_s) = \Phi(W, M^*(\tau, s)f_s)$ if G is orthogonal (and $\Phi^*(W, \phi, f_s) = \Phi(W, \phi, M^*(\tau, s)f_s)$ otherwise) related to $\Phi(W, f_s)$ (or $\Phi(W, \phi, f_s)$) by a functional equation

$$\begin{cases} \gamma(s, \pi \times \tau, \psi)\Phi(W, f_s) = \Phi^*(W, f_s) & \text{if } \; G = SO_{2\ell+1} \\ \gamma(s, \pi \times \tau, \psi)\Phi(W, f_s) = c(s, \ell, \tau, \gamma)\Phi^*(W, f_s) & \text{if } \; G = SO_{2\ell} \\ \gamma(s, \pi \times \tau, \psi)\Phi(W, \phi, f_s) = \Phi^*(W, \phi, f_s) & \text{if } \; G = Sp_{2\ell} \end{cases}$$

For an accessible paper discussing all of these functional equations, we refer the reader to the recent work of Kaplan [K14ii].

The term $\gamma(s, \pi \times \tau, \psi)$ is known as the *gamma factor* of $\pi \times \tau$, and is the key ingredient in determining φ_π in this paper. To determine φ_π, one must locate the poles of the gamma factors $\gamma(s, \pi \times \tau, \psi)$ where τ ranges over the supercuspidal representations of various general linear groups. The location of poles will determine the supercuspidal support of π (see [M98, §3, Conjecture 3.2]), hence will determine a specific functorial lift Π to a general linear group GL, the lift corresponding to the standard L-homomorphism ${}^L G \to {}^L GL$ (see [ACS14, §2]). In the case of simple supercuspidal representations π of a classical group G, it is expected that the supercuspidal support of Π contains tamely ramified characters of GL_1 and/or simple supercuspidal representations. We could then use the explicit local Langlands correspondence for simple supercuspidal representations of general linear groups (see [AL14, BH10, BH14]) to determine φ_π.

While describing the Langlands correspondence explicitly is in general a difficult task, in the simple supercuspidal setting it turns out to be tractable. We now give more details in the case of $G = SO_{2\ell+1}$, which is the central concern of this paper. The simple supercuspidal representations of G are parameterized by two pieces of data: a choice of a uniformizer ϖ in F, and a sign. More explicitly (see §5), let χ be an affine generic character of the pro-unipotent radical I^+ of an Iwahori I. The choice of a uniformizer ϖ in F determines an element g_χ in G which normalizes I and stabilizes χ. We can extend χ to $K = \langle g_\chi \rangle I^+$ in two different ways, since $g_\chi^2 = 1$, and $\pi = \mathrm{Ind}_K^G \chi$ is simple supercuspidal.

Let τ be a tamely ramified character of GL_1. Our main result (Theorem 7.2) is

Theorem 1.1.

$$\gamma(s, \pi \times \tau, \psi) = \chi(g_\chi)\tau(\varpi)q^{1/2-s}.$$

In particular, since there are no poles in $\gamma(s, \pi \times \tau, \psi)$, it is expected that φ_π is an irreducible representation of W_F (see [M98, Conjecture 3.2]). To prove this, one could again locate the poles of a certain family of gamma factors. However, it turns out that in this special case, one can avoid this computation by using results of Kaletha [K15] as well as some standard results about functorial lifting of classical groups. Using these ideas, in Theorem 8.3 we prove that φ_π is an irreducible representation of W_F under the assumption that p is sufficiently large (see Remark 8.5).

To summarize the method, we first note that the work of [K15] shows that the character of π is stable, and so by results of Moeglin [M14], π forms an L-packet by itself. Its corresponding Langlands parameter is therefore an irreducible representation of the Weil-Deligne group, and so transfers to a discrete series representation Π of $GL_{2\ell}$. Using some results of [LR03], together with Theorem 1.1, we may deduce that Π is in fact supercuspidal, showing that φ_π is an irreducible representation of W_F. We may moreover precisely describe φ_π as follows.

By the results of [AL14, BH14] (see in particular [AL14, Remark 3.18] or [BH14, Lemma 2.2, Proposition 2.2]), there exists a unique irreducible 2ℓ-dimensional representation $\varphi : W_F \to GL(2\ell, \mathbb{C})$ with trivial determinant, whose gamma factor $\gamma(s, \varphi \times \tau, \psi)$ equals $\chi(g_\chi)\tau((-1)^{\ell+1})\tau(\varpi)q^{1/2-s}$ (the $\tau((-1)^{\ell+1})$ factor is introduced here because of a normalization of the gamma factor that we will need later; see Theorem 3.1) for every tamely ramified character τ (we give more details on this in §4). The Langlands parameter φ has been explicitly described in [AL14, BH10, BH14]. In the case that $p \nmid 2\ell$, let us briefly recall this description (here we follow [AL14]).

Let ϖ_E be a $2\ell^{\text{th}}$ root of $(-1)^{\ell+1}\varpi$, and set $E = F(\varpi_E)$. Relative to the basis

$$\varpi_E^{2\ell-1}, \varpi_E^{2\ell-2}, \cdots, \varpi_E, 1$$

of E/F, we have an embedding

$$\iota : E^\times \hookrightarrow GL_{2\ell}.$$

Define a character ξ of $E^\times$ by setting $\xi|_{1+\mathfrak{p}_E} = \lambda \circ \iota$, where $\lambda(A) := \psi(A_{12} + A_{23} + ... + A_{2\ell-1,2\ell} + \frac{1}{\varpi} A_{2\ell,1})$, for $A = (A)_{ij} \in GL_{2\ell}$. Moreover, we define $\xi|_{k_F^\times} \equiv (\varkappa_{E/F}|_{k_F^\times})^{-1}$, where $\varkappa_{E/F} = \det(\mathrm{Ind}_{W_E}^{W_F}(1_E))$, 1_E denotes the trivial character of W_E, and k_F is the residue field of F. Finally, we set

$$\xi(\varpi_E) = \chi(g_\chi)\lambda_{E/F}(\psi)^{-1},$$

where $\lambda_{E/F}(\psi)$ is the Langlands constant (see [BH06, §34.3]). Then $\varphi = \mathrm{Ind}_{W_E}^{W_F}\xi$.

