THE LANGLANDS PARAMETER OF A SIMPLE SUPERCUSPIDAL REPRESENTATION: SYMPLECTIC GROUPS

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Abstract. Let π be a simple supercuspidal representation of the symplectic group $\mathrm{Sp}_{2l}(F)$, over a p-adic field F. In this work, we explicitly compute the Rankin–Selberg γ-factor of rank-1 twists of π. We then completely determine the Langlands parameter of π, if $p \neq 2$. In the case that $F = \mathbb{Q}_2$, we give a conjectural description of the functorial lift of π, with which, using a recent work of Bushnell and Henniart, one can obtain its Langlands parameter.

1. introduction

Let G be a reductive group defined over a p-adic field. Supercuspidal representations are a fundamental object of study in the category of representations of G. These are the building blocks of all irreducible representations. The construction of these representations in increasing generality has been the goal of a long line of works, including [MP96, BK98, Mor99, Yu01, Ste02, Ste05, Kim07, Ste08]. The depth of an irreducible representation of G was defined by Moy and Prasad [MP94, MP96], and supercuspidal representations of minimal depth are called simple supercuspidal. Simple supercuspidal representations have been constructed recently by Gross and Reeder [GR10] and Reeder and Yu [RY14].

The local Langlands Correspondence plays a central role in modern theory of representations. This is a correspondence, conjectural in parts, between equivalence classes of irreducible (smooth) representations of G and equivalence classes of Langlands parameters. The correspondence should satisfy a list of natural properties, e.g., preserve L- and ϵ-factors (see e.g., [Bor79, Vog93, Cog03]). The Langlands correspondence for GL_n was proved by Harris and Taylor [HT01] and also by Henniart [Hen00] (over local fields of characteristic p by [LRS93]).

Recently, under a mild assumption on the residual characteristic, Kaletha [Kal13, Kal] has constructed a correspondence for simple supercuspidal representations (in the context of tamely ramified connected reductive p-adic groups), and has verified that his correspondence satisfies many of the expected properties of the Langlands correspondence. The Langlands correspondence for simple supercuspidal representations was also studied in [GR10, RY14].

Let π be a supercuspidal generic representation of a split connected classical group G, where in this work, supercuspidal representations are irreducible by definition. In this case by the works of Cogdell *et. al.* [CKPSS01, CKPSS04] there exists a unique generic representation Π of GL_N, for a suitable N, such that the γ-factors of $\pi \times \tau$ and $\Pi \times \tau$ are identical for every supercuspidal representation τ of GL_n ([CKPSS01, Proposition 7.2]). The γ-factor for $\pi \times \tau$ was defined by Shahidi [Sha90]. This lift corresponds to the standard L-homomorphism

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${}^LG \to {}^L\mathrm{GL}_N$. Then the Langlands correspondence for GL_N can in theory be used to determine the Langlands parameter φ_π of π, from the parameter of Π, but the construction in [CKPSS01] does not provide sufficient information on Π for this purpose.

While describing the Langlands correspondence explicitly is in general a difficult task, in the simple supercuspidal setting it turns out to be tractable. The first observation is that simple supercuspidal representations are always ψ-generic, for a certain choice of a character ψ. The supercuspidal support $\mathrm{supp}(\Pi)$ of Π is expected to contain only tamely ramified characters of GL_1 or simple supercuspidal representations of GL_n, $n \le N$ (see [Mui98, §3, Conjecture 3.2]). With an understanding of $\mathrm{supp}(\Pi)$, we can use the explicit local Langlands correspondence for simple supercuspidal representations of general linear groups to determine φ_π (see also [BH10, BH14, AL16]).

Let $\gamma(s, \Pi \times \tau, \psi)$ be the $\mathrm{GL}_N \times \mathrm{GL}_n$ γ-factor as defined by Jacquet *et. al.* [JPSS83]. Since Π is in our case self-dual, and when τ is supercuspidal, the results of [JPSS83], in particular the multiplicative properties, imply that $\gamma(s, \Pi \times \tau, \psi)$ has a pole at $s = 1$ if and only if τ appears in $\mathrm{supp}(\Pi)$. In this manner we can construct $\mathrm{supp}(\Pi)$, up to multiplicities. Of course, in practice we work with $\gamma(s, \pi \times \tau, \psi)$, because π is the only explicit object.

The γ-factor for $\pi \times \tau$ was first defined and studied by Shahidi [Sha90], via his method of local coefficients. An alternative definition using the method of Rankin–Selberg integrals, was given in [Sou93, Sou95, Sou00, Kap13, Kap15]. This definition is based on a global and local theory of integral representations, with strong local uniqueness properties. The γ-factor can then be defined as the proportionality factor between two integrals, related by an application of an intertwining operator. Moreover, it was proved in [Kap15] that this factor coincides with the γ-factor of Shahidi. Other works on Rankin–Selberg integrals, or Shimura type integrals when G is a symplectic group, for generic representations of $G \times \mathrm{GL}_n$, include [GPSR87, Gin90, GRS98, BAS09, GRS11, BAS16]; for a survey on the Rankin–Selberg method see Bump [Bum05].

In this work we deal with the symplectic group, a follow-up to [Adr16], where the first named author considered odd orthogonal groups. In both works, our main tool is the description of the γ-factor by virtue of the Rankin–Selberg theory. However, the computations in the symplectic case are far more subtle than the analogous computations for odd orthogonal groups. The main reason is that here, as opposed to the odd orthogonal case, the γ-factors may contain poles. In addition, the Shimura type integrals are in general more subtle to compute, because their construction involves a covering group and the Weil representation (see below). Another difference is in the case $p = 2$, which must be treated separately here. We turn to describe our results.

Let F be a local p-adic field with a maximal ideal $\mathfrak{p}_F$, residue field κ_F, and residue cardinality q. Fix a non-trivial additive character ψ of F, and for $a \in F^*$, $\psi_a(x) = \psi(ax)$. Denote the Weil index of $x \mapsto \psi(x^2)$ by $\gamma(\psi)$, and let $\gamma_\psi(a) = \gamma(\psi_a)/\gamma(\psi)$ (see [Rao93, Appendix], $\gamma_\psi(a)$ is $\gamma_F(a, \psi)$ there). Let $l > 1$ be an integer and consider the symplectic group on $2l$ variables, $\mathrm{Sp}_{2l} = \mathrm{Sp}_{2l}(F)$. Fix the Borel subgroup of upper triangular invertible matrices in Sp_{2l}, and let N be its unipotent radical. Extend ψ to a generic character of N by $u \mapsto \psi(\sum_{i=1}^{l} u_{i,i+1})$.

The simple supercuspidal representations of Sp_{2l} are parameterized by triplets (ϖ, ω, α), where $\varpi \in \mathfrak{p}_F$ is a uniformizer, i.e., $|\varpi| = q^{-1}$, ω is a character of the (finite) center Z of Sp_{2l}, and $\alpha \in \kappa_F^\times/(\kappa_F^\times)^2$. Specifically, ϖ and α give rise to an affine generic character χ of the pro-unipotent radical I^+ of the Iwahori subgroup, and the representation compactly induced from ZI^+ and $\omega \otimes \chi$ to Sp_{2l} is simple supercuspidal. This construction is exhaustive. See Section 2.2 for more details.

We denote the simple supercuspidal representation corresponding to (ϖ, ω, α) by π_α^ω, suppressing the dependence on ϖ. Our first step is to identify the characters in $\mathrm{supp}(\Pi_\alpha^\omega)$. The representation π_α^ω is ψ_α^{-1}-generic. Then the γ-factor of $\pi_\alpha^\omega \times \tau$ is defined, and denoted by $\gamma(s, \pi_\alpha^\omega \times \tau, \psi_\alpha)$.

Theorem 1.1. *Assume $p \neq 2$. Let τ be a tamely ramified quadratic character of $F^\times$. Then $\gamma(s, \pi_\alpha^\omega \times \tau, \psi_\alpha)$ has a pole at $s = 1$ if and only if τ is the unique non-trivial quadratic character of $F^\times$ such that $\tau(\varpi) = \gamma_{\psi_\alpha}(\varpi)^{-1} \frac{|G(\psi_\alpha^{-1})|}{G(\psi_\alpha^{-1})}$ and $\tau|_{\mathfrak{o}^\times} \equiv \gamma_{\psi_\alpha}|_{\mathfrak{o}^\times}$. Here $G(\psi)$ is a certain Gauss sum (see Claim 2.6).*

According to the theorem, $\mathrm{supp}(\Pi_\alpha^\omega)$ contains a unique character $\tau = \tau_\alpha$ (which may appear with some finite multiplicity). By a recent result of Blondel *et. al.* [BHS], φ_π decomposes as a direct sum $\varphi_1 \oplus \varphi_2$, where φ_1 is an irreducible $2l$-dimensional representation of the Weil group W_F, and φ_2 is a 1-dimensional representation of W_F; in particular, $\varphi_2 = \tau_\alpha$. As we will show (see Section 2.6), the properties of the γ-factor and the work of [BHS] imply that φ_π is determined by τ_α and by the computation of $\gamma(s, \pi_\alpha^\omega, \psi_\alpha)$. We emphasize here that it is not enough to locate the pole in $\gamma(s, \pi_\alpha^\omega \times \tau, \psi_\alpha)$ and use the work of [BHS]; one needs additional information, and this can be obtained from $\gamma(s, \pi_\alpha^\omega, \psi_\alpha)$.

We turn to describe the parameter, first in the case that $p \nmid 2l$; for more details and precise notation see Section 2.6. Assume the conductor of ψ is $\mathfrak{p}_F$. Let ζ be a $2l$-th root of $\frac{1}{(-1)^{l+1}4\alpha}\varpi$, and set $E = F(\zeta)$. This is a totally ramified extension of F, thus $\kappa_E = \kappa_F$. Relative to the basis $\zeta^{2l-1}, \zeta^{2l-2}, ..., \zeta, 1$, this gives rise to an embedding $\iota : E^\times \hookrightarrow \mathrm{GL}_{2l}$. Let $\lambda_{E/F}(\psi)$ be the Langlands constant ([Lan70]).

To define φ_π we construct a character $\xi = \xi_\alpha^\omega$ of $E^\times = \langle \zeta \rangle \times \kappa_F^\times \times (1 + \mathfrak{p}_E)$. For $x \in 1 + \mathfrak{p}_E$, $\iota(x) = (\iota(x)_{i,j}) \in \mathrm{GL}_{2l}$ and we take

$$\xi(x) = \psi(\sum_{i=1}^{2l-1} \iota(x)_{i,i+1} + \frac{(-1)^{l+1}4\alpha}{\varpi}\iota(x)_{2l,1}).$$

On $\kappa_F^\times$, ξ is given by

$$\tau_\alpha|_{\kappa_F^\times} \otimes \det(\mathrm{Ind}_{W_E}^{W_F}(1_E))^{-1},$$

and

$$\xi(\zeta) = \omega(-I_{2l})\gamma(\psi_\alpha)^{-1}\gamma_{\psi_\alpha}^{-1}(\varpi)\gamma(s, \tau_\alpha, \psi_\alpha)^{-1}\lambda_{E/F}(\psi_\alpha)^{-1}.$$

Here is our main result:

Theorem 1.2. *Assume p is odd, and does not divide the rank of* Sp_{2l}. *Then*

$$\varphi_1 = \mathrm{Ind}_{W_E}^{W_F} \xi_\alpha^\omega, \qquad \varphi_\pi = \mathrm{Ind}_{W_E}^{W_F} \xi_\alpha^\omega \oplus \tau_\alpha.$$

In the case that p is odd and p divides $2l$, the Langlands parameter of π can be explicitly written down using our work, the work of [BHS], and the work of [BH14]. See §2.6 for more details.

Thus far we have excluded dyadic fields. For $F = \mathbb{Q}_2$, there is a unique simple supercuspidal representation π of Sp_{2l}. As above, the conductor of ψ is $\mathfrak{p}_F$. We prove:

Theorem 1.3. *Assume τ is a tamely ramified character of $F^{\times}$. Then $\gamma(s, \pi \times \tau, \psi) = \tau(2)2^{1/2-s}$.*

In this case it is expected that the functorial lift of π to GL_{2l+1} is supercuspidal. Therefore $\mathrm{supp}(\Pi) = \{\Pi\}$, and in particular $\gamma(s, \pi\times\tau, \psi)$ should be entire (as we show). Taking this for granted, the results of [AL16] imply that this lift is already determined by Theorem 1.3. More specifically, the functorial lift of π to GL_{2l+1} is the unique simple supercuspidal representation Π with trivial central character, such that $\gamma(s, \Pi, \psi) = 2^{1/2-s}$; one can write explicit inducing data for Π, see [AL16]. One may then obtain φ_π using [BH14] (in fact, assuming the lift is supercuspidal, the weaker version of the theorem with just $\tau = 1$ suffices).

As mentioned above, our main tool in this work is Shimura type integrals. The integrals for a pair of generic representations of Sp_{2l} and GL_n (in a global or local context), were introduced by Ginzburg *et. al.* [GRS98], who developed the global theory and used the integrals to study the partial L-function of $\mathrm{Sp}_{2l} \times \mathrm{GL}_n$. In particular, they computed the local integrals with unramified data. The γ-factor $\gamma(s, \pi \times \tau, \psi)$, for a pair of generic representations π and τ, over a local field, was defined in [Kap15] as the proportionality factor between two integrals $\Psi(W, \phi, f_s)$ and $\Psi^*(W, \phi, f_s)$. The data for the integrals consist of a Whittaker function W in the Whittaker model of π, a Schwartz–Bruhat function ϕ in the space of a Weil representation, and a section f_s in the space of the representation parabolically induced (essentially) from $\tau \otimes |\det|^{s-1/2}$ to the double cover $\widetilde{\mathrm{Sp}}_{2n}$ of Sp_{2n} (the metaplectic group). The integral $\Psi^*(W, \phi, f_s)$ is obtained from $\Psi(W, \phi, f_s)$ by applying a normalized intertwining operator $M^*(\tau, s)$ to f_s. The equation defining $\gamma(s, \pi \times \tau, \psi)$ is essentially

$$\gamma(s, \pi \times \tau, \psi)\Psi(W, \phi, f_s) = \Psi^*(W, \phi, f_s).$$

By the general theory of Rankin–Selberg integrals, one may always choose data such that $\Psi(W, \phi, f_s)$ is constant. The analytic behavior of $\gamma(s, \pi \times \tau, \psi)$ is therefore controlled by $\Psi^*(W, \phi, f_s)$. When the representations are supercuspidal (of any depth), and $l > n$, the integral is holomorphic, and therefore any pole must already be a pole of the intertwining operator. Of course, the poles of $M^*(\tau, s)$ are independent of π. It is therefore the zeros of the integral that we seek. For precise definitions see Section 2.1.

In our setting $\pi = \pi_\alpha^\omega$ and $n = 1$. The operator $M^*(\tau, s)$ is normalized such that it is holomorphic at $s = 1$ unless τ is quadratic, and if it has a pole, it must be simple. The bulk part of the work is then to relate the existence of a zero to the relation between π_α^ω and τ.

The main idea in the proof is to choose the section f_s such that $M^*(\tau, s)f_s$ can be computed succinctly. The computation for the case of $p = 2$ is more delicate. Even though the γ-factor is in this case entire, we are interested in obtaining its precise value. The precise normalization factor of the intertwining operator becomes crucial for this result, and we benefited from its computation by Sweet [Swe] (reproduced in [GS16, Appendix], see also [Szp10, Szp11]).

The rest of this work is organized as follows. In Section 2.1 we review the construction of the Shimura type integrals for $\mathrm{Sp}_{2l} \times \mathrm{GL}_n$. The construction of simple supercuspidal representations of Sp_{2l} is described in Section 2.2. The proof of Theorem 1.1 is carried out in Section 2.4. The Langlands parameter is computed in Section 2.6. Finally, the case of $\mathbb{Q}_2$ is handled in Section 2.7.

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2. PRELIMINARIES

2.1. **The groups and the Shimura type integral.** Let F be a p-adic field with a ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p}$ and a uniformizer ϖ. Denote the residue field $\mathfrak{o}/\mathfrak{p}$ by κ_F and set $q = |\kappa_F|$. Let ψ be a non-trivial additive character of F. Denote the Weil index of $x \mapsto \psi(x^2)$ by $\gamma(\psi)$, and $\gamma_\psi(a) = \gamma(\psi_a)/\gamma(\psi)$ is the Weil factor, where $\psi_a(x) = \psi(ax)$ for $a \in F^*$ (see [Rao93, Appendix]). We fix a Haar measure dx on F, which is self dual with respect to ψ_2. For a measurable subset of F, $\mathrm{vol}(\cdots)$ denotes its measure assigned by dx. Then $\mathrm{vol}(\mathfrak{o}) = q^{\frac{1}{2}}$ (see e.g., [BH06, § 23]). Also fix a Haar measure $d^\times x$ on $F^\times$ such that $\mathrm{vol}^\times(\mathfrak{o}^\times) = 1$, where $\mathrm{vol}^\times(\cdots)$ is the measure assigned by $d^\times x$.

Let $\mathrm{Sp}_{2l} = \mathrm{Sp}_{2l}(F)$ be the symplectic group on $2l$ variables, defined with respect to the symplectic bilinear form of $(x, y) \mapsto {}^t x \left(\begin{smallmatrix} & J_l \\ -J_l & \end{smallmatrix}\right) y$, where ${}^t x$ is the transpose of x and J_l is the $l \times l$ matrix having 1 on the anti-diagonal and 0 elsewhere. Fix the Borel subgroup $B_l = T_l \ltimes N_l$ of upper triangular matrices in Sp_{2l}, where T_l is the torus and N_l is the unipotent radical. For $1 \le k \le l$, let $Q_k < \mathrm{Sp}_{2l}$ be the standard maximal parabolic subgroup whose Levi part is isomorphic to $\mathrm{GL}_k \times \mathrm{Sp}_{2(l-k)}$, and δ_{Q_k} be its modulus character. For any $a \in \mathrm{GL}_k$, put $a^* = J_k({}^t a^{-1})J_k$. The Levi subgroup of Q_k is then $\{\mathrm{diag}(a, h, a^*) : a \in \mathrm{GL}_k, h \in \mathrm{Sp}_{2(l-k)}\}$. Also put $w_l = \left(\begin{smallmatrix} & I_l \\ -I_l & \end{smallmatrix}\right) \in \mathrm{Sp}_{2l}$ and for any $x, y \in \mathrm{Sp}_{2l}$, ${}^x y = x^{-1} y x$.

Let $\widetilde{\mathrm{Sp}}_{2l}$ denote the metaplectic group, i.e., the double cover of Sp_{2l}. We write its elements as pairs $\langle g, \epsilon\rangle$ where $g \in \mathrm{Sp}_{2l}$ and $\epsilon = \pm 1$. The action is realized using the cocycle of Rao [Rao93]. We identify Sp_{2l} with its image under the map $g \mapsto \langle g, 1\rangle$. Also if $Q < \mathrm{Sp}_{2l}$, $\widetilde{Q}$ denotes its inverse image in $\widetilde{\mathrm{Sp}}_{2l}$.

For $l = 1$, the cocycle of Rao coincides with the cocycle of Kubota [Kub67]:

$$\sigma(g, g') = \Big(\frac{\mathbf{x}(gg')}{\mathbf{x}(g)}, \frac{\mathbf{x}(gg')}{\mathbf{x}(g')}\Big), \qquad \mathbf{x}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{cases} c & c \neq 0, \\ d & c = 0. \end{cases} \tag{2.1}$$

Here $(,)$ is the quadratic Hilbert symbol.

Consider the parabolic subgroup Q_n of Sp_{2n}. If we identify GL_n with the Levi part of Q_n by $a \mapsto \mathrm{diag}(a, a^*)$, the covering $\widetilde{\mathrm{GL}}_n$ we obtain is simple, in the sense that the cocycle is given by the quadratic Hilbert symbol. Consequently, its genuine representations are in bijection with the representations of GL_n, which can be made explicit by fixing a Weil symbol. For any representation τ of GL_n, the genuine representation $\tau \otimes \gamma_\psi$ of $\widetilde{\mathrm{GL}}_n$ is defined by $\tau \otimes \gamma_\psi(\langle a, \epsilon\rangle) = \epsilon \gamma_\psi(\det a)\tau(a)$, where $a \in \mathrm{GL}_n$ and $\epsilon = \pm 1$.

Let H_r be the $(2r + 1)$-dimensional Heisenberg group, realized as the group of matrices

$$\left\{ (x, y, z) = \begin{pmatrix} 1 & x & y & z \\ & I_r & & y' \\ & & I_r & x' \\ & & & 1 \end{pmatrix} \in \mathrm{Sp}_{2(r+1)} \right\}.$$

The subgroup obtained by taking $y = z = 0$ (resp., $x = z = 0$) is denoted X_r (resp., Y_r).

As mentioned in the introduction, our main tool is the computation of Shimura type integrals for $\mathrm{Sp}_{2l} \times \mathrm{GL}_n$. The integrals were developed in [GRS98]. The second named author defined the corresponding local γ-factors and established their fundamental properties in

[Kap15]. For the construction in [Kap15], the representation π of Sp_{2l} was assumed to afford a Whittaker model with respect to the character of N_l given by $u \mapsto \psi^{-1}(\sum_{i=1}^{l-1} u_{i,i+1} - \frac{1}{2}u_{l,l+1})$. This was compatible with [ILM17]. Here π will afford a Whittaker model with respect to the character $u \mapsto \psi^{-1}(\sum_{i=1}^{l} u_{i,i+1})$, which was used in [GRS98, § 4]. The changes between the versions are minor and were (partially) described in [Kap15, Remark 4.4]. To be explicit, and because we correct certain typos from [GRS98, GRS11, Kap15], we provide a complete definition of the integrals, even though we will only be using the $\mathrm{Sp}_{2l} \times \mathrm{GL}_1$ integrals, with $l > 1$.

Let $\mathcal{S}(F^r)$ be the space of Schwartz–Bruhat functions on the row space F^r. Define the Fourier transform $\widehat{\phi}$ of $\phi \in \mathcal{S}(F^r)$ by

$$\widehat{\phi}(y) = \int_{F^r} \phi(x)\psi(2xJ_r{}^t y)dx,$$

where y and x are rows. The measure on F^r is the product of Haar measures on F fixed above, then $\widehat{\widehat{\phi}}(y) = \phi(-y)$ (see e.g., [BH06, § 24]).

Let ω_ψ denote the Weil representation of $\widetilde{\mathrm{Sp}}_{2r} \ltimes H_r$, realized on $\mathcal{S}(F^r)$. It satisfies the following formulas (see [Ral82] and [BS98, Chap. 2][1]): for $\phi \in \mathcal{S}(F^r)$,

$$\omega_\psi((x,0,z))\phi(\xi) = \psi(z)\phi(\xi + x),$$
$$\omega_\psi((0,y,0))\phi(\xi) = \psi(2\xi J_r{}^t y)\phi(\xi),$$
$$\omega_\psi(\langle diag(a,a^*),\epsilon\rangle)\phi(\xi) = \epsilon\gamma_\psi^{-1}(\det a)|\det a|^{\frac{1}{2}}\phi(\xi a), \tag{2.2}$$
$$\omega_\psi(\langle \left(\begin{smallmatrix} I_r & u \\ & I_r \end{smallmatrix}\right), \epsilon\rangle)\phi(\xi) = \epsilon\psi(\xi J_r{}^t u {}^t\xi)\phi(\xi), \tag{2.3}$$
$$\widehat{\phi}(\xi) = \beta_\psi \omega_\psi(w_r)\phi(\xi). \tag{2.4}$$

Here β_ψ is a certain fixed eighth root of unity.

Let π and τ be a pair of generic representations of Sp_{2l} and GL_n $(l, n \geq 1)$. Assume that π is realized in its Whittaker model $\mathcal{W}(\pi, \psi^{-1})$, where $\psi(u) = \psi(\sum_{i=1}^{l} u_{i,i+1})$ $(u \in N_l)$. For $s \in \mathbb{C}$, consider the space $V(\tau, s)$ of the normalized induced representation

$$\mathrm{Ind}_{\widetilde{Q}_n}^{\widetilde{\mathrm{Sp}}_{2n}}((\tau \otimes \gamma_\psi)|\det|^{s-1/2}).$$

Here we use γ_ψ, instead of γ_ψ^{-1} as in [GRS98, Kap15], this is implied by the correction to (2.2) (to keep the formulas for the symplectic and metaplectic cases of [Kap15] as stated there, one simply has to regard γ_ψ as the inverse of the Weil factor). The elements of this space are regarded as smooth genuine functions on $\widetilde{\mathrm{Sp}}_{2n} \times \mathrm{GL}_n$, where for any $g \in \widetilde{\mathrm{Sp}}_{2n}$ and $b \in \mathrm{GL}_n$,

$$f_s(\left(\begin{smallmatrix} b & u \\ & b^* \end{smallmatrix}\right) g, a) = \gamma_\psi(\det b)\delta_{Q_n}^{1/2}\left(\begin{smallmatrix} b & \\ & b^* \end{smallmatrix}\right)|\det b|^{s-1/2} f_s(g, ab),$$

and the mapping $a \mapsto f_s(g,a)$ belongs to $\mathcal{W}(\tau, \psi)$, where ψ is the character of the subgroup of upper triangular unipotent matrices in GL_n given by $z \mapsto \psi(\sum_{i=1}^{n-1} z_{i,i+1})$.

[1]Note the typo in [GRS98, § 6, (1.3)], and also in [GRS11, (1.4)] and [Kap15, § 3.3.1], in their definition of the action of $\omega_\psi(\langle diag(a,a^*),\epsilon\rangle)$: γ_ψ should be γ_ψ^{-1} as in [Ral82] or [BS98, Chap. 2].

