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Formal Degrees and Adjoint γ-factors

(joint with F. Hiraga, T. Ikeda, Wee Teck Gan)

Motivating Example: G = cpt Lie group

π an irred rep'n of G.

Want to understand π (or its invariants)

We can do this using highest weights.

e.g. $\dim \pi$ = Weyl dimension formula

Want to do this for p-adic groups.

Now let G = connected reductive group / F,

F = finite extension of $\mathbb{Q}_p$.

Assume G is semisimple.

let π be an irred. rep of G $(= G(F))$

Today: Look at $\deg(\pi)$ = formal degree.

$\deg(\pi) \in \mathbb{R}_{>0}$

$\subset \mathbb{C}$

$\deg(\tilde{\pi})$ is defined by

$$\int_G (\tilde{\pi}(g)v_1, v_2)(\tilde{\pi}(g)v_3, v_4)\, dg$$

$$= \frac{1}{\deg(\tilde{\pi})} (v_1, v_3)\overline{(v_2, v_4)}$$

for $v_1, v_2, v_3, v_4 \in \tilde{\pi}$

Here $(,)$ is a Hermitian inner product on $\tilde{\pi}$.

dg is a Haar measure on G.

$\deg(\tilde{\pi})$ depends on choice of dg.

If G is cpt, and choose $\mathrm{vol}(G, dg) = 1$, then

$\deg(\tilde{\pi}) = \dim(\tilde{\pi})$.

Conjecture: (Rough version) (Hiraga) Assume LLC.

$\deg(\tilde{\pi})$ approximately equal to $\sim |\gamma(0, \tilde{\pi}, Ad, \psi)|$ for

$\tilde{\pi}$ is a discrete series

~~where $\gamma(s, \tilde{\pi}, Ad, \psi) = \varepsilon($~~

where

$$\gamma(s, \tilde{\pi}, Ad, \psi) = \varepsilon(s, \tilde{\pi}, Ad, \psi) \frac{L(1-s, \check{\tilde{\pi}}, Ad)}{L(s, \tilde{\pi}, Ad)}$$

$Ad : {}^L G \longrightarrow GL(Lie(\hat{G}))$ adjoint representation.

$\psi \in \hat{F}$

Remark:
- Where does it come from?
- This is a natural extension of Langlands conjecture in 1970's, on Plancherel measure.
- If $F = \mathbb{R}$ and G = compact, then conjecture follows from Weyl dimension formula.
- Precise version: Assume G is simply connected, quasi split.

Measures (Gan-Gross) Fix

$G \xrightarrow{\sim} G_0$ isomorphism over $\bar{F}$.

G_0 = split form of G.

Take a differential form ω_0 of top degree, defined over $\mathcal{O}_F$ with non-zero reduction.

$$\begin{array}{ccc} G & \xrightarrow{\sim} & G_0 \\ \omega & \longleftarrow & \omega_0 \end{array}$$

ω is a self dual measure on F wrt ψ determines a Haar measure dg.

LLC:

$$\Pi_{\text{tempered}}(G) = \coprod_{\phi} \Pi_\phi(G)$$

(ϕ ← L-packet)

a partition with reasonable conditions.

$\phi : W_F' \longrightarrow {}^L G$ tempered L-parameter

(stability): $S_\phi = \{ s \in \hat{G} : \mathrm{Int}(s)\phi = \phi \}$ ($\hat{G}$ ← dual group)

$$\mathcal{S}_\phi = S_\phi / S_\phi^\circ z(G)^\Gamma \qquad \Gamma = \mathrm{Gal}(\bar{F}/F).$$

Then $\Pi_\phi(G) \longleftrightarrow \Pi(\mathcal{S}_\phi) = \{ \text{irred. reps of } \mathcal{S}_\phi \}$

$\pi \longleftrightarrow \rho_\pi$ non canonical

s.t. $\sum_{\pi \in \Pi_\phi(G)} \dim \rho_\pi \, \Theta_\pi$ is stable.

* endoscopic character relations . .

- . . .

Conjecture: (Precise Version) Assume LLC.

Let $\phi : W_F' \longrightarrow {}^L G$ ~~packet~~

an elliptic L-parameter.

Then $\deg(\bar{\Pi}) = \dfrac{\dim \rho_{\bar{\Pi}}}{|S_\phi|} \, |\gamma(0, \bar{\Pi}, Ad, \psi)|$

$\gamma(0, \bar{\Pi}, Ad, \psi)$

for $\bar{\Pi} \in \Pi_\phi(G)$.

G semisimple + ϕ elliptic $\Rightarrow$ S_ϕ is finite

Remark: • Stability + endoscopic character relations + a stable Kazhdan's density

$\Rightarrow$ $\dfrac{\deg \bar{\Pi}}{\dim \rho_{\bar{\Pi}}}$ is constant on $\Pi_\phi(G)$

• OK if $G = GL(n)$ (Silberger - Zink)
or $SL(n)$ (Hiraga - Saito)

• Compatible with Kottwitz, Gross, Reeder, DeBacker/Reeder

Gross - Reeder $\rightarrow$ refined the conjecture by removing the absolute value.

How to prove this?

- Hard.

We can check the compatibility with functoriality.