The cases of $SO_{2\ell}$ and $Sp_{2\ell}$ are in some sense similar. But in these cases, if π is simple supercuspidal of such a group, poles are expected to arise (depending on p) in the computations of $\gamma(s, \pi \times \tau, \psi)$ when τ is a tamely ramified character of GL_1. Thus the situation has a different flavor, and is the subject of work in progress.

We now summarize the contents of the sections of this paper. In §3, we recall the functional equation for odd orthogonal groups as in [K14ii]. In §4, we recall a construction of simple supercuspidal representations of GL_n as well as their tamely ramified GL_1-twisted gamma factors. In §5, we give a construction of simple supercuspidal representations of odd orthogonal groups that may be used in computing the Rankin-Selberg integrals from §3. In §6 and §7, we compute a family of Rankin-Selberg integrals that allow us to explicitly compute the tamely ramified GL_1-twisted gamma factors of a simple supercuspidal representation of $SO_{2\ell+1}$. The main result on the values of these gamma factors is Theorem 7.2, which allows us to predict the Langlands parameter of a simple supercuspidal representation of an odd orthogonal group. Finally, in §8, we show that if p is sufficiently large, then our prediction for the Langlands parameter is indeed the correct Langlands parameter.

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2. Notation

Let F be a p-adic field, with ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p}$, and residue field $k_F = \mathfrak{o}/\mathfrak{p}$. Let q denote the cardinality of k_F, and fix a uniformizer ϖ. Throughout, we fix a nontrivial additive character $\psi : F \to \mathbb{C}^\times$ that is trivial on $\mathfrak{p}$ but nontrivial on $\mathfrak{o}$. A character η of $F^\times$ is said to be *tamely ramified* if it is trivial on $1+\mathfrak{p}$. Given a connected reductive group $\mathbf{G}$ defined over F, let $G = \mathbf{G}(F)$. We also let W_F denote the Weil group of F.

3. The local functional equation for odd orthogonal groups

In this section we recall the functional equation for odd orthogonal groups, as in [K14ii, §3.1].

Let J_r denote the $r \times r$ matrix $\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$. Let $J'_{2\ell+1} = diag(I_\ell, 2, I_\ell)J_{2\ell+1}$.

We define

$$SO_{2\ell+1} = \left\{ g \in GL_{2\ell+1} : det(g) = 1, {}^t g J'_{2\ell+1} g = J'_{2\ell+1} \right\}, \tag{3.1}$$

$$SO_{2n} = \left\{ g \in GL_{2n} : det(g) = 1, {}^t g J_{2n} g = J_{2n} \right\}. \tag{3.2}$$

Let $T_{SO_{2n}}$ be the diagonal split maximal torus of SO_{2n}. Let $\Delta_{SO_{2n}}$ be the standard set of simple roots of SO_{2n}, so that $\Delta_{SO_{2n}} = \{\epsilon_1 - \epsilon_2, ..., \epsilon_{n-1}-\epsilon_n, \epsilon_{n-1}+\epsilon_n\}$, where $\epsilon_i(t) = t_i$ is the i-th coordinate function of $t \in T_{SO_{2n}}$. Let Q_n be the standard maximal parabolic subgroup of SO_{2n} corresponding to $\Delta \setminus \{\epsilon_{n-1} + \epsilon_n\}$. The Levi part of Q_n is isomorphic to GL_n, and we denote its unipotent radical by U_n.

Let $T_{SO_{2\ell+1}}$ be the split maximal torus of $SO_{2\ell+1}$. let $\Delta_{SO_{2\ell+1}}$ be the standard set of simple roots of $SO_{2\ell+1}$, so that $\Delta_{SO_{2\ell+1}} = \{\epsilon_1 - \epsilon_2, ..., \epsilon_{\ell-1} - \epsilon_\ell, \epsilon_\ell\}$. The highest root is $\epsilon_1 + \epsilon_2$. Let $U_{SO_{2\ell+1}}$ denote the subgroup of upper triangular unipotent matrices.

Let ψ be a nontrivial additive character of F. We let U_{GL_n} denote the maximal unipotent subgroup of GL_n consisting of upper triangular matrices. We also denote by ψ the standard nondegenerate Whittaker character of

$$\psi(u) = \psi\left(\sum_{i=1}^{n-1} u_{i,i+1}\right),$$

for $u = (u_{ij}) \in U_{GL_n}$.

Let τ be a generic representation of GL_n. We denote by $\mathcal{W}(\tau, \psi^{-1})$ the Whittaker model of τ with respect to ψ^{-1}.

For $s \in \mathbb{C}$, define

$$V_{Q_n}^{SO_{2n}}(\mathcal{W}(\tau, \psi^{-1}), s) = \mathrm{Ind}_{Q_n}^{SO_{2n}}(\mathcal{W}(\tau, \psi^{-1})|\det|^{s-1/2}).$$

Thus, a function f_s in $V_{Q_n}^{SO_{2n}}(\mathcal{W}(\tau, \psi^{-1}), s)$ is a smooth function on $SO_{2n} \times GL_n$, where for any $g \in SO_{2n}$, m in the Levi part of Q_n, and u in the unipotent radical U_n, we have

$$f_s((mug, a) = \delta_{Q_n}^{1/2}(m)|\det(m)|^{s-\frac{1}{2}} f_s(g, am), \tag{3.3}$$

and the mapping $a \mapsto f_s(g, a)$ belongs to $\mathcal{W}(\tau, \psi^{-1})$.

We now let $\ell, n \in \mathbb{N}$ such that $\ell \geq n$. Let $U_{SO_{2\ell+1}}$ denote the subgroup of upper triangular unipotent matrices in $SO_{2\ell+1}$. We use the notation ψ again to define a non-degenerate character on $U_{SO_{2\ell+1}}$ by

$$\psi(u) = \psi\left(\sum_{i=1}^{\ell-1} u_{i,i+1} + \frac{1}{2} u_{\ell,\ell+1}\right).$$

We will also need the "standard" non-degenerate character on $U_{SO_{2\ell+1}}$:

$$\psi_{\mathrm{std}}(u) = \psi\left(\sum_{i=1}^{\ell} u_{i,i+1}\right).$$

Let π be a generic representation of $SO_{2\ell+1}$ with respect to ψ. We shall denote by $\mathcal{W}(\pi,\psi)$ the Whittaker model of π with respect to ψ. Let τ be a generic representation of GL_n with respect to ψ^{-1}.