Let $W \in \mathcal{W}(\pi, \psi^{-1})$, $\phi \in \mathcal{S}(F^{\min(l,n)})$ and $f_s \in V(\tau, s)$. The Shimura type integral for $\pi \times \tau$ is defined by

$$\Psi(W,\phi,f_s) = \begin{cases} \displaystyle\int_{N_l\backslash Sp_{2l}} \int_{Y_l\backslash H_l} \int_{R_{l,n}} W(g)\omega_\psi(hg)\phi(\xi_0) f_s(\gamma r h g, I_n)\, dr\, dh\, dg & l < n, \\ \displaystyle\int_{N_l\backslash Sp_{2l}} W(g)\omega_\psi(g)\phi(\xi_0) f_s(g, I_l)\, dg & l = n, \\ \displaystyle\int_{N_n\backslash Sp_{2n}} \int_{R^{l,n}} \int_{X_n} W({}^{\omega_{l-n,n}}(rxg))\omega_\psi(g)\phi(x) f_s(g, I_n)\, dx\, dr\, dg & l > n. \end{cases}$$

Here

$$R_{l,n} = \left\{ \begin{pmatrix} I_{n-l-1} & 0 & r_1 & 0 & r_2 & r_3 \\ & 1 & & & & r_2' \\ & & I_l & & & 0 \\ & & & I_l & & r_1' \\ & & & & 1 & 0 \\ & & & & & I_{n-l-1} \end{pmatrix} \in \mathrm{Sp}_{2n} \right\} \qquad (l \geq 0),$$

$$\xi_0 = (0, \ldots, 0, 1) \in F^l \qquad \text{(a row vector)},$$

$$\gamma_{l,n} = \begin{pmatrix} & 2I_l & & \\ & & & -I_{n-l} \\ I_{n-l} & & & \\ & & \frac{1}{2}I_l & \end{pmatrix} \qquad (n \geq l \geq 0),$$

$$R^{l,n} = \left\{ \begin{pmatrix} I_{l-n-1} & 0 & r_1 & & & \\ & 1 & & & & \\ & & I_n & & & \\ & & & I_n & & r_1' \\ & & & & 1 & 0 \\ & & & & & I_{l-n-1} \end{pmatrix} \in \mathrm{Sp}_{2l} \right\},$$

$$\omega_{l-n,n} = \begin{pmatrix} & I_{l-n} & & \\ I_n & & & \\ & & & I_n \\ & & I_{l-n} & \end{pmatrix}.$$

The integral is absolutely convergent for $\mathrm{Re}(s) \gg 0$, and admits meromorphic continuation to $\mathbb{C}(q^{-s})$. Moreover, there is a choice of data for which it becomes a nonzero constant, for all s (see [GRS98]).

Set $m = \min(n, l)$ and $m_0 = \max(n, l)$. The integrals can be regarded as trilinear forms on $\mathcal{W}(\pi, \psi^{-1}) \times \mathcal{S}(F^l) \times V(\tau, s)$ satisfying the following equivariance properties:

$$(2.5) \qquad \begin{cases} \Psi(g \cdot W, (h\langle g, \epsilon\rangle) \cdot \phi, (vh\langle g, \epsilon\rangle) \cdot f_s) = \psi^{-1}(v)\Psi(W, \phi, f_s) & l < n, \\ \Psi(g \cdot W, \langle g, \epsilon\rangle \cdot \phi, \langle g, \epsilon\rangle \cdot f_s) = \Psi(W, \phi, f_s) & l = n, \\ \Psi((vhg) \cdot W, (h\langle g, \epsilon\rangle) \cdot \phi, \langle g, \epsilon\rangle \cdot f_s) = \psi^{-1}(v)\Psi(W, \phi, f_s) & l > n. \end{cases}$$

Here $g \in \mathrm{Sp}_{2m}$, $\epsilon = \pm 1$, $h \in H_m$,

$$v = \begin{pmatrix} z & u & v \\ & I_{2(m+1)} & u' \\ & & z^* \end{pmatrix} < N_{m_0}, \qquad \psi(v) = \psi(\sum_{i=1}^{m_0-m-2} z_{i,i+1} + u_{m_0-m-1,1}).$$

Let $M(\tau, s) : V(\tau, s) \to V(\tau^*, 1-s)$ be the standard intertwining operator, given by the meromorphic continuation of the following integral:

$$[M(\tau, s)f_s](g, a) = \int_{U_n} f_s(w_n^{-1}ug, d_n a^*)du,$$

where $g \in \widetilde{\mathrm{Sp}}_{2n}$, $a \in \mathrm{GL}_n$, and $d_n = diag(-1, 1, \dots, (-1)^n) \in \mathrm{GL}_n$. Note that by definition τ^* is still regarded as a genuine representation $\widetilde{\mathrm{GL}}_n$ by taking $\tau^* \otimes \gamma_\psi$. When τ is irreducible, τ^* is the representation contragredient to τ. The normalized intertwining operator is defined by $M^*(\tau, s) = C(s, \tau, \psi)M(\tau, s)$, where $C(s, \tau, \psi)$ is the analog of Shahidi's local coefficient, and is defined via the functional equation

$$\int_{U_n} f_s(d_n w_n u, I_n)\psi(u_{n,n+1})\, du = C(s, \tau, \psi) \int_{U_n} [M(\tau, s)f_s](d_n w_n u, I_n)\psi(u_{n,n+1})\, du. \tag{2.6}$$

Also let $c(s, l, \tau) = \tau(2I_n)^{-2}|2|^{-2n(s-\frac{1}{2})}$ if $l \geq n$, otherwise $c(s, l, \tau) = 1$.

The space of trilinear forms satisfying (2.5) is, outside a finite set of values of q^{-s}, one-dimensional. Hence the integrals $\Psi(W, \phi, f_s)$ and $\Psi(W, \phi, M^*(\tau, s)f_s)$ are proportional. The γ-factor is defined by

$$\gamma(s, \pi \times \tau, \psi) = \pi(-I_{2l})^n \tau(-I_n)^l \gamma(s, \tau, \psi) c(s, l, \tau) \frac{\Psi(W, \phi, M^*(\tau, s)f_s)}{\Psi(W, \phi, f_s)}, \tag{2.7}$$

whenever $\Psi(W, \phi, f_s)$ is not identically 0. For the minimal case of $l = 0$ (i.e., Sp_{2l} is the trivial group), we must define $\gamma(s, \pi \times \tau, \psi) = \gamma(s, \tau, \psi)$, essentially because the Langlands dual group of Sp_{2l} is $\mathrm{SO}_{2l+1}(\mathbb{C})$.

Remark 2.1. *In* [Kap15], *$\gamma(s, \pi \times \tau, \psi)$ was defined as the ratio*

$$\Psi(W, \phi, M^*(\tau, s)f_s)/\Psi(W, \phi, f_s),$$

or

$$c(s, l, \tau)\Psi(W, \phi, M^*(\tau, s)f_s)/\Psi(W, \phi, f_s)$$

depending on the convention for the integrals ([Kap15, Remark 4.4]*). Then a normalized version $\Gamma(s, \pi \times \tau, \psi)$ was defined by*

$$\Gamma(s, \pi \times \tau, \psi) = \pi(-I_{2l})^n \tau(-I_n)^l \gamma(s, \pi \times \tau, \psi).$$

The case of $l = 0$ for Sp_{2l} was overlooked in [Kap15], *$\Gamma(s, \pi \times \tau, \psi)$ was defined to be trivial instead of $\gamma(s, \tau, \psi)$, and in the definition of $\Gamma(s, \pi \times \tau, \psi)$ in general* [Kap15, p. 408], *one must again multiply by $\gamma(s, \tau, \psi)$ as we do here in* (2.7) *(see also* [LR05]*).*

With this correction, in the multiplicative formula [Kap15, Theorem 1, (2)] *when π' is a representation of Sp_0, $\Gamma(s, \pi', \psi) = \gamma(s, \tau, \psi)$ appears on the right hand side of the identity; we get the correct L-function when data are unramified (*[Kap15, Theorem 1, (3)]*); in the formula describing the dependence on ψ (*[Kap15, Theorem 1, (6)]*) $2l$ changes into $2l+1$, which is the expected dependence since Sp_{2l} naturally lifts to GL_{2l+1}; and of course the minimal case (*[Kap15, Theorem 1, (7)]*) is now $\gamma(s, \tau, \psi)$. This modification only applies to Sp_{2l}, and not to $\widetilde{\mathrm{Sp}}_{2l}$ (the dual group of $\widetilde{\mathrm{Sp}}_{2l}$ is taken to be $\mathrm{GL}_{2l}(\mathbb{C})$).*

In this work we defined the γ-factor directly as the normalized one (2.7)*, so we do not use the convention* $\Gamma(\cdots)$*, but* (2.7) *should be regarded as the corrected version of* $\Gamma(s,\pi\times\tau,\psi)$ *from* [Kap15].

We have the following twisting formula: for $a \in F^{\times}$, if $W(\pi,\psi_a^{-1}) \neq 0$,

$$\gamma(s,\pi\times\tau,\psi_a) = \tau(aI_n)^{2l+1}|a|^{(2l+1)n(s-1/2)}\gamma(s,\pi\times\tau,\psi). \tag{2.8}$$

Indeed if a is a square, this was proved in [Kap15, Theorem 1 (6)] (with the correction described in the remark above); in general this follows because $\gamma(s,\pi\times\tau,\psi_a)$ is identical with Shahidi's corresponding γ-factor [Kap15, Corollary 1]. Note that a direct verification of (2.8) when $l = m$ (and a is not a square) appeared in [Zha, Proposition 3.5].

As mentioned above, in this work $l > 1$ and $n = 1$. Then τ is a quasi-character of $F^{\times}$ and the Shimura type integral for $\pi \times \tau$ is defined by

$$\Psi(W,\phi,f_s) = \int_{N_1\backslash Sp_2}\int_{R^{l,1}}\int_{X_1} W({}^{\omega_{l-1,1}}(rxg))\omega_{\psi}(g)\phi(x)f_s(g,1)\;dx\;dr\;dg,$$

with

$$R^{l,1} = \left\{\begin{pmatrix} I_{l-2} & 0 & r_1 & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & r_1' \\ & & & & 1 & 0 \\ & & & & & I_{l-2}\end{pmatrix} \in \mathrm{Sp}_{2l}\right\}, \qquad \omega_{l-1,1} = \begin{pmatrix} & I_{l-1} & & \\ 1 & & & \\ & & & 1 \\ & & I_{l-1} & \end{pmatrix}.$$

For certain computations, we will need to use the precise value of the normalizing factor $C(s,\tau,\psi)$ of the intertwining operator. This value was computed in [Swe, Szp11, GS16], albeit with slightly different conventions. Specifically, in [Szp11] the character appearing on both sides of (2.6) was ψ^{-1}, the representation τ of GL_1 was identified with a genuine representation of $\widetilde{\mathrm{GL}}_1$ by $\tau\otimes\gamma_{\psi^{-1}}$, and the intertwining operator was given by $\int_{U_1} f_s(w_1u)\,du$. Under these conventions the normalizing factor was shown in [Szp11] to be equal to the principal value of

$$\int_{F^{\times}} \tau(x)|x|^s\gamma_{\psi}(x)^{-1}\psi(x)d^{\times}x.$$

According to Sweet [Swe] (see [GS16, Appendix]), this integral equals

$$\gamma(\psi_{-1})\tau(-1)\frac{\gamma(2s-1,\tau^2,\psi_2)}{\gamma(s,\tau,\psi)}.$$

Since

$$[M(\tau,s)f_s](g,1) = \int_{U_1} f_s(w_1^{-1}ug,d_1)\,du = \gamma_{\psi}(-1)^{-1}\int_{U_1} f_s(w_1ug,1)\,du$$

and $\psi_{-1} = \psi^{-1}$,

$$C(s,\tau,\psi) = \gamma_{\psi}(-1)\gamma(\psi)\tau(-1)\frac{\gamma(2s-1,\tau^2,\psi_2^{-1})}{\gamma(s,\tau,\psi^{-1})} = \gamma_{\psi}(-1)\gamma(\psi)\frac{\gamma(2s-1,\tau^2,\psi_2)}{\gamma(s,\tau,\psi)}. \tag{2.9}$$

In particular, we can rewrite (2.7) for $n = 1$ in the form

(2.10)
$$\gamma(s, \pi \times \tau, \psi) = \pi(-I_{2l})\tau(-1)^l c(s, l, \tau)\gamma_\psi(-1)\gamma(\psi)\gamma(2s-1, \tau^2, \psi_2)\frac{\Psi(W, \phi, M(\tau, s)f_s)}{\Psi(W, \phi, f_s)}.$$

We will also use the fact that $\gamma(2s-1, \tau^2, \psi)$ has a pole at $s = 1$ if and only if τ is quadratic (a simple computation).

2.2. **The simple supercuspidal representations of** Sp_{2l}**.** The simple supercuspidal representations of Sp_{2l} are those with smallest depth $\frac{1}{2l}$. We briefly recall their construction. Let $K = \mathrm{Sp}_{2l}(\mathfrak{o})$, $T_0 = T \cap K$, Z be the (finite) center of Sp_{2l}, and I be the standard Iwahori subgroup, which is the preimage in K of the Borel subgroup $B_l(\kappa_F)$. Also define the subgroup I^+ of I, by replacing $\mathfrak{o}$ with $\mathfrak{p}$ in the coordinates $(i, i+1), (l+i, l+i+1)$ for $1 \le i < l$, and replacing $\mathfrak{p}$ with $\mathfrak{p}^2$ in the coordinate $(2l, 1)$.

Additionally denote

$$T(q) = \{t \in T_0 | t^q = t\}, \qquad Z(q) = Z \cap T(q).$$

Fix an additive character ψ^* of F whose conductor is $\mathfrak{p}$. For any $\alpha \in F^\times$, consider the following character $\psi^{*,\alpha}$ of N_l, given by $\psi^{*,\alpha}(u) = \psi^*(\sum_{i=1}^{l-1} u_{i,i+1} + \alpha u_{l,l+1})$.

Following the works of Gross and Reeder [GR10], and Reeder and Yu [RY14], the affine generic characters $\chi : ZI^+ \to \mathbb{C}^*$ are of the form

$$\chi(zk) = \chi^\omega_{\underline{t}}(zk) = \omega(z)\psi^*(\sum_{i=1}^{l} t_i k_{i,i+1} + t_{l+1}k_{2l,1}\varpi^{-1}) \qquad (z \in Z,\ k \in I), \tag{2.11}$$

where ω is a character of Z and $\underline{t} = (t_1, \dots, t_{l+1})$ with $t_i \in \kappa_F^\times$ for all i.

In fact, a complete set of representatives of the $T(q)$-orbits of affine generic characters of ZI^+ is given by $\chi^\omega_{\underline{t}}$, $\underline{t} = (1, \dots, 1, \alpha, t)$ where α varies over the square classes in $\kappa_F^\times$ and $t \in \kappa_F^\times$. Denote such a character by $\chi^\omega_{\alpha,t}$. By [GR10, Proposition 9.3], the simple supercuspidal representations of Sp_{2l} are thus parameterized by a choice of central character ω, a choice $t \in \kappa_F^\times$, and a choice of $\alpha \in \kappa_F^\times/(\kappa_F^\times)^2$. We denote this representation by $\pi^\omega_{\alpha,t}$,

$$\pi^\omega_{\alpha,t} = \mathrm{ind}_{ZI^+}^{\mathrm{Sp}_{2l}} \chi^\omega_{\alpha,t}.$$

Here ind denotes the compact induction.

Instead of using the above parametrization with ω, α and t, we set $t = 1$ and let the affine generic characters be parameterized by central characters ω, $\alpha \in \kappa_F^\times/(\kappa_F^\times)^2$ and the various choices of uniformizer ϖ in F. Since we have already fixed an (arbitrary) uniformizer ϖ at the beginning of Section 2.1, given (ω, α) we denote the corresponding affine generic character and simple supercuspidal representation by χ^ω_α and π^ω_α.

It is easy to see that any representation π^ω_α is $\psi^{*,\alpha}$-generic. Moreover, the following is a Whittaker function in $W(\pi^\omega_\alpha, \psi^{*,\alpha})$:

$$W_{\alpha,\omega}(g) = \begin{cases} \psi^{*,\alpha}(u)\chi^\omega_\alpha(zk) & \text{if } g = uzk \in UZI^+, \\ 0 & \text{otherwise.} \end{cases}$$

2.3. **Computations on $\overline{B}_1$.** For the computation of the integrals, it will be convenient to rewrite the dg-integral over the Borel subgroup $\overline{B}_1$ of lower triangular matrices in Sp_2. In this section we work out several auxiliary computations that will be used repeatedly below.

For $a, c \in F^\times$, put

$$b = \begin{pmatrix} a & \\ a^{-1}c & a^{-1} \end{pmatrix}. \tag{2.12}$$

Note that the assumption $c \neq 0$ excludes a subset of $\overline{B}_1$ of zero measure, hence will not affect our computations. Recall that we identify Sp_2 with its image in $\widetilde{\mathrm{Sp}}_2$ under $g \mapsto \langle g, 1\rangle$. Here

$$\langle b, 1\rangle = (a^{-1}, c)\langle \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, 1\rangle\langle \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}, 1\rangle. \tag{2.13}$$

Claim 2.2. *Let $\varphi \in \mathcal{S}(F)$ and b be given by* (2.12). *Then*

$$\omega_\psi(b)\phi(x) = (a^{-1}, c)\beta_\psi^{-2}\gamma_\psi^{-1}(a)\gamma_\psi(-1)\int_F \psi(2axy)\psi(-cy^2)\int_F \phi(z)\psi(-2yz)\ dz\ dy.$$

Proof. Using (2.1) together with (2.2)–(2.4) we see that

$$\begin{aligned}
\omega_\psi(b)\phi(x) =& \ (a^{-1}, c)\gamma_\psi^{-1}(a)\omega_\psi(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix})\phi(ax) \\
&= (a^{-1}, c)\gamma_\psi^{-1}(a)\omega_\psi(\langle w_1, 1\rangle\langle \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}, 1\rangle\langle w_1^{-1}, 1\rangle)\phi(ax) \\
&= \beta_\psi^{-1}(a^{-1}, c)\gamma_\psi^{-1}(a)(\omega_\psi(\langle \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}, 1\rangle\langle w_1^{-1}, 1\rangle)\phi)\widehat{}(ax) \\
&= \beta_\psi^{-1}(a^{-1}, c)\int_F \gamma_\psi^{-1}(a)\omega_\psi(\langle \begin{pmatrix} 1 & -c \\ 0 & 1 \end{pmatrix}, 1\rangle\langle w_1^{-1}, 1\rangle)\phi(y)\psi(2axy)dy \\
&= (-1,-1)\cdot\beta_\psi^{-1}(a^{-1}, c)\int_F \gamma_\psi^{-1}(a)\psi(-cy^2)\omega_\psi(\langle -I_2, 1\rangle\langle w_1, 1\rangle)\phi(y)\psi(2axy)dy \\
&= (-1,-1)\cdot\beta_\psi^{-2}\gamma_\psi^{-1}(a)\gamma_\psi^{-1}(-1)(a^{-1}, c)\int_F \psi(2axy)\psi(-cy^2)\int_F \phi(z)\psi(-2yz)dzdy.
\end{aligned}$$

Note that we used $\langle -I_2w_1, 1\rangle = (-1,-1)\langle -I_2, 1\rangle\langle w_1, 1\rangle$, one equality before the last. □

Claim 2.3. *Let $f_s \in V(\tau, s)$ and b be given by* (2.12), *with $a \in 1 + \mathfrak{p}$. Then*

$$[M(\tau, s)f_s](b, 1) = \frac{q-1}{q^{\frac{1}{2}}}\int_{F^\times}(-uc, a)\tau(u^{-1})|u|^{\frac{1}{2}-s}\gamma_\psi^{-1}(u^{-1}a)f_s\left(\begin{pmatrix} 1 & 0 \\ a^2u^{-1}+c & 1 \end{pmatrix}, -1\right)d^\times u.$$

Proof. By definition, for any $g \in \widetilde{\mathrm{Sp}}_2$,

$$[M(\tau, s)f_s](g, 1) = \int_{U_1} f_s(w_1^{-1}ug, -1)du.$$

Let $u = \left(\begin{smallmatrix} 1 & u \\ & 1 \end{smallmatrix}\right)$. Then

$$\langle w_1^{-1}, 1\rangle\langle u, 1\rangle\langle b, 1\rangle = \left\langle \begin{pmatrix} -a^{-1}c & -a^{-1} \\ a + uca^{-1} & ua^{-1} \end{pmatrix}, (-ac, a + ua^{-1}c)\right\rangle$$

as long as $a + ua^{-1}c \neq 0$ (in which case there would be no cocycle). Then,

$$[M(\tau, s)f_s](b, 1) = \int_F (-ac, a + ua^{-1}c) f_s\left(\begin{pmatrix} -a^{-1}c & -a^{-1} \\ a + uca^{-1} & ua^{-1} \end{pmatrix}, -1\right) du.$$

As above we may assume $u \neq 0$. Since f_s is left-invariant by U_1, we may row reduce upwards to obtain

$$[M(\tau, s)f_s](b, 1) = \int_{F^\times} (-ac, a + ua^{-1}c) f_s\left(\begin{pmatrix} u^{-1}a & 0 \\ a + uca^{-1} & ua^{-1} \end{pmatrix}, -1\right) du.$$

Recall that du is a Haar measure on F, not $F^\times$. We change the additive Haar measure to a multiplicative one by writing $du = (q-1)q^{-1/2}|u|d^\times u$, then obtain

$$\begin{aligned} [M(\tau, s)f_s](b, 1) = & \int_{F^\times} (-ac, a + ua^{-1}c)(ua^{-1}, a^2u^{-1} + c) \cdot (-ua^{-1}(a^2u^{-1} + c), a + uca^{-1}) \\ & \times f_s\left(\langle\begin{pmatrix} u^{-1}a & 0 \\ 0 & ua^{-1} \end{pmatrix}, 1\rangle\langle\begin{pmatrix} 1 & 0 \\ a^2u^{-1} + c & 1 \end{pmatrix}, 1\rangle, -1\right) \frac{q-1}{q^{\frac{1}{2}}}|u|\, d^\times u \\ = & \int_{F^\times} (-ac, a + ua^{-1}c)(ua^{-1}, a^2u^{-1} + c) \cdot (-ua^{-1}(a^2u^{-1} + c), a + uca^{-1}) \\ & \times \frac{q-1}{q^{\frac{1}{2}}}\tau(u^{-1})|u|^{\frac{1}{2}-s}\gamma_\psi(u^{-1}a) f_s\left(\begin{pmatrix} 1 & 0 \\ a^2u^{-1} + c & 1 \end{pmatrix}, -1\right) d^\times u, \end{aligned}$$

since τ is tamely ramified and hence $\tau(a) = 1$ for $a \in 1 + \mathfrak{p}$. Note that

$$(-ua^{-1}(a^2u^{-1} + c), a + uca^{-1}) = (-a - uca^{-1}, a + uca^{-1}) = 1,$$

since $(z, -z) = 1$ for any $z \in F^\times$. Moreover, one computes that

$$(-ac, a + ua^{-1}c)(ua^{-1}a^2u^{-1} + c) = (-uc, a)(ua^{-1}, -1).$$

The required formula now follows using $\gamma_\psi(u^{-1}a) = \gamma_\psi^{-1}(u^{-1}a)(-1, u^{-1}a)$. □

2.4. **The computation of the γ-factor.** We study the γ-factor using a direct computation of integrals. We begin by defining the data for the computation. Fix an additive character ψ of F, whose conductor is $\mathfrak{p}_F$. As in Section 2.1, this defines the character ψ^{-1} of N_l.

Using the notation of Section 2.2, put $\psi^* = \psi^{-1}$ and $\alpha = 1$. Throughout, we will set $\pi = \pi_1^\omega$ and $\chi = \chi_1^\omega$. Now π admits the Whittaker model $W(\pi, \psi^{-1})$. Our Whittaker function is defined by

$$W(g) = \begin{cases} \psi^{-1}(u)\omega(z)\chi(k) & \text{if} \quad g = uk \in U_l Z I^+, \\ 0 & \text{otherwise.} \end{cases}$$

We will see that once we obtain our result for π_1^ω, we can easily conclude the analogous result for π_α^ω, where α is the non-trivial class in $\kappa_F^\times/(\kappa_F^\times)^2$ (since $p > 2$, there are only two classes). See the end of Section 2.4.

2.5. **The case when $p \neq 2$.** In this section, we compute $\gamma(s, \pi \times \tau, \psi)$ when τ is quadratic and tamely ramified. This implies in particular that the restriction of τ to $\mathfrak{o}^\times$ coincides with either γ_ψ or $(\varpi, \cdot)\gamma_\psi$. Most of the computations do not require τ to be quadratic, but this condition will eventually be used.

The Whittaker function W was defined in Section 2.4. To define f_s, consider the subgroup

$$\mathcal{N} = \left\{ \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{pmatrix} \in \mathrm{Sp}_2 \right\}.$$

Since $p > 2$, according to [Kub69, Theorem 2] the section $\langle x, \vartheta(x)\rangle$ is a splitting (a homomorphism) of $\mathrm{Sp}_2(\mathfrak{o})$ where

$$\vartheta \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) = \begin{cases} 1 & c = 0 \text{ or } |c| = 1, \\ (c,d) & \text{otherwise.} \end{cases}$$

In particular it is a splitting of $\mathcal{N}$. However, because $1+\mathfrak{p}_F \subset (F^\times)^2$, ϑ is trivial on $\mathcal{N}$, thus the trivial section $g \mapsto \langle g, 1\rangle$ is a homomorphism of $\mathcal{N}$.

We define f_s to be the function supported on $\widetilde{B}_1\mathcal{N}$, such that

$$f_s(\langle \left(\begin{smallmatrix} b & u \\ & b^{-1} \end{smallmatrix} \right), \epsilon\rangle \langle v, 1\rangle, a) = \epsilon |b|^{s+1/2} \gamma_\psi(b)\tau(ba), \qquad v \in \mathcal{N}.$$

Also let ϕ be the characteristic function of $\mathfrak{p}$.

Claim 2.4. $\Psi(W,\phi,f_s) = \beta_\psi^{-2}\gamma_\psi^{-1}(-1)\mathrm{vol}(\mathfrak{p})^l \mathrm{vol}(\mathfrak{o})\mathrm{vol}^\times(1+\mathfrak{p})\mathrm{vol}(\mathfrak{p}^2)$.

Proof. Writing the dg-integration over $\overline{B}_1$,

$$\Psi(W,\phi,f_s) = \int_{\overline{B}_1} \int_{R^{l,1}} \int_F W({}^{\omega_{l-1,1}}(rxg))\omega_\psi(g)\phi(x) f_s(g,1)\delta_B(b)\, dx\, dr\, db.$$

The support of W is contained in $U_l Z I^+$. Looking at ${}^{\omega_{l-1,1}}(rxg)$ with $b = \left(\begin{smallmatrix} a & 0 \\ a^{-1}c & a^{-1} \end{smallmatrix} \right) \in \overline{B}_1$, we see that W vanishes unless $a \in 1+\mathfrak{p}$, $x \in \mathfrak{p}$, $c \in \mathfrak{p}$ and the coordinates of r_1 belong in $\mathfrak{p}$. In particular ${}^{\omega_{l-1,1}}(rxg) \in U_l I^+$ (see the definition of W). We may also assume $c \neq 0$. Then

$$W({}^{\omega_{l-1,1}}(rxg)) = \psi^{-1}(a^{-1}c\varpi^{-1}). \tag{2.14}$$

Using our definition of ϕ, the double integral from Claim 2.2 becomes

$$\int_F \psi(2axy)\psi(-cy^2) \int_{\mathfrak{p}} \psi(-2yz)\ dz\ dy.$$

This vanishes unless $y \in \mathfrak{o}$, then since $a \in 1+\mathfrak{p}$ and $x, c \in \mathfrak{p}$,

$$\omega_\psi(b)\phi(x) = \beta_\psi^{-2}\gamma_\psi^{-1}(-1)\mathrm{vol}(\mathfrak{p})\mathrm{vol}(\mathfrak{o}). \tag{2.15}$$

Note that $\gamma_\psi^{-1}(a) = 1$ because $a \in 1+\mathfrak{p} \subset F^{\times 2}$. Also for $a \in 1+\mathfrak{p}$, $f_s(b,1) = 0$ unless $c \in \mathfrak{p}^2$, in which case by (2.13),

$$f_s(b,1) = (a^{-1},c)\gamma_\psi(a) f_s(\langle \left(\begin{smallmatrix} 1 & \\ c & 1 \end{smallmatrix} \right), 1\rangle) = f_s(\langle \left(\begin{smallmatrix} 1 & \\ c & 1 \end{smallmatrix} \right), 1\rangle) = 1. \tag{2.16}$$

Combining (2.14)–(2.16) gives the result. ∎

Claim 2.5. *Let $c \in \mathfrak{p}$, and write $c \in c_0\varpi + \mathfrak{p}^2$ with $c_0 \in \mathfrak{o}$. Then*

$$[M(\tau,s)f_s]\left(\left(\begin{smallmatrix} 1 & \\ c & 1 \end{smallmatrix} \right), 1\right) = \begin{cases} A(\tau,\psi,s)\tau(-1)(q-1)(L(2s-1,\tau^2)-1) & c_0 = 0, \\ q^{-s}\tau(c_0\varpi)\gamma_\psi^{-1}(-c_0\varpi) & c_0 \in \mathfrak{o}^*. \end{cases}$$

Here

$$A(\tau,\psi,s) = \begin{cases} q^{-s}\gamma_\psi^{-1}(\varpi)\tau(\varpi) & \tau|_{\mathfrak{o}^\times} \equiv (\varpi,\cdot)\gamma_\psi|_{\mathfrak{o}^\times}, \\ q^{-1/2} & \tau|_{\mathfrak{o}^\times} \equiv \gamma_\psi|_{\mathfrak{o}^\times}. \end{cases}$$

Proof. First assume $c \in \mathfrak{p}^2$. Thus, c can be ignored and we see that the integrand vanishes unless $u^{-1} \in \mathfrak{p}^2$. Hence by Claim 2.3,

$$\begin{aligned}
&[M(\tau,s)f_s]\left(\left(\begin{smallmatrix}1&\\c&1\end{smallmatrix}\right),1\right)\\
&=\frac{q-1}{q^{1/2}}\sum_{\ell=2}^{\infty}q^{(1/2-s)\ell}\tau^{\ell}(\varpi)\gamma_{\psi}^{-1}(\varpi^{\ell})\int_{\mathfrak{o}^*}(\varpi^{\ell},u)\gamma_{\psi}^{-1}(u)\tau(u)\tau(-1)\,d^{\times}u\\
&=\tau(-1)\frac{q-1}{q^{1/2}}\sum_{\ell=1}^{\infty}q^{(\frac{1}{2}-s)(2\ell+1)}\tau^{2\ell+1}(\varpi)\gamma_{\psi}^{-1}(\varpi)\int_{\mathfrak{o}^{\times}}\gamma_{\psi}^{-1}(u)(\varpi,u)\tau(u)\,d^{\times}u\\
&\quad+\tau(-1)\frac{q-1}{q^{1/2}}\sum_{\ell=1}^{\infty}q^{(\frac{1}{2}-s)(2\ell)}\tau^{2\ell}(\varpi)\int_{\mathfrak{o}^{\times}}\gamma_{\psi}^{-1}(u)\tau(u)\,d^{\times}u\\
&=\begin{cases}\tau(-1)\frac{q-1}{q^{1/2}}\gamma_{\psi}^{-1}(\varpi)q^{\frac{1}{2}-s}\tau(\varpi)\left(\frac{1}{1-q^{1-2s}\tau^2(\varpi)}-1\right) & \tau|_{\mathfrak{o}^{\times}}\equiv(\varpi,\cdot)\gamma_{\psi}|_{\mathfrak{o}^{\times}},\\ \tau(-1)\frac{q-1}{q^{1/2}}\left(\frac{1}{1-q^{1-2s}\tau^2(\varpi)}-1\right) & \tau|_{\mathfrak{o}^{\times}}\equiv\gamma_{\psi}|_{\mathfrak{o}^{\times}}.\end{cases}
\end{aligned}$$

Here we used the fact that γ_ψ is a character of $\mathfrak{o}^*$, which is non-trivial because the conductor of ψ is $\mathfrak{p}$ (as opposed to $\mathfrak{o}$).

Assume now that $c \in c_0\varpi + \mathfrak{p}^2$ for some $c_0 \in \mathfrak{o}^*$. We see that the integrand vanishes unless $u^{-1} + c \in \mathfrak{p}^2$, so $u^{-1} = -u_0c_0\varpi$ with $u_0 \in 1 + \mathfrak{p}$ and we obtain

$$\tau(-1)\frac{q-1}{q^{1/2}}q^{1/2-s}\int_{1+\mathfrak{p}}\tau(-u_0c_0\varpi)\gamma_{\psi}^{-1}(-u_0c_0\varpi)\,d^{\times}u_0.$$

Since τ and γ_ψ are trivial on $1+\mathfrak{p}$, $(1+\mathfrak{p},\cdot) = 1$ and $\mathrm{vol}^{\times}(1+\mathfrak{p}) = (q-1)^{-1}$, the result follows. □

Claim 2.6.

$$\Psi(W,\phi,M(\tau,s)f_s)=\Psi(W,\phi,f_s)A(\tau,\psi,s)\tau(-1)(q-1)(L(2s-1,\tau^2)-1-\frac{\Lambda(\tau,\psi,s)}{q-1}).$$

Here

$$\Lambda(\tau,\psi,s)=\begin{cases}1 & \tau|_{\mathfrak{o}^{\times}}\equiv(\varpi,\cdot)\gamma_{\psi}|_{\mathfrak{o}^{\times}},\\ -\tau(\varpi)\gamma_{\psi}(\varpi)q^{1/2-s}G(\psi^{-1}) & \tau|_{\mathfrak{o}^{\times}}\equiv\gamma_{\psi}|_{\mathfrak{o}^{\times}},\end{cases}$$

and $G(\psi^{-1}) = \sum_{x\in\mathbb{F}_q^{\times}}\psi^{-1}(x)(\varpi,x)$ *is the Gauss sum over the multiplicative group of the finite field* $\mathbb{F}_q$ *with* q *elements.*

Proof. We begin as in the proof of Claim 2.4. In particular, W vanishes unless $a \in 1+\mathfrak{p}$, $x, c \in \mathfrak{p}$. We obtain

$$\begin{aligned}
&\Psi(W,\phi,M(\tau,s)f_s)\\
&=\beta_{\psi}^{-2}\gamma_{\psi}^{-1}(-1)\mathrm{vol}(\mathfrak{p})^{l}\mathrm{vol}(\mathfrak{o})\mathrm{vol}^{\times}(1+\mathfrak{p})\\
&\times\left(\int_{\mathfrak{p}^2}\psi^{-1}(c\varpi^{-1})[M(\tau,s)f_s]\left(\left(\begin{smallmatrix}1&\\c&1\end{smallmatrix}\right),1\right)\,dc+\int_{\mathfrak{p}\setminus\mathfrak{p}^2}\psi^{-1}(c\varpi^{-1})[M(\tau,s)f_s]\left(\left(\begin{smallmatrix}1&\\c&1\end{smallmatrix}\right),1\right)\,dc\right).
\end{aligned}$$

Now we apply Claim 2.5. Since $\psi^*(c\varpi^{-1})$ is trivial when $c \in \mathfrak{p}^2$, the first integral becomes

$$\mathrm{vol}(\mathfrak{p}^2)A(\tau,\psi,s)\tau(-1)(q-1)(L(2s-1,\tau^2)-1). \tag{2.17}$$

For the second integral, because $\psi^{-1}(c\varpi^{-1}) = \psi^{-1}(c_0)$, it equals

$$\begin{aligned}&q^{-s}\int_{\mathfrak{p}\backslash\mathfrak{p}^2}\psi^{-1}(c\varpi^{-1})\tau(c_0\varpi)\gamma_\psi^{-1}(-c_0\varpi)\,dc\\&=q^{-s}\int_{\mathfrak{p}\backslash\mathfrak{p}^2}\psi^{-1}(c_0\varpi^{-1})\tau(c_0\varpi)\gamma_\psi^{-1}(-c_0\varpi)\,dc\\&=q^{-s}\tau(\varpi)\gamma_\psi^{-1}(-\varpi)\int_{\mathfrak{p}\backslash\mathfrak{p}^2}\psi^{-1}(c_0\varpi^{-1})\tau(c_0)\gamma_\psi^{-1}(c_0)(c_0,-\varpi)\,dc.\end{aligned}$$

Using the integration formula

$$\int_{\mathfrak{p}\backslash\mathfrak{p}^2}\xi(c)\,dc = q^{-1}\int_{\mathfrak{o}^\times}\xi(v\varpi)\,dv = q^{-1}(q^{1/2}-q^{-1/2})\int_{\mathfrak{o}^\times}\xi(v\varpi)\,d^\times v,$$

and the fact that the Hilbert symbol is trivial on $\mathfrak{o}^\times\times\mathfrak{o}^\times$ (since p is odd), we reach

$$q^{-s}\tau(\varpi)\gamma_\psi^{-1}(-\varpi)\mathrm{vol}(\mathfrak{p}^2)(q-1)\int_{\mathfrak{o}^\times}\psi^{-1}(v)\tau(v)\gamma_\psi^{-1}(v)(\varpi,v)\,d^\times v. \tag{2.18}$$

Now if $\tau|_{\mathfrak{o}^\times}\equiv(\varpi,\cdot)\gamma_\psi|_{\mathfrak{o}^\times}$, the last $d^\times v$-integral becomes $(1-q)^{-1}$. Combining (2.17) with (2.18), and recalling that in this case $A(\tau,\psi,s)=q^{-s}\tau(\varpi)\gamma_\psi^{-1}(\varpi)$, the sum of integrals in the expression for $\Psi(W,\phi,M(\tau,s)f_s)$ equals

$$\mathrm{vol}(\mathfrak{p}^2)A(\tau,\psi,s)\tau(-1)(q-1)(L(2s-1,\tau^2)-1-\frac{1}{q-1}\tau(-1)\gamma_\psi^{-1}(-1)(-1,\varpi)).$$

Then our assumption on $\tau|_{\mathfrak{o}^\times}$ and Claim 2.4 imply

$$\Psi(W,\phi,M(\tau,s)f_s)=\Psi(W,\phi,f_s)A(\tau,\psi,s)\tau(-1)(q-1)(L(2s-1,\tau^2)-1-\frac{1}{q-1}).$$

For the case $\tau|_{\mathfrak{o}^\times}\equiv\gamma_\psi|_{\mathfrak{o}^\times}$, we observe that the integral in (2.18) is invariant under $1+\mathfrak{p}$, hence (2.18) equals

$$q^{-s}\tau(\varpi)\gamma_\psi^{-1}(-\varpi)\mathrm{vol}(\mathfrak{p}^2)\int_{\mathfrak{o}^\times/1+\mathfrak{p}}\psi^{-1}(w)(\varpi,w)\,d^\times w = q^{-s}\tau(\varpi)\gamma_\psi^{-1}(-\varpi)\mathrm{vol}(\mathfrak{p}^2)G(\psi^{-1}).$$

Together with (2.17) and since now $A(\tau,\psi,s)=q^{-1/2}$, the sum of two integrals equals

$$\mathrm{vol}(\mathfrak{p}^2)A(\tau,\psi,s)\tau(-1)(q-1)(L(2s-1,\tau^2)-1+\frac{1}{q-1}\tau(-\varpi)\gamma_\psi^{-1}(-\varpi)q^{1/2-s}G(\psi^{-1})).$$

As above, the result now follows from Claim 2.4. □

We can now deduce Theorem 1.1 for the case $\alpha=1$, i.e., α is a representative of the trivial square class in $\kappa_F^\times/(\kappa_F^\times)^2$:

Theorem 2.7. *Let τ be a tamely ramified quadratic character of $F^\times$, and recall that $\pi=\pi_1^\omega$. Then $\gamma(s,\pi\times\tau,\psi)$ has a pole at $s=1$ if and only if τ is the unique non-trivial quadratic character of $F^\times$ such that $\tau|_{\mathfrak{o}^\times}\equiv\gamma_\psi|_{\mathfrak{o}^\times}$ and $\tau(\varpi)=\gamma_\psi^{-1}(\varpi)\frac{|G(\psi^{-1})|}{G(\psi^{-1})}$.*

Proof. By Claim 2.4, $\Psi(W,\phi,f_s)$ is in particular nonvanishing, hence the zeros and poles of $\gamma(s,\pi\times\tau,\psi)$ may be read off the numerator of (2.10). Since $\tau^2=1$, $\gamma(2s-1,\tau^2,\psi_2)$ has a simple pole at $s=1$. Also note that $L(1,\tau^2)=(1-q^{-1})^{-1}$.

Assume the restriction of τ to $\mathfrak{o}^\times$ coincides with $(\varpi,\cdot)\gamma_\psi$. Then $\Lambda(\tau,\psi,s)=1$ hence by Claim 2.6, $\Psi(W,\phi,M(\tau,s)f_s)$ has a simple zero at $s=1$. This zero cancels with the pole of $\gamma(2s-1,\tau^2,\psi_2)$, and we deduce that $\gamma(s,\pi\times\tau,\psi)$ does not have a pole at $s=1$.

Now consider the case $\tau|_{\mathfrak{o}^\times}\equiv\gamma_\psi|_{\mathfrak{o}^\times}$. Then by Claim 2.6, $\Psi(W,\phi,M(\tau,s)f_s)$ has a zero at $s=1$ if and only if

$$\tau(\varpi)=-q^{1/2}\gamma_\psi^{-1}(\varpi)G(\psi^{-1})^{-1}.$$

Since $G(\psi^{-1})^2=(-1,\varpi)q$ ([BH06, (23.6.3)]), $G(\psi^{-1})=\zeta q^{1/2}$ for some fourth root of unity ζ and $G(\psi^{-1})/|G(\psi^{-1})|=\zeta$. Hence the last condition is equivalent to

$$\tau(\varpi)=-\gamma_\psi^{-1}(\varpi)\frac{|G(\psi^{-1})|}{G(\psi^{-1})}.$$

When $\tau(\varpi)$ satisfies this condition, the pole of $\gamma(1,\tau^2,\psi_2)$ is canceled. The remaining option is $\tau(\varpi)=\gamma_\psi^{-1}(\varpi)\frac{|G(\psi^{-1})|}{G(\psi^{-1})}$ (because $\tau^2(\varpi)=1$), then $\Psi(W,\phi,M(\tau,s)f_s)$ does not have a zero, so that $\gamma(s,\pi\times\tau,\psi)$ has a pole at $s=1$. □

Remark 2.8. *Note that for $\tau(\varpi)=\pm\gamma_\psi^{-1}(\varpi)\frac{|G(\psi^{-1})|}{G(\psi^{-1})}$, indeed $\tau^2(\varpi)=1$ (see* [Szp10, Lemma 3.4]*).*

Thus far we computed $\gamma(s,\pi_1\times\tau,\psi)$. In general, recall the representation π_α^ω, where α is a non-trivial class in $\kappa_F^\times/(\kappa_F^\times)^2$, constructed in Section 2.2. This representation is $\psi^{*,\alpha}$-generic, but $\psi^{*,\alpha}$ is conjugate to the character $u\mapsto\psi(\alpha\sum_{i=1}^{l}u_{i,i+1})$ of N_l, via the image of $\mathrm{diag}(\alpha^{l-1},\ldots,\alpha,1)\in\mathrm{GL}_l$ in T_l. Therefore $\gamma(\pi_\alpha^\omega\times\tau,\psi_\alpha)$ is defined.

Repeating all of the above computations with ψ_α instead of ψ, the only dependency on ψ_α is that it must have the same conductor as ψ. In particular, we need $\psi_\alpha(a^{-1}c\varpi^{-1})$ to be trivial precisely when $\psi(a^{-1}c\varpi^{-1})$ is. Since $|\alpha|=1$, this assumption holds and we deduce that $\gamma(s,\pi_\alpha^\omega\times\tau,\psi_\alpha)$ has a pole at $s=1$ if and only if τ is the unique non-trivial quadratic character of $F^\times$ such that $\tau(\varpi)=\gamma_{\psi_\alpha}^{-1}(\varpi)\frac{|G(\psi_\alpha^{-1})|}{G(\psi_\alpha^{-1})}$ and $\tau|_{\mathfrak{o}^\times}\equiv\gamma_{\psi_\alpha}^{-1}|_{\mathfrak{o}^\times}$. This completes the proof of Theorem 1.1.

2.6. **The Langlands parameter.** In this section we prove Theorem 1.2. In order to describe the Langlands parameter of a simple supercuspidal representation of Sp_{2l}, we need the parameterization of simple supercuspidal representations of general linear groups. Such representations are parameterized by a central character ω, a uniformizer, and a particular root of the evaluation of ω on the particular uniformizer. For the details see [AL16, § 3.1].

Recall that $\pi=\pi_\alpha^\omega$ is a simple supercuspidal representation of Sp_{2l} containing the character $\chi=\chi_\alpha^\omega$. Then χ is the restriction to Sp_{2l} of a unique character $\tilde\chi$ of $I^+_{\mathrm{GL}_{2l}}$, which is invariant under the involution θ defining Sp_{2l}. Let φ be the Langlands parameter of π. By [BHS], $\varphi=\varphi_1\oplus\varphi_2$, where φ_1 is an irreducible $2l$-dimensional representation of the Weil group W_F, which corresponds to a simple supercuspidal representation Π_1 of GL_{2l}, which on $I^+_{GL_{2l}}$, is given by $(\tilde\chi)^2|_{I^+_{\mathrm{GL}_{2l}}}$, and φ_2 is a 1-dimensional representation of GL_1, which equals τ_α by Theorem 1.1.

The involution θ is given by

$$\theta(g)=\begin{pmatrix} & J_l\\ -J_l & \end{pmatrix}^{-1}({}^tg^{-1})\begin{pmatrix} & J_l\\ -J_l & \end{pmatrix}.$$

A general character λ of $I^+_{\mathrm{GL}_{2l}}$ takes the form

$$\lambda(g) = \psi^{-1}(\sum_{i=1}^{2l-1} s_i v_{i,i+1} + s_{2l} v_{2l,1} \varpi^{-1}).$$

One can check, by root group calculations, that

$$\lambda(\theta(g)) = \psi^{-1}(-s_{2l-1}v_{1,2} - ... - s_{l+1}v_{l-1,l} + s_l v_{l,l+1} - s_{l-1}v_{l+1,l+2} - ... - s_1 v_{2l-1,2l} + s_{2l}v_{2l,1}\varpi^{-1}).$$

Then λ is θ invariant if and only if

$$s_{2l-1} = -s_1, \quad s_{2l-2} = -s_2, \quad \dots, \quad s_{l+1} = -s_{l-1}.$$

Assume that λ is θ invariant. Thus

$$\lambda(g) = \psi^{-1}(s_1 v_{1,2} + ... + s_{l-1}v_{l-1,l} + s_l v_{l,l+1} - s_{l-1}v_{l+1,l+2} + ... - s_1 v_{2l-1,2l} + s_{2l}v_{2l,1}\varpi^{-1}).$$

Restricting this to $I^+_{\mathrm{Sp}_{2l}}$, we get (since $v_{2l-1,2l} = -v_{1,2}$, $v_{2l-2,2l-1} = -v_{2,3}$, etc., in Sp_{2l})

$$\lambda|_{I^+_{\mathrm{Sp}_{2l}}}(g) = \psi^{-1}(2s_1 v_{1,2} + \ldots + 2s_{l-1}v_{l-1,l} + s_l v_{l,l+1} + s_{2l}v_{2l,1}\varpi^{-1}).$$

Therefore, setting

$$s_1 = \ldots = s_{l-1} = 1/2, \quad s_l = \alpha, \quad s_{2l} = 1,$$

we obtain a character λ_s, whose restriction to $I^+ = I^+_{\mathrm{Sp}_{2l}}$ coincides with that of χ. Moreover, λ_s is θ-invariant. We are interested in the square of λ_s, which is given by the tuple $(1, 1, \ldots, 1, 2\alpha, -1, -1, \ldots, -1, 2)$, where 2α is in the l-th position. More precisely, this character is given on $I^+_{\mathrm{GL}_{2l}}$ by

$$k \mapsto \psi(-k_{1,2} - k_{2,3} - \ldots - k_{l-1,l} - 2\alpha k_{l,l+1} + k_{l+1,l+2} + k_{l+2,l+3} + \ldots + k_{2l-1,2l} - \frac{2}{\varpi}k_{2l,1}).$$

Following the description in [AL16, § 3.1], we can conjugate this character by an element of $T(\mathfrak{o})$, to get the tuple $(1, 1, \ldots, 1, 1, (-1)^{l+1}4\alpha)$, which yields the character $k \mapsto \psi(k_{1,2} + k_{2,3} + \ldots + k_{2l-1,2l} + \frac{(-1)^{l+1}4\alpha}{\varpi}k_{2l,1})$ on $I^+_{\mathrm{GL}_{2l}}$. We call this character $\widetilde{\lambda}$. Since the uniformizer ϖ for π is already fixed, we conclude that the uniformizer for Π_1 is $\frac{1}{(-1)^{l+1}4\alpha}\varpi$.

Now assume $p \nmid 2l$. Then $\varphi_1 = \mathrm{Ind}_{W_E}^{W_F}(\xi)$ for some tamely ramified extension E/F of degree $2l$, and some character ξ of $E^\times$ (see [BH06]). Our goal now is to determine E and ξ. Let ζ be a $2l$-th root of $\frac{1}{(-1)^{l+1}4\alpha}\varpi$, and set $E = F(\zeta)$. Then ξ is a character of $E^\times = \langle\zeta\rangle \times \kappa_F^\times \times (1+\mathfrak{p}_E)$.

Relative to the basis

$$\zeta^{2l-1}, \zeta^{2l-2}, \cdots, \zeta, 1$$

of E/F, we have an embedding

$$\iota : E^\times \hookrightarrow \mathrm{GL}_n(F).$$

By [BHS], $\xi(x) = \widetilde{\lambda}(\iota(x))$, for all $x \in 1 + \mathfrak{p}_E$.

Since the determinant of φ is trivial (by [CKPSS04]), the central character ω_{Π_1} of Π_1 must satisfy $\omega_{\Pi_1} = \tau_\alpha^{-1}|_{F^\times} = \tau_\alpha|_{F^\times}$. Therefore, since $\omega_{\Pi_1} = \det(\varphi_1) = \det(\mathrm{Ind}_{W_E}^{W_F} 1_E) \otimes \xi|_{F^\times}$ (see [BH06]), we deduce $\xi|_{\kappa_F^\times} = \tau_\alpha|_{\kappa_F^\times} \otimes (\det(\mathrm{Ind}_{W_E}^{W_F} 1_E))^{-1}$. Thus, we have determined $\xi|_{1+\mathfrak{p}_E}$ and $\xi|_{\kappa_F^\times}$. It remains to find $\xi(\zeta)$.

According to [AL16, Corollary 3.12, § 3.4, § 3.6],

$$\xi(\zeta) = \delta \cdot \lambda_{E/F}(\psi_\alpha)^{-1},$$

where $\lambda_{E/F}(\psi_\alpha)$ is the Langlands constant ([Lan70]), and δ is the coefficient of $q^{1/2-s}$ in $\gamma(s, \Pi_1, \psi_\alpha)$. By the Langlands correspondence for general linear groups, this is precisely the coefficient of $q^{1/2-s}$ in $\gamma(s, \varphi_1, \psi_\alpha)$. Now, since

$$\gamma(s, \pi, \psi_\alpha) = \gamma(s, \varphi, \psi_\alpha) = \gamma(s, \varphi_1, \psi_\alpha)\gamma(s, \varphi_2, \psi_\alpha) = \gamma(s, \Pi_1, \psi_\alpha)\gamma(s, \tau_\alpha, \psi_\alpha),$$

we can compute δ using $\gamma(s, \pi, \psi_\alpha) = \gamma(s, \pi \times 1, \psi_\alpha)$ and $\gamma(s, \tau_\alpha, \psi_\alpha)$.

Theorem 2.9. $\gamma(s, \pi_\alpha^\omega, \psi_\alpha) = \omega(-I_{2l})^n \gamma(\psi_\alpha)^{-1}\gamma_{\psi_\alpha}^{-1}(\varpi)q^{1/2-s}$.

Proof. By Claim 2.6, (2.10) and since $\pi_\alpha^\omega(-I_{2l}) = \omega(-I_{2l})$,

$$\begin{aligned}
&\gamma(s, \pi_\alpha^\omega \times 1, \psi_\alpha) \\
&= \omega(-I_{2l})^n c(s, l, \tau)\gamma_{\psi_\alpha}(-1)\gamma(\psi_\alpha)\gamma(2s-1, \tau^2, (\psi_\alpha)_2)\frac{\Psi(W, \phi, M(\tau, s)f_s)}{\Psi(W, \phi, f_s)} \\
&= \omega(-I_{2l})^n \gamma_{\psi_\alpha}(-1)\gamma(\psi_\alpha)\epsilon(2s-1, 1, \psi_\alpha)\frac{1-q^{1-2s}}{1-q^{2s-2}}\frac{\Psi(W, \phi, M(\tau, s)f_s)}{\Psi(W, \phi, f_s)} \\
&= \omega(-I_{2l})^n \gamma_{\psi_\alpha}(-1)\gamma(\psi_\alpha)q^{-s}\gamma_{\psi_\alpha}^{-1}(\varpi)\epsilon(2s-1, 1, \psi_\alpha)\frac{1-q^{1-2s}}{1-q^{2s-2}}(q-1) \\
&\quad \times [(1-q^{1-2s})^{-1} - 1 - (q-1)^{-1}] \\
&= \omega(-I_{2l})^n \gamma(\psi_\alpha)^{-1}\gamma_{\psi_\alpha}^{-1}(\varpi)q^{1/2-s},
\end{aligned}$$

where we used $\epsilon(2s-1, 1, \psi_\alpha) = q^{2s-3/2}$ ([BH06, Proposition § 23.5]). □

We conclude:

- $\xi(x) = \widetilde{\lambda}(\iota(x))$, for all $x \in 1 + \mathfrak{p}_E$.
- $\xi|_{\kappa_F^\times} = \tau_\alpha|_{\kappa_F^\times} \otimes (\det(\mathrm{Ind}_{W_E}^{W_F} 1_E))^{-1}$,
- $\xi(\zeta) = \omega(-I_{2l})\gamma(\psi_\alpha)^{-1}\gamma_{\psi_\alpha}^{-1}(\varpi)\gamma(s, \tau_\alpha, \psi_\alpha)^{-1}\lambda_{E/F}(\psi_\alpha)^{-1}$,

This character ξ depends on ω and α (and ϖ, though we have suppressed the notation ϖ throughout), so we denote it ξ_α^ω. Then the Langlands paramter for $\pi = \pi_\alpha^\omega$ is $\varphi = \mathrm{Ind}_{W_E}^{W_F}\xi_\alpha^\omega \oplus \tau_\alpha$.

Now suppose that $p \neq 2$ and $p \mid 2l$. Then much of the above computations did not depend on whether or not p divided $2l$. The following conditions then characterize Π_1 uniquely (following again the notation of [AL16]): Π_1 is a simple supercuspidal representation of GL_{2l}, the uniformizer for Π_1 is $\frac{1}{(-1)^{l+1}4\alpha}\varpi$, the central character satisfies $\omega_{\Pi_1} = \tau_\alpha|_{F^\times}$, and $\gamma(s, \Pi_1, \psi_\alpha) = \omega(-I_{2l})^n\gamma(\psi_\alpha)^{-1}\gamma_{\psi_\alpha}^{-1}(\varpi)\gamma(s, \tau_\alpha, \psi_\alpha)^{-1}$. Then the work of [BH14] gives the parameter of Π_1 explicitly, and together with τ_α we get the full parameter of π_α^ω.

2.7. **The case $F = \mathbb{Q}_2$.**

In this case there is only one simple supercuspidal representation up to isomorphism, since the central element -1 is already contained in I^+ (see Section 2.2) and $\kappa_F^\times/(\kappa_F^\times)^2$ is trivial. We denote this representation by π. Moreover, for convenience in the computations, we will fix a specific uniformizer $\varpi = 2$. The end result does not depend on our choice of uniformizer, because of the aforementioned uniqueness. Also note that since in $\mathbb{Q}_2$, $\mathfrak{o}^\times = 1 + \mathfrak{p}$, a tamely ramified quasi-character is in fact unramified. Throughout this section, we will regularly refer to [Szp10, Lemma 3.5] for computations.

We still take the function W of Section 2.4, and we define f_s as in the $p \neq 2$ case except that we use the subgroup

$$\mathcal{N} = \left\{ \begin{pmatrix} 1+\mathfrak{p}^3 & \mathfrak{p}^2 \\ \mathfrak{p}^3 & 1+\mathfrak{p}^3 \end{pmatrix} \in \mathrm{Sp}_2 \right\}.$$

As opposed to the case of odd p, here we must prove directly that the cover is split over $\mathcal{N}$, in fact the trivial section provides a splitting:

Claim 2.10. *For any $v, v' \in \mathcal{N}$, $\langle v, 1\rangle\langle v', 1\rangle = \langle vv', 1\rangle$.*

Proof. Write $v = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $v' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$. Since $d, a' \in F^{\times 2}$, the result $\langle v, 1\rangle\langle v', 1\rangle = \langle vv', 1\rangle$ is clear from (2.1) when $c = 0$ or $c' = 0$. Now assume $c, c' \neq 0$.

Consider the case $ca' + dc' \neq 0$. Using $(x, -x) = 1$, We can rewrite (2.1) in the form

$$\sigma(v, v') = (c, c')(-cc', ca' + dc') = (c, c')(-cc', c(a' + dc'c^{-1})) = (-cc', a' + dc'c^{-1}).$$

By symmetry, we may assume $c' = ct$ for $|t| \leq 1$. Then $(-cc', a' + dc'c^{-1}) = (-t, a' + dt)$ and we can write $a' + dt = 1 + t + r$ for $r \in \mathfrak{p}^3$. Now $1 + r \in F^{\times 2}$ hence

$$(-t, a' + dt) = (-t(1 + r), a' + dt) = (1 - (a' + dt), a' + dt) = 1,$$

where we used $(1 - x, x) = 1$.

Now assume $ca' + dc' = 0$, then $c = -dc'a'^{-1} = -c't$ for $t \in F^{\times 2}$ (because $d, a' \in F^{\times 2}$), and also $cb' + dd' \in F^{\times 2}$. We obtain

$$\sigma(v, v') = (c, c')(-cc', cb' + dd') = (c, c') = (-c', c') = 1.$$

This completes the proof. ∎

Let ϕ be the characteristic function of $\mathfrak{o}$.

Claim 2.11. $\Psi(W, \phi, f_s) = \beta_\psi^{-2}\gamma_\psi(-1)\mathrm{vol}(\mathfrak{o})^2\mathrm{vol}^\times(1 + \mathfrak{p})\mathrm{vol}(\mathfrak{p}^3)\mathrm{vol}(\mathfrak{p})^{l-1}$.

Proof. Writing the dg-integration over $\overline{B}_1$,

$$\Psi(W, \phi, f_s) = \int_{\overline{B}_1} \int_{R^{l,1}} \int_F W({}^{\omega_{l-1,1}}(rxg))\omega_\psi(g)\phi(x)f_s(g, 1)\delta_{B_1}(b)\, dx\, dr\, db.$$

The support of W is contained in $U_l Z I^+$, which equals $U_l I^+$ since $F = \mathbb{Q}_2$. Looking at ${}^{\omega_{l-1,1}}(rxg)$ with $b = \left(\begin{smallmatrix} a & 0 \\ a^{-1}c & a^{-1} \end{smallmatrix}\right) \in \overline{B}_1$, we see that W vanishes unless $a \in 1 + \mathfrak{p}$, $x, c \in \mathfrak{p}$ and the coordinates of r_1 belong in $\mathfrak{p}$. We may also assume $c \neq 0$. Then $\delta_{B_1}(b) \equiv 1$ and

$$W({}^{\omega_{l-1,1}}(rxg)) = \psi^{-1}(a^{-1}c2^{-1}), \tag{2.19}$$

recalling that $\varpi = 2$. By Claim 2.2 and since ϕ is the characteristic function of $\mathfrak{o}$,

$$\omega_\psi(b)\phi(x) = (a^{-1}, c)\beta_\psi^{-2}\gamma_\psi^{-1}(a)\gamma_\psi(-1)\mathrm{vol}(\mathfrak{o})^2. \tag{2.20}$$

Also for $a \in 1 + \mathfrak{p}$, $f_s(b, 1) = 0$ unless $c \in \mathfrak{p}^3$, in which case

$$f_s(\langle \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right), 1\rangle, 1) = (a^{-1}, c)\gamma_\psi(a). \tag{2.21}$$

Therefore by (2.19)–(2.21),

$$\Psi(W, \phi, f_s) = \beta_\psi^{-2}\gamma_\psi(-1)\mathrm{vol}(\mathfrak{o})^2\mathrm{vol}^\times(1 + \mathfrak{p})\mathrm{vol}(\mathfrak{p}^3)\mathrm{vol}(\mathfrak{p})^{l-1},$$

as claimed. ∎

Claim 2.12. *Let* $a \in 1+\mathfrak{p}$, $0 \neq c \in \mathfrak{p}$, *and write* $c \in c_0 2 + c_1 2^2 + \mathfrak{p}^3$ *with* $c_0, c_1 \in \mathfrak{o}$. *Then*

$$[M(\tau,s)f_s]\left(\left(\begin{smallmatrix} a & \\ a^{-1}c & a^{-1}\end{smallmatrix}\right),1\right) = \begin{cases} 2\cdot A(\tau,\psi,s)\gamma_\psi^{-1}(-ac) & c_0 \neq 0, \\ 0 & c_0 = 0, c_1 \neq 0,\end{cases}$$

and if $c_0 = c_1 = 0$,

$$\begin{aligned}&[M(\tau,s)f_s]\left(\left(\begin{smallmatrix} a & \\ a^{-1}c & a^{-1}\end{smallmatrix}\right),1\right)\\ &= A(\tau,\psi,s)\gamma_\psi^{-1}(2)(-c,a)\gamma_\psi^{-1}(a)2(1+\psi(2^{-1}))(L(2s-1,\tau^2)-1).\end{aligned}$$

Here

$$A(\tau,\psi,s) = 2^{-s}\tau(2)\mathrm{vol}^\times(1+\mathfrak{p}^3).$$

Proof. By Claim 2.3, we need to compute

$$\frac{q-1}{q^{\frac{1}{2}}}\int_{F^\times}(-uc,a)\tau(u^{-1})|u|^{\frac{1}{2}-s}\gamma_\psi^{-1}(u^{-1}a)f_s\left(\left(\begin{pmatrix}1 & 0\\ a^2u^{-1}+c & 1\end{pmatrix},-1\right)\,d^\times u.\tag{2.22}$$

We recall again that $F = \mathbb{Q}_2$, so that $q = 2$. We may assume $a^2u^{-1}+c \in \mathfrak{p}^3$, since otherwise f_s vanishes. Since $a \in 1+\mathfrak{p}$, one can see that the condition $a^2u^{-1}+c \in \mathfrak{p}^3$ then implies that $u^{-1} \equiv -c \pmod{\mathfrak{p}^3}$.

First assume that $c \in \mathfrak{p}\setminus\mathfrak{p}^2$. Then $u^{-1} = -c\cdot v$, with $v \in 1+\mathfrak{p}^2$. One can see that (2.22) simplifies to

$$2^{-1/2}\tau(-1)2^{1/2-s}\int_{1+\mathfrak{p}^2}\tau(-cv)\gamma_\psi^{-1}(-cva)(v^{-1},a)\,d^\times v.$$

Since τ is tamely ramified, we have $\tau(-1) = 1$ whence $\tau(-cv) = \tau(2)$, and we get

$$2^{-s}\tau(2)\int_{1+\mathfrak{p}^2}\gamma_\psi^{-1}(-cva)(v,a)\,d^\times v.$$

Also because $\gamma_\psi^{-1}(-cva) = \gamma_\psi^{-1}(-ca)\gamma_\psi^{-1}(v)(-ca,v)$, this last integral equals

$$2^{-s}\tau(2)\gamma_\psi^{-1}(-ca)\int_{1+\mathfrak{p}^2}\gamma_\psi^{-1}(v)(-c,v)\,d^\times v.$$

The $d^\times v$-integral is invariant under $1+\mathfrak{p}^3$, and since $(1+\mathfrak{p}^2)/(1+\mathfrak{p}^3) = \{1,5\}$, using [Szp10, Lemma 3.5] we see that it equals $2\mathrm{vol}^\times(1+\mathfrak{p}^3)$. In summary, (2.22) equals

$$2^{-s}\tau(2)\gamma_\psi^{-1}(-ca)\cdot 2\mathrm{vol}^\times(1+\mathfrak{p}^3).$$

We now assume that $c \in \mathfrak{p}^2\setminus\mathfrak{p}^3$. Then $u^{-1} = -c\cdot v$, with $v \in 1+\mathfrak{p}$. The same analysis as in $\mathfrak{p}\setminus\mathfrak{p}^2$ gives us that (2.22) simplifies to

$$2^{-1/2}2^{1-2s}\tau^2(2)\gamma_\psi^{-1}(-ca)\int_{1+\mathfrak{p}}\gamma_\psi^{-1}(v)(-c,v)\,d^\times v,$$

Again the $d^\times v$-integral is invariant under $1+\mathfrak{p}^3$, but now $(1+\mathfrak{p})/(1+\mathfrak{p}^3) = \{\pm 1, \pm 5\}$, and by [Szp10, Lemma 3.5] this integral vanishes.

Finally assume $c \in \mathfrak{p}^3$. Then we must have $u^{-1} \in \mathfrak{p}^3$ and (2.22) equals

$$2^{-1/2}\tau(-1)(-c,a)\sum_{n=3}^{\infty}\int_{\mathfrak{o}^\times}\tau(2^n v)|2^{-n}v^{-1}|^{\frac{1}{2}-s}\gamma_\psi^{-1}(2^n va)(2^{-n}v^{-1},a)\,d^\times v.$$

Since τ is tamely ramified and $F = \mathbb{Q}_2$, we obtain

$$2^{-1/2}(-c,a)\sum_{n=3}^{\infty}\tau^n(2)2^{n(1/2-s)}\int_{1+\mathfrak{p}}\gamma_{\psi}^{-1}(2^n va)(2^{-n}v^{-1},a)\,d^{\times}v$$
$$= 2^{-1/2}(-c,a)\gamma_{\psi}^{-1}(a)\sum_{n=3}^{\infty}\tau^n(2)2^{n(1/2-s)}\gamma_{\psi}^{-1}(2^n)\int_{1+\mathfrak{p}}\gamma_{\psi}^{-1}(v)(2^n,v)\,d^{\times}v.$$

As above, we compute the $d^{\times}v$-integral using the representatives $\{\pm1,\pm5\}$. If n is even, by [Szp10, Lemma 3.5],

$$\int_{1+\mathfrak{p}}\gamma_{\psi}^{-1}(v)(2^n,v)\,d^{\times}v = \mathrm{vol}^{\times}(1+\mathfrak{p}^3)(\gamma_{\psi}^{-1}(1)+\gamma_{\psi}^{-1}(-1)+\gamma_{\psi}^{-1}(5)+\gamma_{\psi}^{-1}(-5)) = 0.$$

For odd n, by *loc. cit.* the integral equals $\mathrm{vol}^{\times}(1+\mathfrak{p}^3)(2+2\psi(2^{-1}))$. Also $\gamma_{\psi}(2)^{2m+1} = \gamma_{\psi}(2)$, for any integer m. Therefore (2.22) equals

$$2^{-1/2}\gamma_{\psi}^{-1}(2)(-c,a)\gamma_{\psi}^{-1}(a)\mathrm{vol}^{\times}(1+\mathfrak{p}^3)(2+2\psi(2^{-1}))\tau(2)2^{1/2-s}\left(\frac{1}{1-\tau^2(2)2^{1-2s}}-1\right).$$

Because τ is unramified, the factor in parentheses is $L(2s-1,\tau^2)-1$. This completes the proof. □

Claim 2.13.

$$\Psi(W, M(\tau,s)f_s) = 2\Psi(W,\phi,f_s)A(\tau,\psi,s)(1+\psi(2^{-1}))\gamma_{\psi}^{-1}(2)\frac{-1+\tau^2(2)2^{2-2s}}{1-\tau^2(2)2^{1-2s}}.$$

Proof. As in the proof of Claim 2.11, we compute the integral using $\overline{B}_1$, then for b given by (2.12), the support of W implies $a \in 1+\mathfrak{p}$ and $0 \neq c \in \mathfrak{p}$, the choice of ϕ implies $x \in \mathfrak{p}$, and we deduce (2.19) and (2.20). Since $\varpi = 2$ as our uniformizer, and by virtue of Claim 2.12, the integral $\Psi(W, M(\tau,s)f_s)$ becomes a constant

$$2A(\tau,\psi,s)\mathrm{vol}(\mathfrak{p})^{l-1}\gamma_{\psi}(-1)\mathrm{vol}(\mathfrak{o})^2\beta_{\psi}^{-2} \tag{2.23}$$

multiplied by the sum of integrals

$$\int_{1+\mathfrak{p}}\int_{\mathfrak{p}\setminus\mathfrak{p}^2}\psi^{-1}(a^{-1}c2^{-1})(a^{-1},c)\gamma_{\psi}^{-1}(a)\gamma_{\psi}^{-1}(-ac)\,dc\,da$$
$$= \int_{1+\mathfrak{p}}\int_{\mathfrak{p}\setminus\mathfrak{p}^2}\psi^{-1}(a^{-1}c2^{-1})\gamma_{\psi}^{-1}(-c)\,dc\,da \tag{2.24}$$

and

$$\gamma_{\psi}^{-1}(2)(1+\psi(2^{-1}))(L(2s-1,\tau^2)-1)\int_{1+\mathfrak{p}}\int_{\mathfrak{p}^3}\psi^{-1}(a^{-1}c2^{-1})\,dc\,da. \tag{2.25}$$

Note that we used $\gamma_{\psi}^{-1}(-ac) = \gamma_{\psi}^{-1}(-c)\gamma_{\psi}(a)(c,a)$ to obtain (2.24), and $(-1,a)\gamma_{\psi}^{-1}(a) = \gamma_{\psi}(a)$ for (2.25).

Regarding (2.24), Since ψ is of level one and $|a| = 1$, $\psi^{-1}(a^{-1}c2^{-1}) = -1$ for all c. Moreover, since the additive volume of $\mathfrak{p}\setminus\mathfrak{p}^2$ equals $(2^{-1/2}-2^{-3/2})$ and the multiplicative volume of $\mathfrak{o}^\times$ is 1,

$$\int_{\mathfrak{p}\backslash\mathfrak{p}^2} \gamma_\psi^{-1}(-c)\,dc = (2^{-1/2}-2^{-3/2})\gamma_\psi^{-1}(2)\mathrm{vol}^\times(1+\mathfrak{p}^3)(\sum_{v\in\{\pm1,\pm5\}}\gamma_\psi^{-1}(-v)(2,-v))$$
$$= (2^{-1/2}-2^{-3/2})\gamma_\psi^{-1}(2)\mathrm{vol}^\times(1+\mathfrak{p}^3)2(1+\psi(2^{-1})),$$

where the second equality follows using [Szp10, Lemma 3.5]. It follows that (2.24) equals

$$-2\mathrm{vol}^\times(1+\mathfrak{p})(2^{-1/2}-2^{-3/2})\gamma_\psi^{-1}(2)\mathrm{vol}^\times(1+\mathfrak{p}^3)(1+\psi(2^{-1})). \tag{2.26}$$

Also note that in (2.25) $\psi^{-1}(a^{-1}c2^{-1})$ is identically 1, so that the integral $dc\,da$ is a constant.

Combining (2.23), (2.25) and (2.26), we see that $\Psi(W, M(\tau,s)f_s)$ equals

$$2A(\tau,\psi,s)\mathrm{vol}(\mathfrak{p})^{l-1}\gamma_\psi(-1)\mathrm{vol}(\mathfrak{o})^2\beta_\psi^{-2}\mathrm{vol}^\times(1+\mathfrak{p})(1+\psi(2^{-1}))\gamma_\psi^{-1}(2)$$
$$\times\left(-2(2^{-1/2}-2^{-3/2})\mathrm{vol}^\times(1+\mathfrak{p}^3)+(L(2s-1,\tau^2)-1)\mathrm{vol}(\mathfrak{p}^3)\right).$$

To continue, since $q=2$,

$$\mathrm{vol}^\times(1+\mathfrak{p}^3)=2^{-2},\quad \mathrm{vol}(\mathfrak{p}^3)=2^{-5/2},\quad L(2s-1,\tau^2)=(1-\tau^2(2)2^{1-2s})^{-1}.$$

Thus

$$(-2(2^{-1/2}-2^{-3/2})\mathrm{vol}^\times(1+\mathfrak{p}^3)+(L(2s-1,\tau^2)-1)\mathrm{vol}(\mathfrak{p}^3)) = \mathrm{vol}(\mathfrak{p}^3)\frac{-1+\tau^2(2)2^{2-2s}}{1-\tau^2(2)2^{1-2s}}.$$

Comparing this to the result of Claim 2.11 yields the result. ☐

Theorem 2.14. *Assume that τ is tamely ramified. Then $\gamma(s,\pi\times\tau,\psi)=\tau(2)2^{1/2-s}$.*

Proof. Since by Claim 2.11, $\Psi(W,\phi,f_s)$ is nonzero, by definition

$$\gamma(s,\pi\times\tau,\psi)=\pi(-I_{2l})\tau(-1)^l\gamma(s,\tau,\psi)c(s,l,\tau)\frac{C(s,\tau,\psi)\Psi(W,\phi,M(\tau,s)f_s)}{\Psi(W,\phi,f_s)}.$$

Recall that by (2.9),

$$C(s,\tau,\psi)=\gamma_\psi(-1)\gamma(\psi)\frac{\gamma(2s-1,\tau^2,\psi_2)}{\gamma(s,\tau,\psi)}.$$

We also have $\pi(-I_{2l})=\tau(-1)=1$, because $-1\in1+\mathfrak{p}$ ($F=\mathbb{Q}_2$). Plugging in Claim 2.13 and because $A(\tau,\psi,s)=2^{-1-s}\tau(2)$ (note that $\mathrm{vol}^\times(1+\mathfrak{p}^3)=2^{-2}$) we obtain

$$\gamma(s,\pi\times\tau,\psi)=c(s,l,\tau)\gamma_\psi(-1)\gamma(\psi)\gamma(2s-1,\tau^2,\psi_2)$$
$$\times 2^{-1-s}\tau(2)(1+\psi(2^{-1}))\gamma_\psi^{-1}(2)\frac{-1+\tau^2(2)2^{2-2s}}{1-\tau^2(2)2^{1-2s}}.$$

According to the twisting property of the standard γ-factor (see e.g., [BH06, Lemma 1, §23.5]),

$$\gamma(2s-1,\tau^2,\psi_2)=|2|^{2s-3/2}\tau^2(2)\gamma(2s-1,\tau^2,\psi)=c(s,l,\tau)^{-1}2^{1/2}\gamma(2s-1,\tau^2,\psi).$$

Also by definition,

$$\gamma(2s-1,\tau^2,\psi)=\epsilon(2s-1,\tau^2,\psi)\frac{L(\tau^{-2},2-2s)}{L(\tau^2,2s-1)}=\epsilon(2s-1,\tau^2,\psi)\frac{1-\tau^2(2)2^{1-2s}}{1-\tau^{-2}(2)2^{2s-2}}.$$

Then using the value of $\epsilon(2s-1,\tau^2,\psi)$ given by [BH06, Proposition § 23.5],

$$\epsilon(2s-1,\tau^2,\psi)\frac{1-\tau^2(2)2^{1-2s}}{1-\tau^{-2}(2)2^{2s-2}}\times\frac{-1+\tau^2(2)2^{2-2s}}{1-\tau^2(2)2^{1-2s}}=2^{1/2}.$$

Therefore

$$\gamma(s,\pi\times\tau,\psi)=\gamma_\psi(-1)\gamma(\psi)(1+\psi(2^{-1}))\gamma_\psi^{-1}(2)\tau(2)2^{-s}.$$

To complete the computation note that $\gamma_\psi(2)=1$ (see [Szp10, (3.23)]), $\gamma_\psi(-1)=\psi(-2^{-1})$ ([Szp10, (3.21)]), and

$$\psi(-2^{-1})\gamma(\psi)(1+\psi(2^{-1}))=2^{1/2}.$$

This can be verified for both choices of ψ, which are $\psi(x)=e^{\pm\pi i x}$, using the formulas in [Rao93, p. 370] for $\gamma(\psi)$. □

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