Let H = another reductive connected group/F, and with some $\xi: {}^L H \to {}^L G$

Suppose $Ad \circ \xi \cong Ad_H \oplus r$,

r some rep'n of ${}^L H$.

If π_H on H transfers to π on G by ξ, then we should have

$$\gamma(s, \pi, Ad, \psi) = \gamma(s, \pi_H, Ad, \psi)\, \gamma(s, \pi_H, r, \psi).$$

Thus, by the conjecture,

$$\deg(\pi) \sim \deg(\pi_H)\, |\gamma(0, \pi_H, r, \psi)|.$$

Thm 1 (HII) Let $H = U(3)_{E/F}$,

E = quadratic extension.

Let $G = GL(3, F)$. For these groups, we know LLC. Let π_H be a stable d.s.

$\bar{\Pi} = BC(\bar{\pi}_H)$, a d.s. (Rogawski)

Then $\deg(\bar{\Pi}) = C \cdot \deg(\bar{\pi}_H) \cdot |$ Leading term at $s=0$ of $\gamma(s, \bar{\pi}_H, Ad \otimes \omega_{E/F}, \psi)|$

C: explicit constant

$\omega_{E/F}$: LCFT character of $F^\times$

Remark: let $\xi : {}^L H \longrightarrow {}^L G$, base change L-hom.

Then

$$Ad_G \circ \xi = Ad_H \oplus (Ad_H \otimes \omega_{E/F})$$

Since formal degree conjecture is known for $GL(n)$, the conjecture for stable d.s. of $U(3)$ is also true, and

$$\deg(\bar{\pi}_H) = \frac{1}{2} |\gamma(0, \bar{\pi}_H, Ad, \psi)|$$

(Gan-Ichino)

Thm 2: Let $H = O(2n)$, $G = Sp(2n)$

Suppose $\bar{\pi}_H$, $\bar{\Pi}$ d.s. and $\theta(\bar{\pi}_H) = \bar{\Pi}$. Then $\deg(\bar{\Pi}) = C \cdot \deg(\bar{\pi}_H) \cdot |\gamma^{PSR}(0, \bar{\pi}_H, std)|$ for some C

($\gamma^{PSR}(0, \bar{\pi}_H, std, \psi)$)

θ: Local theta correspondence

where Std = standard representation

PSR means Piatetski-Shapiro Rallis γ-factors. Moreover, $C=1$ if G is unramified.

Remark: ${}^L H \cong \{ h \times w \in O(2n,\mathbb{C}) \times W_F : \det(h) = \chi(w) \}$

where χ : quadratic character of $F^\times \leftrightarrow$ disc. of $O(2n)$

$$\xi : {}^L H \longrightarrow {}^L G = SO(2n+1) \times W_F$$

$$h \times w \longmapsto \begin{pmatrix} h & & \\ & \cdot\chi(w) & \\ & & \chi(w) \end{pmatrix} \times w$$

$$Ad_G \circ \xi = Ad_H \oplus \text{Std}.$$

Pf of Thm 1: Since $\bar{\Pi} = BC(\bar{\Pi}_H)$, $\exists A \in End(V_{\bar{\Pi}})$

s.t. $\bar{\Pi} \circ A = A \circ (\bar{\Pi} \circ \sigma)$ where

$$\sigma(g) = J\, {}^t\bar{g}^{-1} J^{-1} \quad \text{for } g \in GL(3,E)$$

$$J = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix}$$ Normalize so $A^2 = Id$.

A unique up to a sign.

Character Relation (Rogawski):

$$\Theta_{\pi}^{A} := \operatorname{trace}(\pi \circ A) = \operatorname{Transfer}_{H}^{G}(\Theta_{\pi_H})$$

$\deg(\pi_H)$ appears in Θ_{π_H}

$$\Theta_{\pi_H} = \sum_{\mathcal{O}_H \text{ nilpotent orbits of } \mathrm{Lie}(H)} c_{\mathcal{O}_H}(\pi_H)\, \hat{\mu}_{\mathcal{O}_H} \circ \log$$

near 0 zero

$$\Rightarrow c_0(\pi_H) \sim \deg(\pi_H) \qquad *$$

We can do the same for twisted character:

$$\Theta_{\pi}^{A} = \sum_{\mathcal{O} \text{ nilpotent orbits of } \mathrm{Lie}(G^{\sigma})} c_{\mathcal{O}}^{A}(\pi)\, \hat{\mu}_{\mathcal{O}} \circ \log$$

near zero (Clozel)

$$\text{This} \Rightarrow \left| c_{\mathcal{O}}^{A}(\pi) \right| = \left| c_{0}(\pi_H) \right|$$

Twisted analogue of $*$:

$$c_{\mathcal{O}}^{A}(\pi)(v, Av') \sim \deg(\pi)\, TO(1, f)$$

(Twisted orbital integral)

$v, v' \in V_\pi$, $f(g) = (\bar{\pi}(g)v, v')$

Remains to compute $T_0(1,f)$: Use Shahidi - Goldberg:

$$|T_0(1,f)| = |\text{Leading term at } s=0 \text{ of } \gamma(s, \bar{\pi}, Asai, \psi)| \cdot |(v, Av')|$$

Rmk: 1) This proof doesn't work if π_H is endoscopic

2) Proof works for $U(2n+1)$, $SO(2n+1)$ if character relations are available.