In computing γ-factors, we will be interested in the following local integrals (see [K14ii]):

$$\Phi(W, f_s) := \int_{U_{SO_{2n}}\backslash SO_{2n}} \int_{\overline{X}_{(n,\ell)}} W(\overline{x} j_{n,\ell}(h)) f_s(h, I_n) d\overline{x} dh,$$

where $W \in \mathcal{W}(\pi,\psi)$, $f_s \in V_{Q_n}^{SO_{2n}}(\mathcal{W}(\tau,\psi^{-1}),s)$, and where dh is a fixed right-invariant Haar measure on the quotient space $U_{SO_{2n}} \setminus SO_{2n}$, and $d\overline{x}$ is a fixed Haar measure.

Here, $j_{n,\ell} : SO_{2n} \to SO_{2\ell+1}$ is the map

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & & B \\ & I_{2(\ell-n)+1} & \\ C & & D \end{pmatrix} \in SO_{2\ell+1},$$

$$\overline{X}_{(n,\ell)} = \begin{pmatrix} I_n & & & & \\ y & I_{\ell-n} & & & \\ & & 1 & & \\ & & & I_{\ell-n} & \\ & & & y' & I_n \end{pmatrix} \in SO_{2\ell+1}.$$

Note that $y \in F^{\ell-1}$ is a column vector, and $y' \in F^{\ell-1}$ is a row vector. Moreover, the notation y' means that these coordinates are determined by the coordinates of y, according to the matrix defining $SO_{2\ell+1}$. Explicitly, if y is the transpose of the row vector $(y_1, y_2, ...y_{\ell-n-1}, y_{\ell-n})$, then y' is the row vector $(-y_{\ell-n}, -y_{\ell-n-1}, ..., -y_2, -y_1)$.

Now let n be odd. Define a matrix

$$\omega = \begin{pmatrix} I_{n-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{n-1} \end{pmatrix}.$$

We define an intertwining operator

$$M(\tau,s) : V_{Q_n}^{SO_{2n}}(\mathcal{W}(\tau,\psi^{-1}),s) \to V_{{}^{\omega}Q_n}^{SO_{2n}}(\mathcal{W}(\tau^*,\psi),1-s)$$

by

$$M(\tau,s) f_s(h,a) = \int_{\overline{U_n}} f_s(\overline{u} w_n^{-1} h, a^*) d\overline{u},$$

where

$$w_n = \begin{pmatrix} & I_n \\ I_n & \end{pmatrix} \begin{pmatrix} & & 1 \\ & I_{2(n-1)} & \\ 1 & & \end{pmatrix} \in SO_{2n},$$

and where $\overline{U_n}$ is the opposite unipotent radical to U_n. Here, $\tau^*(g) := \tau(g^*)$, where $g^* := J_n{}^t g^{-1} J_n$. We note that in the case that $n = 1$ (the case that we will be interested in throughout the paper), we have that $\tau^*(g) = \tau(g^{-1})$, and also that $M(\tau, s) f_s(h, a) = f_s(h, a)$.

Then we will also be interested in the integrals (see [K14ii]) $\Phi^*(W, f_s)$, where

$$\Phi^*(W, f_s)$$
$$:= \gamma(2s-1, \tau, \wedge^2, \psi) \int_{U_{SO_{2n}} \backslash SO_{2n}} \int_{\overline{X}_{(n,\ell)}} W(\hat{c}_{n,\ell} \overline{x} j_{n,\ell}(h) \delta_o \omega') M(\tau, s) f_s({}^{\omega}h, b_n) d\overline{x} dh.$$

Moreover,

$$\hat{c}_{n,\ell} = diag(I_n, -I_{\ell-n}, 1, -I_{\ell-n}, I_n) \in SO_{2\ell+1},$$
$$\delta_o = diag(I_\ell, -1, I_\ell),$$
$$b_n = diag(1, -1, ..., -1, 1) \in GL_n,$$

$$\omega' = \begin{pmatrix} I_{n-1} & & & & \\ & & & 1 & \\ & & I_{2(\ell-n)+1} & & \\ & 1 & & & \\ & & & & I_{n-1} \end{pmatrix}.$$

Finally, $\gamma(2s-1, \tau, \wedge^2, \psi)$ is defined using the Langlands-Shahidi method (see [K14ii, §3.1]. We then have a functional equation for odd orthogonal groups (see [K14ii, §4].

$$\gamma(s, \pi \times \tau, \psi) \Phi(W, f_s) = \Phi^*(W, f_s). \tag{3.4}$$

It was then shown that in [K14ii] that if one suitably normalizes $\gamma(s, \pi \times \tau, \psi)$, then we can get a gamma factor that is equal to that of Shahidi.

Theorem 3.1. [K14ii, §4, §1 Corollary 1] *If we define*

$$\Gamma(s, \pi \times \tau, \psi) = \omega_\pi(-1)^n \omega_\tau(-1)^\ell \gamma(s, \pi \times \tau, \psi),$$

then $\Gamma(s, \pi \times \tau, \psi)$ *coincides with Shahidi's gamma factor for* $\pi \times \tau$ *defined in* [Sha90].

Remark 3.2. *We will be interested in the case that* $n = 1$. *This is permissible (see* [K14ii, §3.1]*).*

4. The simple supercuspidal representations of GL_n

In this section we review the definition of a simple supercuspidal representation of GL_n, as in [AL14] (see also [KL15]). We then recall the values of a family of its twisted gamma factors, in order to predict the Langlands parameter of a simple supercuspidal of $SO_{2\ell+1}$.

Let $GL_n = \mathbf{GL_n}(F)$, Z the center of G, $K = \mathbf{GL_n}(\mathfrak{o})$ the standard maximal compact subgroup, I the standard Iwahori subgroup, and I^+ its pro-unipotent radical. Fix a nontrivial additive character ψ of F that is trivial on $\mathfrak{p}$ and nontrivial on $\mathfrak{o}$. Let U denote the unipotent radical of the standard upper triangular Borel subgroup of GL_n. For any $u \in U$, we recall from §3 the standard nondegenerate Whittaker character ψ of U:

$$\psi(u) = \psi\left(\sum_{i=1}^{n-1} u_{i,i+1}\right).$$

Set $H = ZI^+$, and fix a character ω of $Z \cong F^\times$, trivial on $1+\mathfrak{p}$. We define a character $\chi : H \to \mathbb{C}^\times$ by $\chi(zk) = \omega(z)\psi(r_1 + r_2 ... + r_{n-1} + r_n)$ for

$$k = \begin{pmatrix} x_1 & r_1 & * & \cdots & \\ * & x_2 & r_2 & \cdots & \\ \vdots & & \ddots & \ddots & \\ * & & & & r_{n-1} \\ \varpi r_n & & \cdots & & x_n \end{pmatrix}$$

The compactly induced representation $\pi_\chi := \mathrm{cInd}_H^G \chi$ is then a direct sum of n distinct irreducible supercuspidal representations of GL_n. They are parameterized by ζ, where ζ is a complex n^{th} root of $\omega(\varpi)$, as follows. Set

$$g_\chi = \begin{pmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ \varpi & & & & 0 \end{pmatrix}$$

and set $H' = \langle g_\chi \rangle H$. Then the summands of π_χ are

$$\sigma_\chi^\zeta := \mathrm{cInd}_{H'}^G \chi_\zeta$$

where $\chi_\zeta(g_\chi^j h) = \zeta^j \chi(h)$, as ζ runs over the complex n^{th} roots of $\omega(\varpi)$. σ_χ^ζ is called a *simple supercuspidal* representation of GL_n. As we vary ϖ, ζ, and ω, we obtain all simple supercuspidal representations of GL_n.

Since we will need to consider the choice of uniformizer later, we denote by $\sigma(\varpi,\zeta,\omega)$ the simple supercuspidal representation of GL_n determined by the triple (ϖ,ζ,ω).

Finally, for the simple supercuspidal representation σ_χ^ζ, we may define a Whittaker function on GL_n by setting

$$W(g) = \begin{cases} \psi(u)\chi_\zeta(h') & \text{if} \quad g = uh' \in UH' \\ 0 & \text{else} \end{cases}$$

This function is well-defined, by definition of ψ and χ_ζ. Moreover, $W \in Ind_U^G \psi \cap \mathrm{cInd}_{H'}^G \chi_\zeta$. Using this Whittaker function, it turns out to be not so difficult to compute the Rankin-Selberg integrals that arise in the definition of the twisted gamma factors $\gamma(s, \sigma_\chi^\zeta \times \tau, \psi)$, where τ is a tamely ramified character of $F^\times$. In §5, we will define an analogous Whittaker function for a simple supercuspidal representation π of $SO_{2\ell+1}$, which will allow us to compute the relevant Rankin-Selberg integrals for π.

In [AL14] (see also [BH14, Lemma 2.2]), we explicitly computed $\gamma(s, \sigma_\chi^\zeta \times \tau, \psi)$:

Lemma 4.1. [AL14, Lemma 3.14] *Let τ be a tamely ramified character of GL_1. Then*

$$\gamma(s, \sigma_\chi^\zeta \times \tau, \psi) = \tau(-1)^{n-1}\tau(\varpi)\chi(g_\chi)q^{1/2-s}$$

5. The simple supercuspidal representations of $SO_{2\ell+1}$

In this section we explicitly construct the simple supercuspidal representations of $SO_{2\ell+1}$, following [RY14, §2]. We then define an explicit Whittaker function for each such simple supercuspidal.

Let $T = T_{SO_{2\ell+1}}$ denote the diagonal split maximal torus of $SO_{2\ell+1}$. Associated to T we have the set of affine roots Ψ. Let $X^*(T)$ denote the character lattice of T, let T_0 be the maximal compact subgroup of T, and set

$$T_1 = \langle t \in T_0 : \lambda(t) \in 1 + \mathfrak{p} \ \forall \lambda \in X^*(T)\rangle.$$

To each $\alpha \in \Psi$ we have an associated affine root group U_α in G. Associated to the standard set of simple roots $\Delta_{SO_{2\ell+1}}$ we have a set of simple affine Π and positive affine roots Ψ^+. Set

$$I = \langle T_0, U_\alpha : \alpha \in \Psi^+\rangle,$$

$$I^+ = \langle T_1, U_\alpha : \alpha \in \Psi^+\rangle,$$

$$I^{++} = \langle T_1, U_\alpha : \alpha \in \Psi^+ \setminus \Pi\rangle.$$

In terms of the Moy-Prasad filtration, if x denotes the barycenter of the fundamental alcove, then $I = G_x = G_{x,0}, I^+ = G_{x+} = G_{x,\frac{1}{2\ell}}$, and $I^{++} = G_{x,\frac{1}{2\ell}+}$.

For $(t_1, t_2, ..., t_{\ell+1}) \in \mathfrak{o}^\times/(1+\mathfrak{p}) \times \mathfrak{o}^\times/(1+\mathfrak{p}) \times \cdots \times \mathfrak{o}^\times/(1+\mathfrak{p})$, we define a character

$$\chi : I^+ \to \mathbb{C}^\times$$

$$h \mapsto \psi(t_1 h_{12} + t_2 h_{23} + \cdots + t_{\ell-1} h_{\ell-1,\ell} + t_\ell x_{\ell,\ell+1} + t_{\ell+1} \frac{h_{2\ell,1}}{\varpi})$$

for $h = (h)_{ij} \in I^+$. These χ's are called the *affine generic characters* of I^+ (see [RY14, §2.6] and [GR10, Lemma 9.2] for more details on this). The orbits of affine generic characters are parameterized by the set of elements in $\mathfrak{o}^\times/(1+\mathfrak{p})$, as follows. $T \cap \mathbf{SO_{2\ell+1}}(\mathfrak{o})$ normalizes I^+, so acts on the set of affine generic characters by conjugation. Every orbit of affine generic characters contains one of the form $(1, 1, ..., 1, t)$, for $t \in \mathfrak{o}^\times/(1+\mathfrak{p})$. Specifically, after conjugating by $diag((t_1 t_2 t_3 \cdots t_\ell)^{-1}, (t_2 t_3 \cdots t_\ell)^{-1} ..., t_\ell^{-1}, 1, t_\ell, ..., t_2 t_3 \cdots t_\ell, t_1 t_2 t_3 \cdots t_\ell)$, the orbit of $(t_1, t_2, ..., t_{\ell+1})$ contains $(1, 1, ..., 1, t)$, where $t = \frac{t_{\ell+1}}{t_1 t_2^2 t_3^2 \cdots t_\ell^2}$.

Instead of viewing the affine generic characters as parameterized by $t \in \mathfrak{o}^\times/(1+\mathfrak{p})$, we will set $t = 1$ and let the affine generic characters be parameterized by the various choices of uniformizer in F. Since we have already fixed an (arbitrary) uniformizer ahead of time in §2, we have therefore fixed an affine generic character χ. The compactly induced representation $\pi_\chi := \mathrm{ind}_H^G \chi$ is a direct sum of 2 distinct irreducible supercuspidal representations of $SO_{2\ell+1}$. They are parameterized $\zeta \in \{-1, 1\}$, as follows. Set

$$g_\chi = \begin{pmatrix} & & \varpi^{-1} \\ & -I_{2\ell-1} & \\ \varpi & & \end{pmatrix}.$$

One can compute that g_χ normalizes I^+, and that $\chi(g_\chi h g_\chi^{-1}) = \chi(h)\ \forall h \in I^+$, as follows. We first note that

$$\chi(g_\chi h g_\chi^{-1}) = \psi(-\varpi^{-1} h_{2\ell+1,2} + h_{23} + h_{34} + ... + h_{\ell,\ell+1} - h_{2\ell,2\ell+1}),$$

$$\chi(h) = \psi(h_{12} + h_{23} + h_{34} + ... + h_{\ell,\ell+1} + \varpi^{-1} h_{2\ell,1}).$$

By explicit computation inside the group $SO_{2\ell+1}$, one then shows that $h_{2\ell+1,2} = -h_{2\ell,1}$ and $h_{2\ell,2\ell+1} = -h_{12}$.

Set $H' = \langle g_\chi \rangle I^+$. Since $g_\chi^2 = 1$, we may extend χ to a character of H' in two different ways by mapping g_χ to $\zeta \in \{-1, 1\}$. Then the summands of π_χ are the compactly induced representations

$$\pi_\chi^\zeta := \mathrm{ind}_{H'}^G \chi_\zeta$$

where $\chi_\zeta(g_\chi h) = \zeta \cdot \chi(h)$, where $\zeta \in \{-1, 1\}$ and $h \in I^+$. π_χ^ζ is called a *simple supercuspidal* representation of $SO_{2\ell+1}$. As we vary ϖ and ζ, we obtain all simple supercuspidal representations of GL_n. Moreover, if one fixes pairs (ϖ_1, ζ_1) and (ϖ_2, ζ_2), where ϖ_i are uniformizers and $\zeta_i \in \{-1, 1\}$, then the associated simple supercuspidals π_1, π_2 of $SO_{2\ell+1}$ are isomorphic if and only if $\varpi_1 \equiv \varpi_2 \mod (1 + \mathfrak{p})$ and $\zeta_1 = \zeta_2$. Since we will need to consider the choice of uniformizer later, we denote by $\pi(\varpi, \zeta)$ the simple supercuspidal representation of $SO_{2\ell+1}$ determined by the pair (ϖ, ζ).

To compute the Rankin-Selberg integrals, we need a Whittaker function for π_χ^ζ with respect to ψ. We note that $\chi_\zeta(u) = \psi\left(\sum_{i=1}^{\ell} u_{i,i+1}\right)$ for $u \in U_{SO_{2\ell+1}} \cap I^+$, so the function

(5.1)

$$W_{\mathrm{std}}(g) := \begin{cases} \psi_{\mathrm{std}}(u)\chi_\zeta(g_\chi^i)\chi_\zeta(k) & \text{if} \quad g = u g_\chi^i k \in U_{SO_{2\ell+1}}\langle g_\chi \rangle I^+ \\ 0 & \text{else} \end{cases}$$

is well-defined and lives in $Ind_U^G \psi_{\mathrm{std}} \cap \mathrm{ind}_{H'}^G \chi_\zeta$, so is therefore a Whittaker function for π_χ^ζ with respect to ψ_{std}. In order to compute the Rankin-Selberg integrals Φ, Φ^*, we need a Whittaker function for π_χ^ζ with respect to ψ. For this, note that there is a canonical isomorphism

$$Ind_U^G \psi_{\mathrm{std}} \xrightarrow{\Omega} Ind_U^G \psi$$

$$f \mapsto {}^\gamma f,$$

where ${}^\gamma f(g) := f(\gamma g \gamma^{-1})$, and $\gamma = diag(I_{\ell-1}, \frac{1}{2}, 1, 2, I_{\ell-1})$. The function $W := {}^\gamma W_{\mathrm{std}}$ is therefore a Whittaker function for the representation $\Omega(\pi_\chi^\zeta)$ with respect to ψ. Since π_χ^ζ and $\Omega(\pi_\chi^\zeta)$ are isomorphic as representations, we will also say that W is a Whittaker function for π_χ^ζ with respect to ψ.

6. The computation of $\Phi(W, f_s)$

In this section, we compute one side of the functional equation (3.4) that defines $\gamma(s, \pi \times \tau, \psi)$, where π is a simple supercuspidal representation of $SO_{2\ell+1}$, and τ is a tamely ramified character of GL_1.

We first note that since τ is a character of GL_1, $\Phi(W, f_s)$ simplifies to

$$\int_{U_{SO_2} \backslash SO_2} \int_{\overline{X}_{(1,\ell)}} W(\overline{x} j_{1,\ell}(h)) f_s(h, 1) d\overline{x} dh,$$

Moreover, note that by (3.2),

$$SO_2 = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in F^\times \right\} \cong GL_1.$$

We will pass between SO_2 and GL_1 when convenient.

We now define $f_s \in V_{Q_1}^{SO_2}(\mathcal{W}(\tau, \psi^{-1}), s)$ (see (3.3)) by

$$f_s(h, a) = |\det(h)|^{s-1/2} \tau(ah), \tag{6.1}$$

where $\tau(h)$ and $\det(h)$, mean τ and det, applied to the upper left entry of h. We remind the reader that $Q_1 \cong SO_2$, so in particular $\delta_{Q_1} \equiv 1$.

Recall the Whittaker function W that we defined earlier (see §5) for $\pi_\chi^\zeta = \mathrm{Ind}_{\langle g_\chi \rangle I^+}^{SO_{2\ell+1}} \chi_\zeta$.

We will need the following computation more than once: if $u = (u)_{ij} \in U_{SO_{2\ell+1}}$ and $k = (k)_{ij} \in I^+$, we get that

$$u g_\chi k = \begin{pmatrix} & & * & \\ \varpi k_{11} & \varpi k_{12} & \cdots & \varpi k_{1n} \end{pmatrix} \tag{6.2}$$

Now let

$$\overline{x} = \begin{pmatrix} 1 & & & & \\ y & I_{\ell-1} & & & \\ & & 1 & & \\ & & & I_{\ell-1} & \\ & & & y' & 1 \end{pmatrix} \in \overline{X}_{1,\ell},$$

with y the transpose of the row vector $(y_1, y_2, ..., y_{\ell-2}, y_{\ell-1})$ (so that $y' = (-y_{\ell-1}, -y_{\ell-2}, ..., -y_2, -y_1)$).

Lemma 6.1. *If $W(\overline{x} j_{n,\ell}(h)) \neq 0$, then $y_i \in \mathfrak{p}$ for $i = 1, 2, ..., \ell - 2$, $y_{\ell-1} \in 2\mathfrak{p}$, and $a \in 1 + \mathfrak{p}$, where $h = diag(a, a^{-1})$. In particular, $\overline{x} \in I^{++}$.*

Proof. Let $h = diag(a, a^{-1}) \in SO_2$. Note that $W(\overline{x} j_{n,\ell}(h)) = W_{\mathrm{std}}(\gamma \overline{x} j_{n,\ell}(h) \gamma^{-1})$. Then $\gamma \overline{x} j_{n,\ell}(h) \gamma^{-1}$ embeds in $SO_{2\ell+1}$ as

$$\begin{pmatrix} a & & & & \\ w & I_{\ell-1} & & & \\ & & 1 & & \\ & & & I_{\ell-1} & \\ & & & w' & a^{-1} \end{pmatrix}.$$

Here, w is the transpose of the row vector $(ay_1, ay_2, ..., ay_{\ell-2}, \frac{1}{2} a y_{\ell-1})$, and $w' = (-\frac{1}{2} y_{\ell-1}, -y_{\ell-2}, -y_{\ell-3}, ..., -y_2, -y_1)$. By comparing lower left matrix entries, one can see that $\gamma \overline{x} j_{n,\ell}(h) \gamma^{-1} \notin U_{SO_{2\ell+1}} g_\chi I^+$ (see (6.2)).

Moreover, by comparing bottom right matrix entries, one can see that if $\gamma \overline{x} j_{n,\ell}(h) \gamma^{-1} \in U_{SO_{2\ell+1}} I^+$, we must have that $a \in 1 + \mathfrak{p}$. Finally, by comparing bottom rows, that $y_i \in \mathfrak{p}$ for $i = 1, 2, ..., \ell - 2$, and that $y_{\ell-1} \in 2\mathfrak{p}$. □

We may now compute $\Phi(W, f_s)$.

Corollary 6.2.

$$\int_{U_{SO_2} \backslash SO_2} \int_{\overline{X}_{(1,\ell)}} W(\overline{x} j_{1,\ell}(h)) f_s(h, 1) d\overline{x} dh = vol(\mathfrak{p})^{\ell-2} vol(2\mathfrak{p}) vol(1 + \mathfrak{p}).$$

Proof. By Lemma 6.1, since $U_{SO_2} = 1$, and since τ is tamely ramified, we obtain

$$\int_{U_{SO_2} \backslash SO_2} \int_{\overline{X}_{(1,\ell)}} W(\overline{x} j_{1,\ell}(h)) f_s(h, 1) d\overline{x} dh$$
$$= \int_{1+\mathfrak{p}} \int_{\overline{X}_{(1,\ell)}} W(\overline{x} j_{1,\ell}(h)) d\overline{x} dh.$$

Moreover, because of the constraints placed on the y_i in the statement of Lemma 6.1, and by definition of W, this last double integral equals $vol(\mathfrak{p})^{\ell-2} vol(2\mathfrak{p}) vol(1 + \mathfrak{p})$, as claimed. □

7. The computation of $\Phi^*(W, f_s)$

In this section, we compute $\Phi^*(W, f_s)$, for the same choice of W and f_s as in §6. Since τ is a character of GL_1, $\Phi^*(W, f_s)$ simplifies to

$$\Phi^*(W, f_s)$$
$$= \gamma(2s - 1, \tau, \wedge^2, \psi) \int_{U_{SO_2} \backslash SO_2} \int_{\overline{X}_{(1,\ell)}} W(\hat{c}_{1,\ell} \overline{x} j_{1,\ell}(h) \delta_o \omega') M(\tau, s) f_s(h^{-1}, 1) d\overline{x} dh.$$

We note in particular that

$$\omega' = \begin{pmatrix} & & 1 \\ & I_{2\ell-1} & \\ 1 & & \end{pmatrix}.$$

Now let

$$\overline{x} = \begin{pmatrix} 1 & & & & \\ y & I_{\ell-1} & & & \\ & & 1 & & \\ & & & I_{\ell-1} & \\ & & & y' & 1 \end{pmatrix} \in \overline{X}_{1,\ell},$$

with y the transpose of the row vector $(y_1, y_2, ..., y_{\ell-2}, y_{\ell-1})$ (so that $y' = (-y_{\ell-1}, -y_{\ell-2}, ..., -y_2, -y_1)$.

Lemma 7.1. *In order that* $W(\hat{c}_{1,\ell}\overline{x}j_{1,\ell}(h)\delta_o\omega') \neq 0$, *it must be that* $y_i \in \mathfrak{p}$ *for* $i = 1, 2, ..., \ell-2$, *that* $y_{\ell-1} \in 2\mathfrak{p}$, *and that* $a^{-1} \in \varpi \cdot (1+\mathfrak{p})$, *where* $h = diag(a, a^{-1})$. *In particular,* $\overline{x} \in I^+$.

Proof. Note first that $W(\hat{c}_{1,\ell}\overline{x}j_{1,\ell}(h)\delta_o\omega') = W_{\mathrm{std}}(\gamma\hat{c}_{1,\ell}\overline{x}j_{1,\ell}(h)\delta_o\omega'\gamma^{-1})$. If $h = diag(a, a^{-1})$, then

$$\gamma\hat{c}_{1,\ell}\overline{x}j_{1,\ell}(h)\delta_o\omega'\gamma^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & a \\ 0 & -I_{\ell-1} & 0 & 0 & w \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -I_{\ell-1} & 0 \\ a^{-1} & 0 & 0 & w' & 0 \end{pmatrix},$$

where w is the transpose of the row vector $(-ay_1, -ay_2, ..., -ay_{\ell-2}, -\frac{1}{2}ay_{\ell-1})$ and $w' = (-\frac{1}{2}y_{\ell-1}, -y_{\ell-2}, -y_{\ell-3}, ..., -y_2, -y_1)$.

We now perform some matrix computations. In order that $W_{\mathrm{std}}(\gamma\hat{c}_{1,\ell}\overline{x}j_{1,\ell}(h)\delta_o\omega'\gamma^{-1}) \neq 0$, we must have that $\gamma\hat{c}_{1,\ell}\overline{x}j_{1,\ell}(h)\delta_o\omega'\gamma^{-1} \in U_{SO_{2\ell+1}}\langle g_\chi\rangle I^+$, by definition of W_{std}.

We first consider the double coset $U_{SO_{2\ell+1}}g_\chi I^+$. Namely, suppose that $\gamma\hat{c}_{1,\ell}\overline{x}j_{1,\ell}(h)\delta_o\omega'\gamma^{-1} = ug_\chi k$, for some $u \in U_{SO_{2\ell+1}}, k \in I^+$. Let us compare the last row of $\gamma\hat{c}_{1,\ell}\overline{x}j_{1,\ell}(h)\delta_o\omega'\gamma^{-1}$ to the last row of $ug_\chi k$. In particular, since $k_{11} \in 1+\mathfrak{p}$, we must have that $a^{-1} \in \varpi \cdot (1+\mathfrak{p})$ (see (6.2)). Since $k_{1j} \in \mathfrak{o}$ for $j \geq 2$, we moreover require that $y_i \in \mathfrak{p}$ for $i = 1, 2, ..., \ell-2$, and $y_{\ell-1} \in 2\mathfrak{p}$.

A similar type of analysis shows that $\gamma\hat{c}_{1,\ell}\overline{x}j_{1,\ell}(h)\delta_o\omega'\gamma^{-1} \notin U_{SO_{2\ell+1}}I^+$. □

We now have our main result.

Theorem 7.2.

$$\gamma(s, \pi_\chi^\zeta \times \tau, \psi) = \chi_\zeta(g_\chi)\tau(\varpi)q^{1/2-s}.$$

Proof. We first note that since τ is a character of GL_1, we have that $\gamma(2s-1, \tau, \wedge^2, \psi) = 1$. This can be explicitly calculated from its definition, which can be found in [K14ii, §3.1].

Now let $h = diag(a, a^{-1}) \in SO_2$, and suppose that $W_{\mathrm{std}}(\gamma\hat{c}_{1,\ell}\overline{x}j_{1,\ell}(h)\delta_o\omega'\gamma^{-1}) \neq 0$. By Lemma 7.1, we have that $a^{-1} \in \varpi \cdot (1+\mathfrak{p})$. In particular, we can write $a = \varpi^{-1}v$, for some $v \in 1+\mathfrak{p}$.

We now decompose $\gamma\hat{c}_{1,\ell}\overline{x}j_{1,\ell}(h)\delta_o\omega'\gamma^{-1}$ in the double coset $U_{SO_{2\ell+1}}g_\chi I^+$ as

$$\gamma\hat{c}_{1,\ell}\overline{x}j_{1,\ell}(h)\delta_o\omega'\gamma^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & a \\ 0 & -I_{\ell-1} & 0 & 0 & w \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -I_{\ell-1} & 0 \\ a^{-1} & 0 & 0 & w' & 0 \end{pmatrix}$$

$$= \begin{pmatrix} & & \varpi^{-1} \\ & -I_{2\ell-1} & \\ \varpi & & \end{pmatrix} \begin{pmatrix} v^{-1} & & & \varpi^{-1}w' & \\ & I_{\ell-1} & & & -w \\ & & 1 & & \\ & & & I_{\ell-1} & \\ & & & & v \end{pmatrix},$$

where w, w' are as in Lemma 7.1. This decomposition implies by Lemma 7.1 that if $W_{\text{std}}(\gamma\hat{c}_{1,\ell}\overline{x}j_{1,\ell}(h)\delta_o\omega'\gamma^{-1}) \neq 0$, then in fact $W_{\text{std}}(\gamma\hat{c}_{1,\ell}\overline{x}j_{1,\ell}(h)\delta_o\omega'\gamma^{-1}) = \chi_\zeta(g_\chi)$.

Finally, one can see that $M(\tau, s)f_s(h^{-1}, 1) = f_s(h^{-1}, 1)$. Since $a^{-1} = \varpi v^{-1}$, we have $f_s(h^{-1}, 1) = |\det(\varpi v^{-1})|^{s-1/2}\tau(\varpi v^{-1})$. Therefore, because of the constraints placed on the y_i in the statement of Lemma 7.1, we have

$$\begin{aligned} &\Phi^*(W, f_s) \\ &= \int_{SO_2}\int_{\overline{X}_{(1,\ell)}} W(\hat{c}_{1,\ell}\overline{x}j_{1,\ell}(h)\delta_o\omega')M(\tau, s)f_s(h^{-1}, 1)d\overline{x}dh \\ &= vol(\mathfrak{p})^{\ell-2}vol(2\mathfrak{p}) \int_{1+\mathfrak{p}} \chi_\zeta(g_\chi)|\det(\varpi v^{-1})|^{s-1/2}\tau(\varpi v^{-1})dh \\ &= vol(\mathfrak{p})^{\ell-2}vol(2\mathfrak{p})\chi_\zeta(g_\chi)|\det(\varpi)|^{s-1/2}\tau(\varpi)vol(1+\mathfrak{p}) \\ &= vol(\mathfrak{p})^{\ell-2}vol(2\mathfrak{p})\chi_\zeta(g_\chi)q^{1/2-s}\tau(\varpi)vol(1+\mathfrak{p}), \end{aligned}$$

since τ is tamely ramified. Combining this computation with Corollary 6.2 and (3.4), we have our result. □

Corollary 7.3.

$$\Gamma(s, \pi_\chi^\zeta \times \tau, \psi) = \chi_\zeta(g_\chi)\tau(-1)^\ell\tau(\varpi)q^{1/2-s}.$$

We now compare $\Gamma(s, \pi_\chi^\zeta \times \tau, \psi)$ with the gamma factors of simple supercuspidal representations of $GL_{2\ell}$. First, we note that we may rewrite π_χ^ζ as $\pi(\varpi, \zeta)$ (see §5). By Corollary 7.3 and Lemma 4.1, we obtain the following matching of gamma factors.

Corollary 7.4. *The representation $\sigma((-1)^{\ell+1}\varpi, \zeta, 1)$ of $GL_{2\ell}$ and the representation $\pi(\varpi, \zeta)$ of $SO_{2\ell+1}$ share the same gamma factors twisted* *by tamely ramified characters of GL_1. Namely,*

$$\Gamma(s, \pi(\varpi, \zeta) \times \tau, \psi) = \gamma(s, \sigma((-1)^{\ell+1}\varpi, \zeta, 1) \times \tau, \psi),$$

for all tamely ramified characters τ of GL_1.

8. Langlands correspondence

In this section we determine the Langlands parameter of a simple supercuspidal representation of $SO_{2\ell+1}$, under the assumption that p is sufficiently large (see Remark 8.5). We first need to recall the definition of local transfer.

Definition 8.1. [ACS14, Definition 3.1] *Let π be an irreducible generic representation of $SO_{2\ell+1}$. Let Π be an irreducible representation of $GL_{2\ell}$. We say that Π is a* local transfer *of π if*

$$L(s, \pi \times \rho) = L(s, \Pi \times \rho) \text{ and } \gamma(s, \pi \times \rho, \psi) = \gamma(s, \Pi \times \rho, \psi)$$

for all irreducible, unitary, supercuspidal representations ρ of $GL_m, 1 \leq m \leq 2\ell - 1$. The L, γ-factors on the left hand side are those of the Langlands-Shahidi method while those on the right hand side are defined via either the Rankin-Selberg method [JPSS83] *or the Langlands-Shahidi methods* [Sha90]*.*

Remark 8.2. *It is not true in general that Rankin-Selberg factors for $GL \times GL$ equal the corresponding Langlands-Shahidi gamma factors. Indeed, this is the crux of Shahidi's paper* [Sha84]*. However, one can show that if a generic representation of a GL arises as a local lift from a classical group, then its twisted Rankin-Selberg gamma factors equal its twisted Langlands-Shahidi gamma factors.*

Recall the Langlands parameter $Ind_{W_E}^{W_F}\xi$ constructed in §1. We may now prove:

Theorem 8.3. *For p sufficiently large, the functorial lift Π of $\pi = \pi_\chi^\zeta$ (under the embedding $Sp(2\ell, \mathbb{C}) \hookrightarrow GL(2\ell, \mathbb{C})$) to $GL_{2\ell}$ is supercuspidal.*

Proof. We need to recall some results of [K15]. Specialized to the group $G = SO_{2\ell+1}$, the work of [K15] implies that the character of π is stable. To see this, let ϕ be the Langlands parameter constructed in [K15] associated to π. The part of the L-packet that Kaletha constructs, on the group $SO_{2\ell+1}$, is in bijection with the characters of the centralizer S_ϕ of ϕ that are trivial on the center of $\hat{G} = Sp_{2\ell}(\mathbb{C})$ (see [K15, (0.0.1)]). In this case the centralizer of ϕ is the subgroup of the maximal torus $\hat{T}$ of $\hat{G}$ consisting of those elements that are fixed by a Coxeter element w ([K15, p. 29, 34]). A computation shows that $(\widehat{T})^w = Z(\hat{G})$.

Therefore, the quotient $S_\phi/Z(\hat{G})$ is trivial. A special case of [K15, Theorem 5.2.1] now shows that the character of π is stable. Therefore, by [M14, Corollary 4.5, Theorem 4.7], we have that π constitutes an entire L-packet by itself.

Since π constitutes an L-packet, the component group of its associated Langlands parameter (the parameter of Arthur [A13]) φ_π is trivial, so that φ_π is an irreducible 2ℓ dimensional representation of the Weil-Deligne group. The associated representation Π of $GL_{2\ell}$ via the local Langlands correspondence is therefore a discrete series representation.

We now note that the functorial lift Π (which comes from Arthur's work) agrees with the lifting constructed by [CKPSS04]. Therefore, by Definition 8.1, we in particular have that

$$\gamma(s, \Pi, \psi) = \Gamma(s, \pi, \psi).$$

Since the conductor of Π can be read off $\gamma(s, \Pi, \psi)$ (see, in particular, [BH14, §1.3]), we conclude that the conductor of Π is $2\ell+1$ by Corollary 7.3.

We now turn to some results of [LR03]. In particular, we follow some arguments in the proof of [LR03, Theorem 3.1]. According to the classification of Bernstein and Zelevinskii, Π is equivalent to the Langlands Quotient $Q(\lambda)$ of the parabolic induction $\mathrm{Ind}_P^H(\sigma)$. Here, P is a standard parabolic subgroup of $H = GL_{2\ell}$ whose Levi subgroup is of the form $GL_a \times ... \times GL_a$, and σ is a representation of P of the form $\lambda \otimes \lambda|\det| \otimes ... \otimes \lambda|\det|^{b-1}$, where λ is a supercuspidal representation of GL_a. Note that $ab = 2\ell$. There are two cases to consider. If $a = 1$ and λ is unramified, then Π is an unramified twist of the Steinberg representation. But an unramified twist of the Steinberg representation has conductor $2\ell - 1$, a contradiction. Otherwise, denoting c by conductor, the proof of [LR03, Theorem 3.1] further shows that $c(\Pi) = c(\lambda)b$. Together with $ab = 2\ell$ and $c(\Pi) = 2\ell + 1$ we obtain immediately that $b = 1$, so that $a = 2\ell$, so that $\Pi = \lambda$ is supercuspidal. □

Corollary 8.4. *For p sufficiently large, the Langlands parameter of $\pi = \pi_\chi^\zeta$ is $Ind_{W_E}^{W_F}\xi$.*

Proof. We have concluded in Theorem 8.3 that the functorial lift Π of π to $GL(2\ell, F)$ is supercuspidal. Π has conductor $2\ell + 1$, so in fact Π is simple supercuspidal. We also note that this lift must have trivial central character since it is a lift from $SO_{2\ell+1}$. By Corollary 7.4 and by [AL14, Remark 3.18] (see also [BH14, Lemma 2.2, Proposition 2.2]), this lift must be $\sigma((-1)^{\ell+1}\varpi, \zeta, 1)$. The main result of [AL14, §3] says that the Langlands parameter of σ_χ^ζ is $Ind_{W_E}^{W_F}\xi$. □

Remark 8.5. *The meaning of p sufficiently large is as follows. Let e be the ramification index of $F/\mathbb{Q}_p$. Kaletha proves the stability results needed in Corollary 8.3 under the assumption that $p \geq (2+e)(2\ell+1)$* [K15, §5.1]*.*

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