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## Freydoon Shahidi Lecture :

I. Functoriality, Trace Formula, Converse Theorems.
endoscopy, packets (L-packets and Arthur packets)
Intertwining operators, Eisenstein series, trace identities.

Let $G, G'$ connected reductive / $k$, $k$ = local or global.
If $k$ = global, have $\mathbb{A}_k$.
Have ${}^L G$, ${}^L G'$. Assume $G'$ is quasisplit.

Functoriality says: IF ${}^L G \longrightarrow {}^L G'$ is an L-hom,
i.e. factors

$$\begin{array}{ccc} {}^L G & \xrightarrow{\rho} & {}^L G' \\ & \searrow \quad \nearrow & \\ & W_k' & \end{array}$$

$({}^L G := {}^L G^\circ \rtimes W_k'.)$ such that

$\rho|_{{}^L G^\circ}$ is complex analytic, then

if $R(G) :=$ irred. adm. rep'ns of $G(k)$
(if $k$ = local) or automorphic rep'ns of $G(\mathbb{A}_k)$ (if $k$ = global)
(i.e. if appears in $L^2(G(k) \backslash G(\mathbb{A}_k))$),

then there should be a canonical dual

$\rho^*: R(G) \longrightarrow R(G')$.

## Local Langlands ~~Conj~~ Conjecture :

Jiang-Soudry proved LLC for $SO(\text{odd})$

An $^L$-parameter is called tempered if its projection onto $^L G^\circ$ (i.e. $\varphi : W_K' \to {}^L G \to {}^L G^\circ$) is bounded.

These parametrize tempered representations.

i.e. $\Phi(G)_{temp} \longleftrightarrow R(G)_{temp}$

The partition of $R(G)$ must be such that the sum of the ~~characters~~ distribution characters in an $L$-packet is a stable ~~character~~ distribution only when the $L$-packet consists solely of tempered representations.

↑ that is minimally stable

## Stable Distributions :

Defn: $\gamma, \gamma' \in G(k)$ are stably conjugate if $\exists\ g \in G(\bar{k})$ such that $g\gamma g^{-1} = \gamma'$.

$\left(G(\bar{k})\text{-conjugacy class of } \gamma\right) \cap G(k) =$ finite union of $G(k)$-conjugacy classes

Assume $\gamma$ is strongly regular semisimple.

Then define $O_\gamma(f) = \int_{G_\gamma(h)\backslash G(h)} f(g\gamma g^{-1})\, dg$

the <u>orbital integral</u> for $f \in C_c^\infty(G(h))$

<u>Stable orbital integral</u> is defined to be

$$SO_\gamma(f) = \sum_{\gamma' \text{ s.t. } \gamma' \text{ is stably conjugate to } \gamma} O_{\gamma'}(f)$$

$f \in C_c^\infty(G(h))$

Defn: Let $T$ = distribution on $G$. