ON THE LANGLANDS PARAMETER OF A SIMPLE SUPERCUSPIDAL REPRESENTATION: EVEN ORTHOGONAL GROUPS

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Abstract. Let π be a simple supercuspidal representation of the split even special orthogonal group. We compute the Rankin–Selberg γ-factors for rank 1-twists of π by quadratic tamely ramified characters of F^*. Our results determine certain 1-dimensional summands of the Langlands parameter. Assuming an expected analogue of a work of Blondel, Henniart, and Stevens, these summands completely determine the Langlands parameter of π.

1. Introduction

Let F be a local p-adic field of characteristic 0. Supercuspidal representations are fundamental objects in the study of representations of p-adic groups, being essentially the building blocks of all irreducible representations. The construction of supercuspidal representations has been the focus of several works, including [How77, Adl98, BK98, Yu01, Ste08]. In this fascinating class of representations, we have the simple supercuspidal representations, i.e., the supercuspidal representations of minimal nonzero depth in the sense of Moy and Prasad [MP94, MP96]. These representations, recently constructed and studied by Gross and Reeder [GR10] and Reeder and Yu [RY14], can be considered as an initial "litmus test" for statements on arbitrary supercuspidal representations. See for example, the work of Kaletha [Kal13] on an explicit correspondence for simple supercuspidal representations of simply connected groups. In this work we study the Langlands parameter for simple supercuspidal representations of even orthogonal groups.

Let $\mathrm{SO}_{2l} = \mathrm{SO}_{2l}(F)$ be the split special even orthogonal group of rank l. We compute a certain family of twisted gamma factors of an arbitrary simple supercuspidal representation π of SO_{2l}. Our results enable us to explicitly identify the quadratic, tamely ramified 1-dimensional summands of the Langlands parameter of π. It is expected that the rest of the parameter corresponds to a simple supercuspidal representation, and this can be verified by proving an analogue of the work of Blondel *et al.* [BHS] for SO_{2l}. Moreover, this analogue determines the restriction of the parameter to the wild inertia subgroup. We will show that our results determine the rest of the parameter. The present work is the follow-up to [Adr16, AK], where the analogous computations were carried out and the theory of Rankin–Selberg integrals was applied, in order to determine the Langlands parameter of odd orthogonal groups and symplectic groups.

Let π be a simple supercuspidal representation of SO_{2l}. Throughout, supercuspidal represenations are assumed to be irreducible. The representation π is known to be generic, for a certain character of a maximal unipotent subgroup of SO_{2l}. Thus π admits a local

2010 *Mathematics Subject Classification.* Primary 11S37, 22E50; Secondary 11F85, 22E55.

Key words and phrases. Simple supercuspidal, Local Langlands Conjecture, Rankin–Selberg method.

Support to Adrian was provided by a grant from the Simons Foundation #422638 and by a PSC-CUNY award, jointly funded by the Professional Staff Congress and The City University of New York. Kaplan was supported by the Israel Science Foundation, grant number 421/17.

functorial lift Π to GL_{2l}, as defined by Cogdell *et. al.* [CKPSS04, Proposition 7.2]. In general, one can then study the Langlands parameter φ_π of π using an explicit local Langlands correspondence for the supercuspidal representations in the support of Π.

In this work we find the quadratic, tamely ramified 1-dimensional summands of the Langlands parameter of π, by identifying the quadratic tamely ramified characters τ of F^* such that the Rankin–Selberg γ-factor $\gamma(s, \pi \times \tau, \psi)$ (defined in [Kap13, Kap15]) has a pole at $s = 1$. More precisely, the simple supercuspidal representations of SO_{2l} are parameterized by four pieces of data: a choice of uniformizer ϖ, an element $\alpha \in \kappa^\times/(\kappa^\times)^2$ ($\kappa = \mathfrak{o}/\mathfrak{p}$), a sign $\epsilon = \pm 1$, and a central character ω. For convenience in later computations, we set $\gamma = -4\alpha$. To be explicit, let χ be an affine generic character of the pro-unipotent radical I^+ of an Iwahori group I, defined by an element γ. The choice of uniformizer ϖ in F determines an element g_χ in SO_{2l} which normalizes I and stabilizes χ. We can extend χ to $\langle g_\chi \rangle I^+$ in two different ways, since $g_\chi^2 = 1$. Further extending χ by ω to the group $K = Z\langle g_\chi \rangle I^+$, and calling the new character χ again, we obtain a simple supercuspidal representation $\pi = \mathrm{Ind}_K^{\mathrm{SO}_{2l}} \chi$. For more details see § 3. The following is our main theorem, which characterizes the tamely ramified quadratic 1-dimensional representations in the support of Π (again, expected to be all of the 1-dimensional summands).

Theorem 1.1. *Let $(\varpi, \gamma, \epsilon, \omega)$ be the parameters uniquely determined by π. Let τ be a quadratic tamely ramified character of F^*. Then $\gamma(s, \pi \times \tau, \psi)$ is holomorphic and nonzero at $s = 1$ if and only if $\tau(\varpi) = -\chi(g_\chi)\tau(\gamma)$.*

The theorem is proved in §4. In fact our result is stronger: we compute $\gamma(s, \pi \times \tau, \psi)$ explicitly for any quadratic tamely ramified character τ, see Corollary 4.7. In particular $\gamma(s, \pi \times \tau, \psi)$ has at most a simple pole at $s = 1$.

The analogue of [BHS] for SO_{2l} is expected to show that the complement of these 1-dimensional summands is a simple supercuspidal representation, which we denote Π', and also expected to determine the restriction of the parameter to the wild inertia subgroup. We note that if $p \neq 2$, we compute two 1-dimensional summands, and if $p = 2$, only one.

The representation Π' can be parameterized by a triplet consisting of a uniformizer ϖ of F, a central character ω, and a certain root of $\omega(\varpi)$. Our gamma factor computations yield not only the quadratic, tamely ramified 1-dimensional summands of the Langlands parameter, but also yield the central character ω of Π', and moreover we subsequently obtain the root of $\omega(\varpi)$ by computing $\gamma(s, \pi, \psi)$. What remains in order to fully describe the parameter is the uniformizer ϖ, and the proof that the complement of the 1-dimensional summands (or summand, in the case that $p = 2$) does indeed correspond to a simple supercuspidal representation. Our method does not provide these two items, but they are obtainable from an analogue of [BHS] for SO_{2l}.

Our main tool in this work is the γ-factor defined by the theory of Rankin–Selberg integrals for $\mathrm{SO}_{2l} \times \mathrm{GL}_n$ in [Kap15], following the development of these integrals in [GPSR87, Kap10, Kap12, Kap13, Kap]. The γ-factor is essentially the proportionality factor between two integrals, related by an application of an intertwining operator. The proof of Theorem 1.1 is based on a direct computation of this factor for $n = 1$, and is among the first few applications of Rankin–Selberg integrals to results of this kind.

A subtle part of the definition of $\gamma(s, \pi \times \tau, \psi)$ is to normalize it properly, in order to obtain precise multiplicative formulas which identify this factor with the corresponding γ-factor of Shahidi (defined in [Sha90]). While this normalization does not play a role in the determination of the poles, it is crucial for the computation of φ_π. Obtaining precise normalization is nontrivial. The equality between these γ-factors, in the context of Shimura-type integrals (proved in [Kap15]) was one of the ingredients in the work of Ichino *et. al.* [ILM17] on the formal degree conjecture. For other works on Rankin–Selberg integrals and their γ-factors, in the context of generic representations of classical groups, see e.g., [Gin90, Sou93, Sou95, GRS98, Sou00].

As mentioned above, this work is a follow-up to [Adr16, AK]. The case of odd orthogonal groups [Adr16] was a bit different in the sense that the lift Π of π was already expected to be simple supercuspidal. Indeed the twisted γ-factors had no poles, and their computation was sufficient to determine the Langlands parameter using, among other result, the works of Mœglin [Mœg14] and Kaletha [Kal15]. For the symplectic case ([AK]) and when $p \neq 2$, according to [BHS], φ_π decomposes into 2 summands, one of them 1-dimensional. This summand was again identified using an analogue of Theorem 1.1, and the computation of $\gamma(s, \pi, \psi)$ was then sufficient to obtain φ_π. When $p = 2$, the symplectic case was similar to the odd orthogonal case, since the twisted γ-factors had no poles.

The rest of this work is organized as follows. The Rankin–Selberg integral is described in §2. The simple supercuspidal representations are defined in §3. The computation of the γ-factor is carried out in §4. In §5, we describe the Langlands parameter. Finally §6 contains the computation of certain normalization factors used for the definition of the γ-factor.

Acknowledgements. We are happy to thank Gordan Savin for helpful conversations. We would like to thank the referee for her/his careful reading of the manuscript and helpful remarks, which helped clarify certain points in the presentation.

2. The groups and the Rankin–Selberg integral

Let F be a p-adic field of characteristic 0, with a ring of integers $\mathfrak{o}$ and maximal ideal $\mathfrak{p}$. Denote $\kappa = \mathfrak{o}/\mathfrak{p}$ and $q = |\kappa|$. Let ϖ be a uniformizer ($|\varpi| = q^{-1}$). Fix the Haar measure dx on F which assigns the volume $q^{1/2}$ to $\mathfrak{o}$, and define a measure $d^\times x$ on $F^\times$ by $d^\times x = \frac{q^{1/2}}{q-1}|x|^{-1}dx$. We use the notation vol (resp., $\mathrm{vol}^\times$) to denote volumes of measurable subsets under dx (resp., $d^\times x$), e.g. $\mathrm{vol}^\times(\mathfrak{o}^\times) = 1$. Let $J_r \in \mathrm{GL}_r(F)$ denote the permutation matrix with 1 along the anti-diagonal. For $g \in \mathrm{GL}_r(F)$, ${}^tg$ denotes the transpose of g, and $g^* = J_r {}^tg^{-1} J_r$.

Fix $\gamma \in F^*$. We define the orthogonal groups appearing in this work:

$$\mathrm{SO}_{2l}(F) = \{g \in \mathrm{SL}_{2l}(F) : {}^tgJ_{2l}g = J_{2l}\},$$
$$\mathrm{SO}_{2n+1}(F) = \{g \in \mathrm{SL}_{2n+1}(F) : {}^tgJ_{2n+1,\gamma}g = J_{2n+1,\gamma}\}, \qquad J_{2n+1,\gamma} = \begin{pmatrix} & & J_n \\ & \gamma/2 & \\ J_n & & \end{pmatrix}.$$

Throughout, we identify linear groups with their F-points, i.e., $\mathrm{SO}_r = \mathrm{SO}_r(F)$.

Fix the Borel subgroup $B_{\mathrm{SO}_r} = T_{\mathrm{SO}_r} \ltimes U_{\mathrm{SO}_r}$ of upper triangular invertible matrices in SO_r, where T_{SO_r} is the diagonal torus. Denote $K_{\mathrm{SO}_r} = \mathrm{SO}_r(\mathfrak{o})$, which is a maximal compact open subgroup in SO_r. Let Z_{SO_r} be the center of SO_r. For a unipotent radical U of a parabolic subgroup $P < \mathrm{SO}_r$, let $\overline{U}$ be the unipotent radical of the parabolic subgroup opposite to P which contains the Levi part of P, $\overline{U}$ is generated by the roots $-\alpha$ where α varies over the roots in U.

We describe the Rankin–Selberg integral for $\mathrm{SO}_{2l}\times\mathrm{GL}_1$, $l\geq 2$, which will be our main tool for the computation of the γ-factor. We follow the definitions and conventions of [Kap15], where the full details of the construction for $\mathrm{SO}_{2l}\times\mathrm{GL}_n$ were given.

Let τ be a quasi-character of F^*. For $s\in\mathbb{C}$, let $V(\tau,s)$ be the space of the representation $\mathrm{Ind}_{B_{\mathrm{SO}_3}}^{\mathrm{SO}_3}(|\det|^{s-1/2}\tau)$ (normalized induction). The elements of $V(\tau,s)$ are complex-valued smooth functions f_s on $\mathrm{SO}_3\times\mathrm{GL}_1$, such that for all $a,m\in F^*$, $u\in U_{\mathrm{SO}_3}$ and $g\in\mathrm{SO}_3$,

$$f_s(\mathrm{diag}(m,1,m^{-1})ug,a)=|m|^sf_s(g,am)=|m|^s\tau(am)f_s(g,1).$$

The right-action of SO_3 on $V(\tau,s)$ is denoted $g\cdot f_s$. A function f_s is called a standard section if its restriction to K_{SO_3} is independent of s, and a holomorphic section if its restriction to K_{SO_3} is a polynomial function in $q^{\mp s}$.

Let $l\geq 2$ and fix a nontrivial additive character ψ of F. Define the following non-degenerate character ψ of $U_{\mathrm{SO}_{2l}}$ by

$$\psi(u)=\psi(\sum_{i=1}^{l-2}u_{i,i+1}+\tfrac{1}{4}u_{l-1,l}-\gamma u_{l-1,l+1}). \tag{2.1}$$

Let π be an irreducible ψ^{-1}-generic representation of SO_{2l}, and denote the corresponding Whittaker model of π by $\mathcal{W}(\pi,\psi^{-1})$.

We turn to describe the embedding of SO_3 in SO_{2l}. Let $Q=M\ltimes N$ be the standard parabolic subgroup of SO_{2l}, whose Levi part M is isomorphic to $\mathrm{GL}_1\times\ldots\times\mathrm{GL}_1\times\mathrm{SO}_4$. For $l\geq 3$, define a character ψ_N of N by

$$\psi_N(u)=\psi(\sum_{i=1}^{l-3}u_{i,i+1}+\tfrac{1}{4}u_{l-2,l}-\gamma u_{l-2,l+1}).$$

The group SO_3 is then embedded in SO_{2l} in the stabilizer of ψ_N in M. When $l=2$, we embed SO_3 in the subgroup of $g\in\mathrm{SO}_4$ such that $g(\frac{1}{4}e_2-\gamma e_3)=\frac{1}{4}e_2-\gamma e_3$, where $(e_1,\ldots,e_4)$ is the standard basis of the column space F^4. In coordinates, the image of $(x_{i,j})_{1\leq i,j\leq 3}\in\mathrm{SO}_3$ in SO_{2l} is given by

$$diag(I_{l-2},\begin{pmatrix}1&&&\\&\frac{1}{4}&\frac{1}{4}&\\&-\gamma&\gamma&\\&&&1\end{pmatrix}\begin{pmatrix}x_{1,1}&&x_{1,2}&x_{1,3}\\&1&&\\x_{2,1}&&x_{2,2}&x_{2,3}\\x_{3,1}&&x_{3,2}&x_{3,3}\end{pmatrix}\begin{pmatrix}1&&&\\&2&-\frac{1}{2}\gamma^{-1}&\\&2&\frac{1}{2}\gamma^{-1}&\\&&&1\end{pmatrix},I_{l-2}).$$

The conjugating matrix is different from the one used in [Kap15]. To explain this, let e belong to the orthogonal complement of $\frac{1}{4}e_2-\gamma e_3$ in F^4 with respect to the bilinear form $(u,v)\mapsto {}^tuJ_4v$. Fixing e in the span of e_2 and e_3, it belongs to the span of $\frac{1}{4}e_2+\gamma e_3$, then SO_3 is defined with respect to (e_1,e,e_4) (for $l>2$, (e_1,e_2,e_3,e_4) is replaced with $(e_{l-1},e_l,e_{l+1},e_{l+2})$). In [Kap15] 2γ was assumed to be a square (in the split case), then e could be scaled to a unit vector and the Gram matrix of (e_1,e,e_4) was J_3. Without this assumption we take here $e=\frac{1}{4}e_l+\gamma e_{l+1}$ and work with $J_{3,\gamma}$. In the general case of an arbitrary $n<l$, the definition of SO_{2n+1} is then using $J_{2n+1,\gamma}$.

Also let

$$R^{l,1}=\left\{\begin{pmatrix}1&&&&\\ r&I_{l-2}&&&\\ &&I_2&&\\ &&&I_{l-2}&\\ &&&r'&1\end{pmatrix}\in\mathrm{SO}_{2l}\right\},\qquad w^{l,1}=\begin{pmatrix}&1&&&\\ I_{l-2}&&&&\\ &&I_2&&\\ &&&&I_{l-2}\\ &&&1&\end{pmatrix}\in\mathrm{SO}_{2l}.$$

We will occasionally refer to $r\in R^{l,1}$ also as a column vector in F^{l-2}. Now we can define the Rankin–Selberg integral for $\pi\times\tau$: for any $W\in\mathcal{W}(\pi,\psi^{-1})$ and a holomorphic section f_s, the integral is defined for $\mathrm{Re}(s)\gg 0$ by

$$\Psi(W,f_s)=\int_{U_{\mathrm{SO}_3}\backslash\mathrm{SO}_3}\int_{R^{l,1}}W(rw^{l,n}h)f_s(h,1)\,dr\,dh. \tag{2.2}$$

It admits meromorphic continuation to a rational function in q^{-s}.

Next consider the intertwining operator

$$M(\tau,s):V(\tau,s)\to V(\tau^{-1},1-s)$$

given by the meromorphic continuation of the integral

$$M(\tau,s)f_s(h,a)=\int_{U_{\mathrm{SO}_3}}f_s(w_1uh,-a^{-1})du,\qquad w_1=\begin{pmatrix}&&1\\&-1&\\1&&\end{pmatrix}.$$

The measure du is the additive measure dx of F, where we identify $u\in U_{\mathrm{SO}_3}$ with F via $u\mapsto u_{1,2}$. The normalized intertwining operator $M^*(\tau,s)=C(s,\tau,\psi)M(\tau,s)$ is defined by the functional equation

$$\int_{U_{\mathrm{SO}_3}}f_s(w_1u,1)\psi^{-1}(u_{1,2})\,du=C(s,\tau,\psi)\int_{U_{\mathrm{SO}_3}}M(\tau,s)f_s(w_1u,1)\psi^{-1}(u_{1,2})\,du. \tag{2.3}$$

Note that we omitted the matrix $d_1=-1$ appearing on both sides of this equation in [Kap15, (3.5)], because $\tau(-1)=\tau^{-1}(-1)$. The constant $C(s,\tau,\psi)$ is essentially Shahidi's γ-factor $\gamma(2s-1,\tau,S^2,\psi)$ defined in [Sha90], up to a factor of the form Bq^{As} where A and B are constants depending only on τ, ψ and F. For our purpose here we need to find the precise value of $C(s,\tau,\psi)$ and we have the following proposition, proved in § 6 below.

Proposition 2.1. *Let $\gamma^{\mathrm{Tate}}(s,\tau^2,\psi)$ be the γ-factor of Tate* [Tat67] *(see § 6). We have*

$$C(s,\tau,\psi)=\tau^4(2)|2|^{4s}\tau^{-1}(\gamma)|\gamma|^{-s-1}\gamma^{\mathrm{Tate}}(2s-1,\tau^2,\psi). \tag{2.4}$$

The integral $\Psi^*(W,f_s)=\Psi(W,M^*(\tau,s)f_s)$ is absolutely convergent in $\mathrm{Re}(s)\ll 0$, and the functional equation is defined by

$$\gamma(s,\pi\times\tau,\psi)\Psi(W,f_s)=\pi(-I_{2l})\tau(-1)^l\left(\tau^2(2)|2|^{2s-1}\tau^{-2}(\gamma)|\gamma|^{-2s+1}\right)\Psi^*(W,f_s). \tag{2.5}$$

Since the definition of SO_3 here and the choice of vector e are different from [Kap15], the normalization factor appearing on the right hand side of (2.5) is different. We compute this factor, i.e., prove (2.5), in § 6.

Remark 2.2. *In the split case in* [Kap15]*, the parameter γ was chosen such that $2\gamma=\rho$ was a square, because the same parameter was used for the embedding of SO_{2l} in SO_{2n+1}. The group SO_{2l} was embedded in SO_{2n+1} in the stabilizer of a character of a unipotent subgroup of SO_{2n+1}. That character depended on γ, and its stabilizer contained either the split or the quasi-split and nonsplit SO_{2l}, depending on ρ. Also note that SO_{2n+1} here is isomorphic to* *the special orthogonal group defined with respect to* J_{2n+1} *in* [Kap15] *(for any* $\gamma \in F^*$*). The factor* $\tau^{-2}(\gamma)|\gamma|^{-2s+1}$ *in* (2.5) *was denoted* $c(s,l,\tau,\gamma)$ *in loc. cit.*

3. The simple supercuspidal representations of SO_{2l}

In this section, we recall the construction of the simple supercuspidal representations of SO_{2l}. Let $\Delta_{\mathrm{SO}_{2l}} = \{\epsilon_1-\epsilon_2,\ldots,\epsilon_{l-1}-\epsilon_l,\epsilon_{l-1}+\epsilon_l\}$ denote the set of simple roots of SO_{2l}, determined by our choice of the Borel subgroup $B_{\mathrm{SO}_{2l}}$. Let $X^*(T_{\mathrm{SO}_{2l}})$ denote the character lattice of $T_{\mathrm{SO}_{2l}}$ and T_0 be the maximal compact subgroup of $T_{\mathrm{SO}_{2l}}$. Set

$$T_1 = \langle t \in T_0 : \lambda(t) \in 1+\mathfrak{p}\ \forall \lambda \in X^*(T_{\mathrm{SO}_{2l}})\rangle.$$

We also have the set of affine roots Ψ, and we denote the subset of simple affine roots by Π and positive affine roots by Ψ^+. For $\psi \in \Psi$, U_ψ is the associated affine root group in SO_{2l}. With our identifications,

$$\Pi = \{\epsilon_1-\epsilon_2,\epsilon_2-\epsilon_3,\ldots,\epsilon_{r-1}-\epsilon_r,\epsilon_{r-1}+\epsilon_r,1-\epsilon_1-\epsilon_2\}.$$

Also define

$$I = \langle T_0, U_\psi : \psi \in \Psi^+\rangle, \qquad I^+ = \langle T_1, U_\psi : \psi \in \Psi^+\rangle.$$

Let ψ be a character of F of level 1. According to [GR10, RY14], the *affine generic characters* of I^+ take the form

$$\chi_{\underline{a}}(h) = \psi(a_1h_{1,2}+a_2h_{2,3}+\cdots+a_{l-1}h_{l-1,l}+a_lh_{l-1,l+1}+a_{l+1}\frac{h_{2l-1,1}}{\varpi}), \quad h \in I^+,$$

where $\underline{a} = (a_1,a_2,...,a_{l+1}) \in (\mathfrak{o}^\times)^{l+1}$, and because the level of ψ is 1, we can further assume $a_i \in \kappa^\times$ for each i. A complete set of representatives of T_0-orbits of affine generic characters of I^+ are given by the tuples $(1,1,\ldots,1,1,\alpha,t)$, where α varies over $\kappa^\times/(\kappa^\times)^2$ and $t \in \kappa^\times$. Instead of viewing the affine generic characters of I^+ as parameterized by $\alpha \in \kappa^*/(\kappa^*)^2$ and $t \in \kappa^*$, we will set $t=1$ and let the affine generic characters be parametrized by $\alpha \in \kappa^*/(\kappa^*)^2$ and the various choices of uniformizer ϖ in F.

Let x be the barycenter of the fundamental alcove and, for simplicity of notation, set $\chi = \chi_\alpha = \chi_{\underline{a}}$, noting that (up to the choice of a uniformizer ϖ), this character depends only on α. Simple supercuspidal representations are constructed using induction from compact subgroups. We describe this construction by explicating [RY14, §2] for SO_{2l}.

Put

$$g_\chi = \begin{pmatrix} &&&&& -\varpi^{-1} \\ & I_{l-2} & 0 & 0 & 0 & \\ & 0 & 0 & \alpha^{-1} & 0 & \\ & 0 & \alpha & 0 & 0 & \\ & 0 & 0 & 0 & I_{l-2} & \\ -\varpi &&&&& \end{pmatrix} \in \mathrm{SO}_{2l},$$

and note that g_χ stabilizes χ since

$$\chi(g_\chi h g_\chi^{-1}) = \psi(-\varpi^{-1}h_{2l,2}+h_{23}+\cdots+h_{l-1,l}+\alpha h_{l-1,l+1}-h_{2l-1,2l})$$

and the form defining SO_{2l} implies $h_{2l,2} = -h_{2l-1,1}$ and $h_{2l-1,2l} = -h_{12}$.

Let $H_{x,\chi} = Z\langle g_\chi\rangle I^+$, where $Z = Z_{\mathrm{SO}_{2l}}$ (the center of SO_{2l}) and $\langle g_\chi\rangle$ is the group with two elements $\{I_{2l}, g_\chi\}$ (for a general definition of $H_{x,\chi}$ see [RY14, §2]). Let ω be a character of Z, thereby extending χ from I^+ to ZI^+. We may extend χ to a character χ_α^ω of $H_{x,\chi}$ by setting $\chi_\alpha^\omega(g_\chi) = \pm 1$, since $g_\chi^2 = 1$. After such an extension, which we denote again by χ, we have that $\pi = \pi_\alpha^\omega = Ind_{H_{x,\chi}}^G \chi$ is a simple supercuspidal representation (see [RY14, §2]).

Define the character

$$\psi_\alpha(u) = \psi(\sum_{i=1}^{l-2} u_{i,i+1} + u_{l-1,l} + \alpha u_{l-1,l+1}), \qquad u \in U_{\mathrm{SO}_{2l}}.$$

The representation π is ψ_α-generic. For the purpose of constructing the integral, put

$$\iota = \mathrm{diag}(I_{l-1}, 1/4, 4, I_{l-1}), \qquad \alpha = -\tfrac{\gamma}{4}.$$

By the definition of affine generic characters, $|\gamma/4| = 1$. Then we have the isomorphic representation π^ι, defined on the space of π by

$$\pi^\iota(g) = \pi({}^\iota g) = \pi(\iota^{-1} g \iota).$$

For $W \in W(\pi, \psi_{-\gamma/4}^{-1})$, define $W^\iota(g) = W({}^\iota g)$. The map $W \mapsto W^\iota$ is an isomorphism $W(\pi, \psi_{-\gamma/4}^{-1}) \cong W(\pi^\iota, \psi^{-1})$, where for the model $W(\pi^\iota, \psi^{-1})$, ψ is defined by (2.1). Since $W(\pi^\iota, \psi^{-1})$ is also a Whittaker model for π, definition (2.5) implies

$$\gamma(s, \pi \times \tau, \psi) = \gamma(s, \pi^\iota \times \tau, \psi).$$

4. The computation of $\gamma(s, \pi \times \tau, \psi)$

Throughout this section and § 5, ψ is taken to be of level 1 and τ is tamely ramified (i.e., τ is trivial on $1+\mathfrak{p}$). In this section, we compute $\gamma(s, \pi\times\tau, \psi)$ using a specific choice of data. Recall $\pi = \mathrm{Ind}_{Z\langle g_\chi\rangle I^+}^{\mathrm{SO}_{2l}} \chi$. Let $I_{\mathrm{SO}_3}^+$ be the pro-unipotent part of the standard Iwahori subgroup of SO_3. For the Whittaker function, consider $W = (w^{l,1})^{-1} \cdot W_0$ where $W_0 \in \mathcal{W}(\pi, \psi_{-\gamma/4}^{-1})$ is given by

$$W_0(g) = \begin{cases} \psi_{-\gamma/4}^{-1}(u)\chi(g_\chi^i)\omega(z)\chi(y) & g = u g_\chi^i z y, \quad u \in U_{\mathrm{SO}_{2l}}, z \in Z, i = 0,1, y \in I^+, \\ 0 & \text{otherwise.} \end{cases}$$

Then $W^\iota \in W(\pi^\iota, \psi^{-1})$. Define f_s by

$$f_s(g,a) = \begin{cases} |m|^s \tau(am) & g = \mathrm{diag}(m, 1, m^{-1}) u y, \quad m \in F^*, u \in U_{\mathrm{SO}_3}, y \in I_{\mathrm{SO}_3}^+, \\ 0 & \text{otherwise.} \end{cases}$$

We can write a general element of the lower Borel subgroup $\overline{B}_{\mathrm{SO}_3}$ in the form

$$b = \begin{pmatrix} a & & \\ & 1 & \\ & & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & & \\ x & 1 & \\ -\frac{\gamma}{4}x^2 & -\frac{\gamma}{2}x & 1 \end{pmatrix}, \qquad a \in F^*, x \in F.$$

Define the right invariant Haar measure db on $\overline{B}_{\mathrm{SO}_3}$ by $db = |a|^{-1} d^*a dx$.

Note that since SO_3 is defined with respect to $J_{\gamma/2}$, if b is such that $a = 1$, then it belongs to $I_{\mathrm{SO}_3}^+$ if and only if $x \in \mathfrak{p}$.

Lemma 4.1. *Assume b as above.*

(1) If ${}^\iota(r w^{l,1} b (w^{l,1})^{-1}) \in UZI^+$, then $a \in 1+\mathfrak{p}$, $|x| < 1$, $r \in \mathfrak{p}^{l-2}$.

(2) If ${}^\iota(r w^{l,1} b (w^{l,1})^{-1}) \in U g_\chi Z I^+$, then for some $k \geq 0$, we have $|a| = q^{2k+1}$, $|x| = q^k$ and $\frac{\gamma}{4}x^2 a^{-1} \in \varpi \cdot (1+\mathfrak{p})$, and also $r \in \mathfrak{p}^{l-2}$.

Proof. The image of b in SO_{2l} is given by

$$\begin{pmatrix} I_{l-2} & & & & & \\ & a & & & & \\ & \frac{1}{4}x & 1 & & & \\ & \gamma x & & 1 & & \\ & -\frac{\gamma}{4}x^2a^{-1} & -\gamma x a^{-1} & -\frac{1}{4}xa^{-1} & a^{-1} & \\ & & & & & I_{l-2} \end{pmatrix}, \tag{4.1}$$

and

$$^{\iota}(rw^{l,1}b(w^{l,1})^{-1}) = \begin{pmatrix} a & & & & & \\ ar & I_{l-2} & & & & \\ x & & 1 & & & \\ \frac{\gamma}{4}x & & & 1 & & \\ & & & & I_{l-2} & \\ -\frac{\gamma}{4}x^2a^{-1} & & -\frac{\gamma}{4}xa^{-1} & -xa^{-1} & r' & a^{-1} \end{pmatrix}.$$

Both assertions follow from this; note first that $^{\iota}(rw^{l,1}b(w^{l,1})^{-1}) \notin U \cdot z \cdot I^+$, where z denotes the nontrivial central element in SO_{2l}. Also note that we are using $|\frac{\gamma}{4}| = 1$, and for the second part of the Lemma, consider the entries of the last row of an element in $Ug_{\chi}ZI^+$. $\square$

Corollary 4.2. *We have* $\Psi(W^{\iota}, f_s) = \mathrm{vol}^{\times}(1+\mathfrak{p})\mathrm{vol}(\mathfrak{p})^{l-1}$.

Proof. We may write the dh-integral of $\Psi(W^{\iota}, f_s)$ over $\overline{B}_{\mathrm{SO}_3}$, then

$$\Psi(W^{\iota}, f_s) = \int_{\overline{B}_{\mathrm{SO}_3}} (\int_{R^{l,1}} W_0(^{\iota}(rw^{l,1}b(w^{l,1})^{-1}))f_s(b,1)\,dr\,db.$$

For b as in Lemma 4.1 (2), $f_s(b,1) = 0$. Hence we are in part (1) of Lemma 4.1, and we conclude that the integrand vanishes unless $a \in 1+\mathfrak{p}$, $|x| < 1$ and $r \in \mathfrak{p}^{l-2}$. In this case the integrand is identically 1 (since τ is tamely ramified) and the result follows. $\square$

We turn to compute $\Psi^*(W^{\iota}, f_s)$. We start with computing $M(\tau, s)f_s$ on the support of W, in the next two lemmas.

Lemma 4.3. *Assume τ is quadratic, $a \in 1+\mathfrak{p}$ and $|x| < 1$. Then*

$$M(\tau, s)f_s(b,1) = |\gamma|^s\tau(\gamma)(q-1)|2|^{1-2s}\frac{q^{1/2-2s}}{(1-q^{1-2s})}.$$

Proof. Since τ is tamely ramified, we can already assume $a = 1$. Write

$$u = \begin{pmatrix} 1 & v & -\gamma^{-1}v^2 \\ & 1 & -2\gamma^{-1}v \\ & & 1 \end{pmatrix} \tag{4.2}$$

and we can already assume $v \neq 0$ for the computation of $M(\tau, s)$ (the measure of a singleton is zero). Then

$$w_1u = \begin{pmatrix} 1 & \gamma v^{-1} & -\gamma v^{-2} \\ & 1 & -2v^{-1} \\ & & 1 \end{pmatrix} \begin{pmatrix} -\gamma v^{-2} & & \\ 2v^{-1} & 1 & \\ 1 & v & -\gamma^{-1}v^2 \end{pmatrix},$$

and because f_s is left-invariant under U_{SO_3},

$$M(\tau,s)f_s(b,1) = \int_{F^\times} f_s(\begin{pmatrix} -\gamma v^{-2} & & \\ 2v^{-1} & 1 & \\ 1 & v & -\gamma^{-1}v^2 \end{pmatrix}\begin{pmatrix} 1 & & \\ x & 1 & \\ -\frac{\gamma}{4}x^2 & -\frac{\gamma}{2}x & 1 \end{pmatrix}, -1)\, dv$$
$$= |\gamma|^s\tau(\gamma)\int_{F^\times} \tau(v^{-2})|v|^{-2s} f_s(\begin{pmatrix} 1 & & \\ 2v^{-1}+x & 1 & \\ -\frac{\gamma}{4}(2v^{-1}+x)^2 & -\frac{\gamma}{2}(2v^{-1}+x) & 1 \end{pmatrix}, 1)\, dv.$$

Since f_s is supported in $B_{\mathrm{SO}_3}I^+_{\mathrm{SO}_3}$, considering the entries in the last row we deduce $2v^{-1}+x \in \mathfrak{p}$, and because $|x| < 1$, we find that $2v^{-1} \in \mathfrak{p}$. We obtain

$$|\gamma|^s\tau(\gamma)\int_{\{v\in F^\times : |2v^{-1}|<1\}} \tau(v^{-2})|v|^{-2s}\, dv.$$

Using $dv = (q-1)q^{-1/2}|v|d^\times v$ and changing $v \mapsto 2v$, we have

$$(q-1)q^{-1/2}\tau(2^{-2})|2|^{1-2s}\int_{\{v\in F^\times : |v|>1\}} \tau(v^{-2})|v|^{1-2s}\, d^\times v.$$

Writing $v = \varpi^{-l}o$ with $|o| = 1$,

$$\int_{\{v\in F^\times : |v|>1\}} \tau(v^{-2})|v|^{1-2s}\, d^\times v = \sum_{l=1}^{\infty} q^{l(1-2s)}\tau(\varpi^{2l})\int_{\mathfrak{o}^\times} \tau^2(o) d^\times o.$$

Now if τ is not quadratic, the $d^\times o$-integral vanishes and the result holds. Otherwise using $\mathrm{vol}^\times(\mathfrak{o}^\times) = 1$ and $\tau(\varpi^{2l}) = 1$, again we obtain the result. □

Lemma 4.4. *Assume τ is quadratic and $|x| = q^k$ with $k \geq 0$. Then*

$$M(\tau,s)f_s(b,1) = |\gamma|^s\tau(\gamma)|a|^{1-s}\tau^{-1}(a)|2|^{1-2s}q^{2k(s-1)}\mathrm{vol}(\mathfrak{p}).$$

Proof. As in the proof of Lemma 4.3 and with the same notation (but for any a),

$$M(\tau,s)f_s(b,1)$$
$$= |\gamma|^s\tau(\gamma)|a|^{1-s}\tau^{-1}(a)\int_{F^\times} \tau(v^{-2})|v|^{-2s} f_s(\begin{pmatrix} 1 & & \\ 2v^{-1}+x & 1 & \\ -\frac{\gamma}{4}(2v^{-1}+x)^2 & -\frac{\gamma}{2}(2v^{-1}+x) & 1 \end{pmatrix}, 1)\, dv.$$

Since τ is quadratic, $\tau(v^{-2}) = 1$. Changing $v \mapsto 2v$ we obtain

$$|\gamma|^s\tau(\gamma)|a|^{1-s}\tau^{-1}(a)|2|^{1-2s}\int_{F^\times} |v|^{-2s} f_s(\begin{pmatrix} 1 & & \\ v^{-1}+x & 1 & \\ -\frac{\gamma}{4}(v^{-1}+x)^2 & -\frac{\gamma}{2}(v^{-1}+x) & 1 \end{pmatrix}, 1)\, dv.$$

Again the integrand vanishes unless $v^{-1}+x \in \mathfrak{p}$, or $v^{-1} \in -x(1+x^{-1}\mathfrak{p})$. Since $|x| \geq 1$, the additive group $x^{-1}\mathfrak{p}$ is contained in $\mathfrak{p}$. Thus $|v| = q^{-k}$ and the dv-integral equals $q^{2k(s-1)}\mathrm{vol}(\mathfrak{p})$. The result follows. □

Lemma 4.5. *Assume ${}^\iota(rw^{l,1}b(w^{l,1})^{-1}) \in Ug_\chi ZI^+$. Then*

$$\int_{R^{l,1}} W_0^\iota(rw^{l,1}b(w^{l,1})^{-1})\, dr = \chi(g_\chi)\mathrm{vol}(\mathfrak{p})^{l-2}.$$

Proof. Write the image of b in SO_{2l} as in (4.1) and consider

$$u = \begin{pmatrix} I_{l-2} & & & & & \\ & 1 & -4ax^{-1} & -\frac{1}{\gamma}ax^{-1} & -\frac{4}{\gamma}a^2x^{-2} & \\ & & 1 & & \frac{1}{\gamma}ax^{-1} & \\ & & & 1 & 4ax^{-1} & \\ & & & & 1 & \\ & & & & & I_{l-2} \end{pmatrix}.$$

Then

$$ub = \begin{pmatrix} I_{l-2} & & & & & \\ & & & & -\frac{4}{\gamma}ax^{-2} & \\ & & & -\frac{1}{4\gamma} & \frac{1}{\gamma}x^{-1} & \\ & & -4\gamma & 0 & 4x^{-1} & \\ & -\frac{\gamma}{4}x^2a^{-1} & -\gamma xa^{-1} & -\frac{1}{4}xa^{-1} & a^{-1} & \\ & & & & & I_{l-2} \end{pmatrix}$$

and $g_\chi w^{l,1}ub(w^{l,1})^{-1} = \iota^{-1}y\iota^{-1}$ where

$$y = \begin{pmatrix} \frac{\gamma}{4}x^2a^{-1}\varpi^{-1} & 0 & \frac{\gamma}{4}xa^{-1}\varpi^{-1} & xa^{-1}\varpi^{-1} & 0 & -a^{-1}\varpi^{-1} \\ 0 & I_{l-2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -\frac{4}{\gamma}x^{-1} \\ 0 & 0 & 0 & 1 & 0 & -x^{-1} \\ 0 & 0 & 0 & 0 & I_{l-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{\gamma}ax^{-2}\varpi \end{pmatrix}.$$

By Lemma 4.1 (2), $|a| = q^{2k+1}$, $|x| = q^k$ for some $k \geq 0$ and $\frac{\gamma}{4}x^2a^{-1} \in \varpi \cdot (1+\mathfrak{p})$, and because $|\frac{4}{\gamma}| = 1$, we deduce $y \in I^+$. Also note that $g_\chi = g_\chi^{-1}$ and $g_\chi\iota^{-1} = \iota g_\chi$. Then

$$b(w^{l,1})^{-1} = u^{-1}(w^{l,1})^{-1}g_\chi\iota^{-1}y\iota^{-1} = u^{-1}(w^{l,1})^{-1}\iota g_\chi y\iota^{-1},$$
$$W_0^\iota(rw^{l,1}b(w^{l,1})^{-1}) = W_0^\iota(rw^{l,1}u^{-1}(w^{l,1})^{-1}\iota g_\chi y\iota^{-1}).$$

Next we see that

$$r(w^{l,1}u^{-1}(w^{l,1})^{-1})r^{-1} = v_{r,u} = \begin{pmatrix} 1 & & 4ax^{-1} & \frac{1}{\gamma}ax^{-1} & * & * \\ & I_{l-2} & 4ax^{-1}r & \frac{1}{\gamma}ax^{-1}r & * & * \\ & & 1 & & * & -\frac{1}{\gamma}ax^{-1} \\ & & & 1 & * & -4ax^{-1} \\ & & & & I_{l-2} & \\ & & & & & 1 \end{pmatrix} \in U_{\mathrm{SO}_{2l}},$$

where the remaining coordinates are determined by a, x, r and the form defining SO_{2l}. Since

$$\psi(v_{r,u}) = \psi(\tfrac{1}{4}(4ax^{-1}r) - \gamma(\tfrac{1}{\gamma}ax^{-1}r)) = 1,$$

W_0^ι is right-invariant under $\iota I^+\iota^{-1}$ and ι commutes with r,

$$W_0^\iota(rw^{l,1}b(w^{l,1})^{-1}) = W_0^\iota(v_{r,u}r\iota g_\chi\iota^{-1}) = W_0^\iota(r\iota g_\chi\iota^{-1}) = W_0(rg_\chi).$$

Now, we have $W_0(rg_\chi) = W_0(g_\chi({}^{g_\chi}r))$, ${}^{g_\chi}r \in I^+$ and $\chi({}^{g_\chi}r) = 1$, hence $W_0(rg_\chi) = \chi(g_\chi)$. $\square$

Corollary 4.6. *$\Psi(W^\iota, M(\tau,s)f_s)$ equals*

$$\Psi(W^\iota, f_s)(q-1)|2|^{1-2s}|\gamma|^s\left(\tau(\gamma)\frac{q^{1/2-2s}}{(1-q^{1-2s})}+\chi(g_\chi)\tau(\varpi)q^{-1/2-s}/(1-q^{-1})\right).$$

Proof. As in the proof of Corollary 4.2, we write the dh-integral over $\overline{B}_{\mathrm{SO}_3}$ and using db. According to the support of W^ι and by Lemma 4.1, $\Psi(W^\iota, M(\tau,s)f_s)$ is the sum of two integrals, each corresponding to one of the cases of the lemma. The first summand is

$$\begin{aligned}&\int_{a\in 1+\mathfrak{p}}\int_{x\in\mathfrak{p}}\int_{r\in\mathfrak{p}^{l-2}} W_0^\iota(rw^{l,1}b(w^{l,1})^{-1})[M(\tau,s)f_s](b,1)\,dr\,da\,dx\\ &=\mathrm{vol}^\times(1+\mathfrak{p})\mathrm{vol}(\mathfrak{p})^{l-1}|\gamma|^s\tau(\gamma)(q-1)|2|^{1-2s}\frac{q^{1/2-2s}}{(1-q^{1-2s})},\end{aligned}$$

by Lemma 4.3.

The second summand is an integral over (a,x) such that

$$|a|=q^{2k+1},\qquad |x|=q^k,\qquad \tfrac{\gamma}{4}x^2a^{-1}\in\varpi\cdot(1+\mathfrak{p}),\qquad k\geq 0.$$

For each such elements, by Lemma 4.4 and Lemma 4.5 the integrand equals

$$|\gamma|^s\tau(\varpi)|2|^{1-2s}q^{1-s}\chi(g_\chi)\mathrm{vol}(\mathfrak{p})^{l-1},$$

where we also used $\tau^{-1}(a)=\tau(\gamma)\tau(\varpi)$ (τ is quadratic and tamely ramified). It remains to compute the measure db in the integral, and because $db=|a|^{-1}da\,dx$, $|xa^{-1}|=q^{-k-1}$ and $\mathrm{vol}^\times(\frac{\gamma}{4}x^2\varpi^{-1}(1+\mathfrak{p}))=\mathrm{vol}^\times(1+\mathfrak{p})$, we have

$$q^{-1}\mathrm{vol}(\mathfrak{o}^\times)\mathrm{vol}^\times(1+\mathfrak{p}))\sum_{k=0}^{\infty}q^{-k}=(q-1)q^{-3/2}\mathrm{vol}^\times(1+\mathfrak{p})\frac{1}{1-q^{-1}}.$$

Thus

$$\begin{aligned}\Psi(W^\iota, M(\tau,s)f_s)=&\mathrm{vol}^\times(1+\mathfrak{p})\mathrm{vol}(\mathfrak{p})^{l-1}(q-1)|2|^{1-2s}|\gamma|^s\\ &\left(\tau(\gamma)\frac{q^{1/2-2s}}{(1-q^{1-2s})}+\chi(g_\chi)\tau(\varpi)q^{-1/2-s}/(1-q^{-1})\right),\end{aligned}$$

since τ is quadratic. The formula follows when we plug Corollary 4.2 into this identity. □

Corollary 4.7. *For any quadratic tamely ramified character τ of F^*,*

$$\begin{aligned}\gamma(s,\pi\times\tau,\psi)=&\pi(-I_{2l})\tau(-1)^l\tau(\gamma)(q-1)\gamma^{\mathrm{Tate}}(2s-1,\tau^2,\psi)\\ &\times\left(\tau(\gamma)\frac{q^{1/2-2s}}{(1-q^{1-2s})}+\chi(g_\chi)\tau(\varpi)q^{-1/2-s}/(1-q^{-1})\right).\end{aligned}$$

Proof. Use (2.5), Corollary 4.2, Corollary 4.6, (2.4), and note that $\tau^2=1$ and $|\gamma|=|4|$. □

Theorem 1.1 now follows from Corollary 4.7. Specifically, since τ is quadratic, $\gamma^{\mathrm{Tate}}(2s-1,\tau^2,\psi)$ has only one pole on the real line: a simple pole at $s=1$. Looking at Corollary 4.7 we see that this pole is cancelled by the factor in parentheses precisely when $\tau(\varpi)=-\chi(g_\chi)\tau(\gamma)$ ($g_\chi^2=I_{2l}$ whence $\chi(g_\chi)^{-1}=\chi(g_\chi)$). This completes the proof.

5. The Langlands parameter

In this section we discuss the Langlands parameter for π. Recall that $\pi = \pi_\alpha^\omega$ is a simple supercuspidal representation of SO_{2l}, corresponding to the character $\chi = \chi_\alpha^\omega$. Let $\varphi = \varphi_\pi$ be the Langlands parameter of π.

First assume $p \neq 2$. The parameter φ is $2l$-dimensional. Our results have both shown the existence of two 1-dimensional summands φ_1 and φ_2 of φ, and have determined them explicitly. It follows that

$$\varphi = \varphi_1 \oplus \varphi_2 \oplus \varphi_3, \tag{5.1}$$

where φ_3 is the $2l-2$ dimensional summand. Now it is expected (and an anlogue of [BHS] for SO_{2l} would show) that φ_3 is irreducible and corresponds, via the local Langlands correspondence, to a simple supercuspidal representation Π' of GL_{2l-2}. We turn to discuss φ_3 (and Π') under these assumptions.

By Theorem 1.1, if τ is a quadratic tamely ramified character such that $\tau(\varpi) \neq -\chi(g_\chi)\tau(\gamma)$, $\gamma(s, \pi \times \tau_i, \psi)$ has a pole at $s = 1$. Note that $\gamma = 4 \cdot u$ for some $u \in \mathfrak{o}^\times$, hence for any quadratic character μ, $\mu(\gamma) = \mu(u)$. We let τ_1 be the unramified character such that $\tau_1(\varpi) = \chi(g_\chi)$, and τ_2 be the character which restricts to the unique nontrivial quadratic character of $\mathfrak{o}^\times$ and satisfies $\tau_2(\varpi) = \chi(g_\chi)\tau_2(\gamma)$. Clearly both characters satisfy the conditions of Theorem 1.1, therefore $\gamma(s, \pi \times \tau_i, \psi)$ has a pole at $s = 1$. Without loss of generality, $\varphi_1 = \tau_1$ and $\varphi_2 = \tau_2$.

Since $\det \varphi = 1$, the central character of Π' equals $\tau_1\tau_2$. Let δ be the coefficient of $q^{1/2-s}$ in $\gamma(s, \Pi', \psi)$.

Proposition 5.1. $\delta = \pi(-I_{2l})\chi(g_\chi)\epsilon(s, \tau_2, \psi)^{-1}$.

Proof. By the local Langlands correspondence for general linear groups, δ is precisely the coefficient of $q^{1/2-s}$ in $\gamma(s, \varphi_1, \psi)$ (see [AK, § 2.6]). By (5.1), the local Langlands correspondence also implies

$$\gamma(s, \pi, \psi) = \gamma(s, \Pi', \psi)\gamma(s, \tau_1, \psi)\gamma(s, \tau_2, \psi).$$

Then by Corollary 4.7 (with $\tau \equiv 1$),

$$\begin{aligned}\gamma(s, \Pi', \psi) =& \pi(-I_{2l})(q-1)q^{2s-3/2}\frac{1-q^{1-2s}}{1-q^{2s-2}}\left(\frac{q^{1/2-2s}}{(1-q^{1-2s})} + \chi(g_\chi)q^{-1/2-s}/(1-q^{-1})\right) \\ &\times \gamma(s, \tau_1, \psi)^{-1}\gamma(s, \tau_2, \psi)^{-1}.\end{aligned} \tag{5.2}$$

Here we used $\gamma^{\mathrm{Tate}}(2s-1, 1, \psi) = q^{2s-3/2}\frac{1-q^{1-2s}}{1-q^{2s-2}}$.

Now by virtue of [BH06, § 23.4, § 23.5] and our choice of characters τ_1 and τ_2,

$$\gamma(s, \tau_1, \psi) = q^{s-1/2}\chi(g_\chi)\frac{1-\chi(g_\chi)q^{-s}}{1-\chi(g_\chi)q^{s-1}}, \qquad \gamma(s, \tau_2, \psi) = \epsilon(s, \tau_2, \psi).$$

Plugging this into (5.2) we obtain

$$\gamma(s, \Pi', \psi) = \pi(-I_{2l})\chi(g_\chi)\epsilon(s, \tau_2, \psi)^{-1}q^{1/2-s},$$

as claimed. ∎

Remark 5.2. *Because τ_2 is tamely ramified and not unramified, the power of q in $\epsilon(s, \tau_2, \psi)$ is zero.*

The final ingredient we need is the restriction of φ_3 to the wild inertia subgroup. As mentioned in the introduction, our method does not provide this information. We expect it to follow from an analogue of [BHS], but note that only the simple supercuspidal case of [BHS] is required here. Such an analogue would both prove that φ does indeed contain a $2l-2$ dimensional summand which corresponds to a simple supercuspidal representation, and also determine the restriction of φ_3 to the wild inertia subgroup. Then by [AL16], the central character of Π', the parameter δ obtained by Proposition 5.1, and the restriction of φ_3 to the wild inertia subgroup, completely determine φ_3. Since we already described φ_1 and φ_2, this completely determines the Langlands parameter φ of π.

Now consider the case $p = 2$. In this case by Theorem 1.1, there is a unique quadratic tamely ramified character τ of F^* such that $\gamma(s, \pi \times \tau, \psi)$ a pole at $s = 1$. This determines a 1-dimensional summand $\varphi_1 = \tau$ of φ, so that $\varphi = \varphi_1 \oplus \varphi_2$ for a $(2l-1)$-dimensional summand φ_2. As above, we expect that φ_2 is irreducible and corresponds to a simple supercuspidal representation Π' of GL_{2l-1} (again, this will follow from an analogue of [BHS] for SO_{2l}).

The central character of Π' is automatically trivial since $-1 \in I^+$ $(p = 2)$. For the same reason $\pi(-I_{2l}) = 1$. Then the computation in the proof of Proposition 5.1 (but without $\gamma(s, \tau_2, \psi)$) implies $\delta = \chi(g_\chi)$, where δ is the coefficient of $q^{1/2-s}$ in $\gamma(s, \Pi', \psi)$. When $F = \mathbb{Q}_2$ this already completely determines the Langlands parameter φ, because in the parameterization of simple supercuspidal representations of general linear groups, the uniformizer may be chosen modulo $1 + \mathfrak{p}$. For more general 2-adic fields, we still need to find the restriction of φ_2 to the wild inertia subgroup, which as mentioned above is expected to be obtained from an analogue of [BHS]. Note that while the results of *loc. cit.* were obtained under the assumption $p \neq 2$, at least for the class of simple supercuspidal representations an extension of their results to $p = 2$ seems possible.

6. The normalization parameters of $\gamma(s, \pi \times \tau, \psi)$

In this section we prove Proposition 2.1, i.e., compute $C(s, \tau, \psi)$, and determine the normalization factor of (2.5). We start with some preliminaries.

Let $\mathcal{S}(F^r)$ be the space of Schwartz–Bruhat functions on the row space F^r and let $(e_1, \ldots, e_r)$ be the standard basis of F^r. Define the Fourier transform $\widehat{\Phi} \in \mathcal{S}(F^r)$ by

$$\widehat{\Phi}(y) = \int_{F^r} \Phi(z)\psi(z({}^t y))dz$$

(here y and z are rows).

We recall the definition of Tate's γ-factor $\gamma^{\mathrm{Tate}}(s, \eta, \psi)$, for a quasi-character η of F^* [Tat67]. For $\Phi \in \mathcal{S}(F)$, consider the zeta integral

$$Z(\Phi, s, \eta) = \int_{F^*} \Phi(x)\eta(x)|x|^s \, d^*x,$$

which is absolutely convergent for $\mathrm{Re}(s) \gg 0$ and admits meromorphic continuation to a function in q^{-s}. The γ-factor is then defined via the functional equation

$$\gamma^{\mathrm{Tate}}(s, \eta, \psi) Z(\Phi, s, \eta) = Z(\widehat{\Phi}, 1 - s, \eta^{-1}). \tag{6.1}$$

We calculate $C(s, \tau, \psi)$ by choosing a special section in $V(\tau, s)$, for which we can succinctly compute its image under $M(\tau, s)$, then compare both sides of (2.3). We argue by adapting parts of the arguments from [Kap15, § 6.1]. To construct the section we use an isomorphism $\imath : Z_{\mathrm{GL}_2} \backslash \mathrm{GL}_2 \to \mathrm{SO}_3$, where Z_{GL_2} is the center of GL_2.

To define $\imath$, it is useful to consider the complex Lie algebras, in order to identify the images of unipotent elements. Let

$$A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \quad B = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \quad C = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \quad D = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

be a basis for the Lie algebra $\mathfrak{gl}_2$ of GL_2 over $\mathbb{C}$. The center $\mathfrak{z}_{\mathfrak{gl}_2}$ is spanned by D. Then

$$[A,B] = C, \qquad [A,C] = -2A, \qquad [B,C] = 2B.$$

Also let

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -\gamma/2 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & \gamma/2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

be a basis for the Lie algebra $\mathfrak{so}_3$ of SO_3 over $\mathbb{C}$,

$$[X,Y] = -\tfrac{\gamma}{2}Z, \qquad [X,Z] = X, \qquad [Y,Z] = -Y.$$

Hence the following defines an isomorphism of Lie algebras $d\imath_0 : \mathfrak{z}_{\mathfrak{gl}_2}\backslash\mathfrak{gl}_2 \to \mathfrak{so}_3$:

$$d\imath_0(A) = X, \quad d\imath_0(B) = \tfrac{4}{\gamma}Y, \quad d\imath_0(C) = -2Z, \quad d\imath_0(D) = 0.$$

In particular

$$\imath_0 \left(\begin{smallmatrix} 1 & u \\ & 1 \end{smallmatrix}\right) = \begin{pmatrix} 1 & & \\ u & 1 & \\ -\frac{\gamma}{4}u^2 & -\frac{\gamma}{2}u & 1 \end{pmatrix}, \qquad \imath_0 \left(\begin{smallmatrix} 1 & \\ u & 1 \end{smallmatrix}\right) = \begin{pmatrix} 1 & 2u & -\frac{4}{\gamma}u^2 \\ & 1 & -\frac{4}{\gamma}u \\ & & 1 \end{pmatrix}.$$

It follows that $\imath$ is determined by

$$\imath \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right) = \begin{pmatrix} a^{-1}b & & \\ & 1 & \\ & & ab^{-1} \end{pmatrix}, \qquad \imath \left(\begin{smallmatrix} 1 & \\ u & 1 \end{smallmatrix}\right) = \begin{pmatrix} 1 & 2u & -\frac{4}{\gamma}u^2 \\ & 1 & -\frac{4}{\gamma}u \\ & & 1 \end{pmatrix}, \qquad \imath \begin{pmatrix} & \frac{4}{\gamma} \\ 1 & \end{pmatrix} = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} = w_1.$$

If $h \in \mathrm{SO}_3$, let $\imath^{-1}(h)$ be an arbitrary pre-image of h in GL_2, under $\imath$.

Proof of Proposition 2.1. For $\Phi \in \mathcal{S}(F^2)$, define $f_{\Phi,\tau,s} \in V(\tau, s)$ by

$$f_{\Phi,\tau,s}(h,a) = \int_{Z_{\mathrm{GL}_2}} \Phi(e_1 z\imath^{-1}(g))\tau(\det z\imath^{-1}(g))|\det z\imath^{-1}(g)|^s\, dz.$$

Recall that $M(\tau,s)f_{\Phi,\tau,s}(h,a) = \int_{U_{\mathrm{SO}_3}} f_{\Phi,\tau,s}(w_1 uh, -a^{-1})du$. Since

$$w_1 u = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} \begin{pmatrix} 1 & u & -\frac{1}{\gamma}u^2 \\ & 1 & -\frac{2}{\gamma}u \\ & & 1 \end{pmatrix} = \imath(\begin{pmatrix} & \frac{4}{\gamma} \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & \\ \frac{1}{2}u & 1 \end{pmatrix}) = \imath \begin{pmatrix} \frac{2}{\gamma}u & \frac{4}{\gamma} \\ 1 & 0 \end{pmatrix},$$

$$M(\tau,s)f_{\Phi,\tau,s}(I_3,1) = \tau(\tfrac{4}{\gamma})|\tfrac{4}{\gamma}|^s \int_F \int_{F^*} \Phi(\tfrac{2}{\gamma}zu, \tfrac{4}{\gamma}z)\tau^2(z)|z|^{2s}d^*z du.$$

Changing variables $z \mapsto \frac{\gamma}{4}z$ and $u \mapsto 2z^{-1}u$, we have

$$M(\tau,s)f_{\Phi,\tau,s}(I_3,1) = \tau(\tfrac{\gamma}{4})|\tfrac{\gamma}{4}|^s|2|Z(\Phi_1, 2s-1, \tau^2), \qquad \Phi_1(z) = \int_F \Phi(u,z)du.$$

This formal step is justified for $\mathrm{Re}(s) \gg 0$ by Fubini's Theorem. According to (6.1), when we multiply $Z(\Phi_1, 2s-1, \tau^2)$ by $\gamma^{\mathrm{Tate}}(2s-1, \tau^2, \psi)$ we get $Z(\widehat{\Phi_1}, 2-2s, \tau^{-2})$ (as meromorphic continuations) and because

$$\widehat{\Phi_1}(z) = \int_F \int_F \Phi(u,y)\psi(yz)\, du\, dy = \widehat{\Phi}(0,z),$$

we have

$$\gamma^{\mathrm{Tate}}(2s-1,\tau^2,\psi)M(\tau,s)f_{\Phi,\tau,s}(h,1) \tag{6.2}$$
$$= \tau(\tfrac{\gamma}{4})|\tfrac{\gamma}{4}|^s|2|\int_{Z_{\mathrm{GL}_2}}(\imath^{-1}(h)\cdot\Phi)\widehat{\ }(e_2z)\tau^{-1}(\det z)\tau(\det\imath^{-1}(h))|\det z|^{1-s}|\det\imath^{-1}(h)|^s\,dz.$$

Now we compute $C(s,\tau,\psi)$ by substituting $f_{\Phi,\tau,s}$ for f_s in (2.3), which becomes

$$\int_{U_{\mathrm{SO}_3}} f_{\Phi,\tau,s}(w_1u,1)\psi^{-1}(u_{1,2})\,du = C(s,\tau,\psi)\int_{U_{\mathrm{SO}_3}} M(\tau,s)f_{\Phi,\tau,s}(w_1u,1)\psi^{-1}(u_{1,2})\,du. \tag{6.3}$$

Changing variables as above, but now also paying attention to ψ^{-1}, the left hand side is

$$\tau(-1)\tau(\tfrac{4}{\gamma})|\tfrac{4}{\gamma}|^s|2|\int_{F^*}\left(\int_F\Phi(u,z)\psi^{-1}(2z^{-1}u)\,du\right)\tau^2(z)|z|^{2s-1}\,d^*z. \tag{6.4}$$

For the right hand side, we use (6.2) with $h=w_1u$ and obtain

$$\gamma^{\mathrm{Tate}}(2s-1,\tau^2,\psi)^{-1}\tau(-1)|2|\int_F\int_{Z_{\mathrm{GL}_2}}(\imath^{-1}(w_1u)\cdot\Phi)\widehat{\ }(e_2z)\tau^{-1}(\det z)|\det z|^{1-s}\,dz\,\psi^{-1}(u)\,du.$$

Using $(g\cdot\Phi)\widehat{\ }(x,y)=|\det g|^{-1}\widehat{\Phi}((x,y)({}^tg^{-1}))$ and changing $u\mapsto 2z^{-1}u$, this equals

$$\gamma^{\mathrm{Tate}}(2s-1,\tau^2,\psi)^{-1}\tau(-1)|\gamma|\int_{F^*}\left(\int_F\widehat{\Phi}(z,u)\psi^{-1}(-2z^{-1}u)\,du\right)\tau^{-2}(z)|z|^{1-2s}\,d^*z. \tag{6.5}$$

Observe that for a fixed z, by partial Fourier inversion,

$$\int_F\widehat{\Phi}(z,u)\psi(2z^{-1}u)\,du = \int_F\int_F\Phi(x,y)\psi(xz)\left(\int_F\psi((y+2z^{-1})u)\,du\right)dx\,dy$$
$$=\int_F\Phi(x,-2z^{-1})\psi(xz)\,dx.$$

Hence (6.5) becomes

(6.6)

$$\gamma^{\mathrm{Tate}}(2s-1,\tau^2,\psi)^{-1}\tau^{-2}(2)|2|^{1-2s}\tau(-1)|\gamma|\int_{F^*}\left(\int_F\Phi(u,z)\psi^{-1}(2z^{-1}u)\,du\right)\tau^2(z)|z|^{2s-1}d^*z.$$

Dividing (6.4) by (6.6), we conclude

$$C(s,\tau,\psi)=\tau^4(2)|2|^{4s}\tau^{-1}(\gamma)|\gamma|^{-s-1}\gamma^{\mathrm{Tate}}(2s-1,\tau^2,\psi).$$

This completes the proof of the proposition. □

To find the normalization factor appearing in (2.5) we must follow the computations from [Kap13, Kap15]. This factor is extracted from the multiplicativity properties [Kap15, (6.1), (6.2)] and from the minimal case of $\mathrm{SO}_2\times\mathrm{GL}_1$, but since here we only consider split SO_{2l}, the multiplicativity properties are sufficient.

Let $Q_r=M_r\ltimes U_r$ be the standard maximal parabolic subgroup of SO_{2l} whose Levi part $M_r=\mathrm{GL}_r\times\mathrm{SO}_{2(l-r)}$ if $r<l$, and $\{\mathrm{diag}(b,b^*):b\in\mathrm{GL}_l\}$ for $r=l$. Let $P_{(n_1,n_2)}$ be a parabolic subgroup of GL_n, $n=n_1+n_2$, containing the subgroup of upper triangular invertible matrices, whose Levi part is isomorphic to $\mathrm{GL}_{n_1}\times\mathrm{GL}_{n_2}$. We could in theory work with $n=1$, but since the multiplicativity properties for the case $\pi=\mathrm{Ind}_{Q_r}^{\mathrm{SO}_{2l}}(\sigma\otimes\pi')$ with $r<l$ and $l>n$ were obtained using the case $r<l<n$ and the multiplicativity for

$\tau = \mathrm{Ind}_{P_{(n_1,n_2)}}^{\mathrm{GL}_n}(\tau_1 \otimes \tau_2)$, we actually need to consider the general $\mathrm{SO}_{2l} \times \mathrm{GL}_n$ construction. In this case for $l \leq n$, SO_{2l} is embedded in SO_{2n+1} which is defined with respect to J_{2n+1}, exactly as in [Kap15]; but for $l > n$, SO_{2n+1} is now defined using $J_{2n+1,\gamma}$.

The functional equation for all l and n takes following form. Define the factor $c(s, l, \tau, \gamma) = \tau^{-2}(\gamma)|\gamma|^{n(-2s+1)}$ if $l > n$, otherwise $c(s, l, \tau, \gamma) = 1$ (as in [Kap13, Kap15]). Then we claim

$$\gamma(s, \pi \times \tau, \psi)\Psi(W, f_s) = \pi(-I_{2l})^n \tau(-1)^l \left(\tau^2(2)|2|^{n(2s-1)} c(s, l, \tau, \gamma)\right) \Psi^*(W, f_s). \tag{6.7}$$

Here $\Psi(W, f_s)$ and $\Psi^*(W, f_s)$ are the $\mathrm{SO}_{2l} \times \mathrm{GL}_n$ integrals, described in § 2 for $n = 1$ and in [Kap15] for all n. Specializing (6.7) to $n = 1$, we obtain (2.5).

Remark 6.1. *The factor $\tau^2(2)|2|^{n(2s-1)}$ in (6.7) is different from the corresponding one in* [Kap15, p. 408] *($|2\gamma|^{n(s-1/2)}\tau(2\gamma)$) because the embedding is different, see § 2.*

Inspecting [Kap13, Kap15], the only multiplicativity property for $\gamma(s, \pi \times \tau, \psi)$ which is affected by the difference in the definition of SO_{2n+1} and choice of embedding (the vector e, see § 2) here is the one for $r = l > n$, which was proved in [Kap13, § 5.4]. This property is replaced by the following result, which implies (6.7) (see [Kap15, § 6]).

Proposition 6.2. *Assume $\pi = \mathrm{Ind}_{\overline{Q}_l}^{\mathrm{SO}_{2l}}(\sigma)$, where $\overline{Q}_l = M_l \ltimes \overline{U}_l$, and τ is an irreducible generic representation of GL_n. Then*

$$\frac{\Psi^*(W, f_s)}{\Psi(W, f_s)} = \sigma(-I_{2l})^n \tau(-1)^l \tau^{-2}(2)|2|^{-2s+1} c(s, l, \tau, \gamma)^{-1} \gamma(s, \sigma \times \tau, \psi)\gamma(s, \sigma^* \times \tau, \psi). \tag{6.8}$$

Here σ^* is the representation on the space of σ acting by $\sigma^*(b) = \sigma(b^*)$, and the γ-factors are the Rankin–Selerg $\mathrm{GL}_l \times \mathrm{GL}_n$ γ-factors of [JPSS83].

Proof. Closely inspecting the proof in [Kap13, § 5.4] (of [Kap13, (5.5)], and see also the top of p. 419 of [Kap15], there $\beta^2 = 2\gamma$), we see that the only change is to [Kap13, Claim 5.6] (this claim appeared as Claim 7.13 in [Kap] where it was proved in detail, but we reproduce the argument below), and we can observe the difference already when $n = 1$. Thus we argue for $n = 1$ (the extension to $n > 1$ is straightforward). We introduce the necessary notation from [Kap13, § 5.4]. Consider the subgroup V_l'' of U_l defined by

$$V_l'' = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & v_4 & 0 \\ & I_{l-2} & 0 & v_3 & v_5 & v_4' \\ & & 1 & 0 & v_3' & 0 \\ & & & 1 & 0 & 0 \\ & & & & I_{l-2} & 0 \\ & & & & & 1 \end{pmatrix} \in U_l \right\}.$$

Put $w' = \left(\begin{smallmatrix} & I_{l-1} \\ 1 & \end{smallmatrix}\right)\left(\begin{smallmatrix} I_{l-1} & \\ & 4 \end{smallmatrix}\right)$. Let φ_ζ belong to the space of $\mathrm{Ind}_{\overline{Q}_l}^{\mathrm{SO}_{2l}}(|\det|^{-\zeta}\sigma)$, where ζ is an auxiliary complex parameter ($\mathrm{Re}(\zeta) \gg 0$) and σ is realized in its Whittaker model with respect to the subgroup of upper triangular unipotent matrices in GL_l and character $z \mapsto \psi^{-1}(\sum_{i=1}^{l-1} z_{i,i+1})$. Consider the function

$$F(h) = \int_{V_l''} \varphi_\zeta(v'' w^{l,1} h, w')\psi_\gamma(v'')\, dv'', \qquad h \in \mathrm{SO}_3.$$

We show $F(uh) = \psi^{-1}(u_{1,2})F(h)$, as opposed to [Kap13, Claim 5.6] where the claim was $F(uh) = \psi^{-1}(\frac{2}{\beta}u_{1,2})F(h)$ under the assumption $2\gamma = \beta^2$ (see Remark 6.3 below). For

$$u = \begin{pmatrix} 1 & & \\ x & 1 & \\ -\frac{\gamma}{4}x^2 & -\frac{\gamma}{2}x & 1 \end{pmatrix}, \quad {}^{(w^{l,1})^{-1}}u = w^{l,1}u(w^{l,1})^{-1} = \begin{pmatrix} 1 & & & & & \\ & I_{l-2} & & & & \\ \frac{1}{4}x & & 1 & & & \\ \gamma x & & & 1 & & \\ & & & & I_{l-2} & \\ -\frac{\gamma}{4}x^2 & & -\gamma x & -\frac{1}{4}x & & 1 \end{pmatrix}.$$

Then for $v'' \in V_l''$, $({}^{(w^{l,n})^{-1}}u)^{-1}v''({}^{(w^{l,n})^{-1}}u) = b_u v_u$ where b_u is the image in M_l of

$$\begin{pmatrix} 1 & & \\ \gamma x v_3 - \frac{\gamma}{4}x^2 v_4' & I_{l-2} & -\gamma x v_4' \\ & & 1 \end{pmatrix} \in \mathrm{GL}_l$$

and

$$v_u = \begin{pmatrix} I_n & 0 & 0 & 0 & v_4 & 0 \\ & I_{l-n-1} & 0 & v_3 - \frac{1}{4}xv_4' & v_5 + \ldots & v_4' \\ & & 1 & 0 & v_3' - \frac{1}{4}xv_4 & 0 \\ & & & 1 & 0 & 0 \\ & & & & I_{l-n-1} & 0 \\ & & & & & I_n \end{pmatrix} \in V_l''.$$

It follows that

$$F(uh) = \int_{V_l''} \varphi_\zeta(({}^{(w^{l,n})^{-1}}u)b_u v_u w^{l,n}h, w')\psi_\gamma(v'')\,dv''.$$

Now on the one hand, changing variables in v_u removes the dependence on u and changes $\psi_\gamma(v'') = \psi(-\gamma(v_3)_{l-2})$ to $\psi_\gamma(v'')\psi(-\frac{\gamma}{4}x(v_4')_{l-2})$. On the other hand, for any $h \in \mathrm{SO}_3$,

$$\varphi_\zeta(({}^{(w^{l,n})^{-1}}u)b_u h, w') = \psi^{-1}(-\tfrac{\gamma}{4}x(v_4')_{l-2} + x)\varphi_\zeta(h, w').$$

We conclude $F(uh) = \psi^{-1}(x)F(h) = \psi^{-1}(u_{1,2})F(h)$ (cf. [Kap13, Claim 5.6]). Plugging this result into [Kap13, § 5.4, p. 340], rewriting the du-integration over U_{SO_3} and changing variables $x \mapsto \frac{2}{\gamma}x$ we obtain the analogue of *loc. cit.* (5.22):

$$|\tfrac{2}{\gamma}| \int_{\overline{U}_{\mathrm{SO}_3}\backslash \mathrm{SO}_3} F(h) \left(\int_{U_{\mathrm{SO}_3}} f_s(w_1 u w_1^{-1}h, 1)\psi^{-1}(\tfrac{2}{\gamma}u_{1,2})\,du \right) dh. \tag{6.9}$$

Now applying (2.3) we obtain the analogue of *loc. cit.* (5.23): as meromorphic continuations (6.9) equals

$$c_{\tau,\beta}|\tfrac{2}{\gamma}| \int_{\overline{U}_{\mathrm{SO}_3}\backslash \mathrm{SO}_3} F(h) \left(\int_{U_{\mathrm{SO}_3}} M^*(\tau, s)f_s(w_1 u w_1^{-1}h, 1)\psi^{-1}(\tfrac{2}{\gamma}u_{1,2})\,du \right) dh. \tag{6.10}$$

Here $c_{\tau,\beta} = |\frac{2}{\gamma}|^{2s-1}\tau^2(\frac{2}{\gamma}) = \tau^2(2)|2|^{2s-1}c(s,l,1,\gamma)$ is calculated by substituting $y \cdot f_s$ for f_s in (2.3) where $y = \mathrm{diag}(\frac{2}{\gamma}, 1, \frac{\gamma}{2})w_1^{-1}h$. Note that the extra factor $|\frac{2}{\gamma}|$ in (6.10) will be canceled when we proceed as in [Kap13, § 5.4] and rewrite the du-integration over $\overline{U}_{\mathrm{SO}_3}$ again. Identity (6.8) now follows as in *loc. cit.* □

Remark 6.3. *Even if $2\gamma = \beta^2$ here as well, for some β, SO_3 is still defined differently, so one can not expect to reproduce the formula of loc. cit. here unless $\gamma = 2$, then $J_{3,\gamma} = J_3$, and if $\beta = 2$ the embedding matches with our embedding. Then indeed $\frac{2}{\beta} = 1$.*

References

[Adl98] J. D. Adler. Refined anisotropic K-types and supercuspidal representations. *Pacific J. Math.*, 185(1):1–32, 1998.

[Adr16] M. Adrian. The Langlands parameter of a simple supercuspidal representation: odd orthogonal groups. *J. Ramanujan Math. Soc.*, 31(2):195–214, 2016.

[AK] M. Adrian and E. Kaplan. The Langlands parameter of a simple supercuspidal representation: symplectic groups. *Ramanujan J.* 50 (2019), no. 3, 589–619.

[AL16] M. Adrian and B. Liu. Some results on simple supercuspidal representations of $\mathrm{GL}_n(F)$. *J. Number Theory*, 160:117–147, 2016.

[BHS] C. Blondel, G. Henniart, and S. Stevens. Jordan blocks of cuspidal representations of symplectic groups. *Algebra Number Theory*, 12 (2018), no. 10, 2327–2386.

[BH06] C. J. Bushnell and G. Henniart. *The local Langlands conjecture for* GL(2), volume 335 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

[BK98] C. J. Bushnell and P. C. Kutzko. Smooth representations of reductive p-adic groups: structure theory via types. *Proc. London Math. Soc. (3)*, 77(3):582–634, 1998.

[CKPSS04] J. W. Cogdell, H.H. Kim, I. I. Piatetski-Shapiro, and F. Shahidi. Functoriality for the classical groups. *Publ. Math. Inst. Hautes Études Sci.*, 99(1):163–233, 2004.

[GPSR87] S. Gelbart, I. Piatetski-Shapiro, and S. Rallis. *L-Functions for $G \times GL(n)$*, volume 1254 of *Lecture Notes in Math.* Springer-Verlag, New York, 1987.

[Gin90] D. Ginzburg. L-functions for $\mathrm{SO}_n \times \mathrm{GL}_k$. *J. Reine Angew. Math.*, 1990(405):156–180, 1990.

[GRS98] D. Ginzburg, S. Rallis, and D. Soudry. L-functions for symplectic groups. *Bull. Soc. math. France*, 126:181—-244, 1998.

[GR10] B. H. Gross and M. Reeder. Arithmetic invariants of discrete Langlands parameters. *Duke Math. J.*, 154(3):431–508, 2010.

[How77] R. E. Howe. Some qualitative results on the representation theory of Gl_n over a p-adic field. *Pacific J. Math.*, 73(2):479–538, 1977.

[ILM17] A. Ichino, E. Lapid, and Z. Mao. On the formal degrees of square-integrable representations of odd special orthogonal and metaplectic groups. *Duke Math. J.*, 166(7):1301–1348, 2017.

[JPSS83] H. Jacquet, I. I. Piatetski-Shapiro, and J. A. Shalika. Rankin–Selberg convolutions. *Amer. J. Math.*, 105(2):367–464, 1983.

[Kal13] T. Kaletha. Simple wild L-packets. *J. Inst. Math. Jussieu*, 12(1):43–75, 2013.

[Kal15] T. Kaletha. Epipelagic L-packets and rectifying characters. *Invent. Math.*, 202(1):1–89, 2015.

[Kap] E. Kaplan. On the local theory of Rankin–Selberg convolutions for $\mathrm{SO}_{2l} \times \mathrm{GL}_n$. Thesis, Tel Aviv University, Israel, available at https://arxiv.org/abs/1506.05773.

[Kap10] E. Kaplan. An invariant theory approach for the unramified computation of Rankin-Selberg integrals for quasi-split $SO_{2n} \times GL_n$. *J. Number Theory*, 130:1801–1817, 2010.

[Kap12] E. Kaplan. The unramified computation of Rankin—Selberg integrals for $SO_{2l} \times GL_n$. *Israel J. Math.*, 191(1):137—-184, 2012.

[Kap13] E. Kaplan. Multiplicativity of the gamma factors of Rankin-Selberg integrals for $SO_{2l} \times GL_n$. *Manuscripta Math.*, 142(3-4):307–346, 2013.

[Kap15] E. Kaplan. Complementary results on the Rankin-Selberg gamma factors of classical groups. *J. Number Theory*, 146:390—-447, 2015.

[Mœg14] C. Mœglin. Paquets stables des séries discrètes accessibles par endoscopie tordue; leur paramètre de Langlands. In *Automorphic forms and related geometry: assessing the legacy of I. I. Piatetski-Shapiro*, volume 614 of *Contemp. Math.*, pages 295–336. Amer. Math. Soc., Providence, RI, 2014.

[MP94] A. Moy and G. Prasad. Unrefined minimal K-types for p-adic groups. *Invent. Math.*, 116(1-3):393–408, 1994.

[MP96] A. Moy and G. Prasad. Jacquet functors and unrefined minimal K-types. *Comment. Math. Helv.*, 71(1):98–121, 1996.

[RY14] M. Reeder and J.-K. Yu. Epipelagic representations and invariant theory. *J. Amer. Math. Soc.*, 27(2):437–477, 2014.

[Sha90] F. Shahidi. A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups. *Ann. of Math. (2)*, 132(2):273–330, 1990.

[Sou93] D. Soudry. Rankin–Selberg convolutions for $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$: local theory. *Mem. Amer. Math. Soc.*, 105(500):vi+100, 1993.

[Sou95] D. Soudry. On the Archimedean theory of Rankin-Selberg convolutions for $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$. *Ann. Sci. École Norm. Sup. (4)*, 28(2):161–224, 1995.

[Sou00] D. Soudry. Full multiplicativity of gamma factors for $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$. *Israel J. Math.*, 120(1):511–561, 2000.

[Ste08] S. Stevens. The supercuspidal representations of p-adic classical groups. *Invent. Math.*, 172(2):289–352, 2008.

[Tat67] J. T. Tate. Fourier analysis in number fields and Hecke's zeta functions. In J. W. S. Cassels and A. Frohlich, editors, *Algebraic number theory*, pages 305–347, Washington, D. C., 1967. Academic Press.

[Yu01] J.K. Yu Construction of tame supercuspidal representations *Journal of the Amer. Math. Soc.*, 14(3):579–622, 2001.

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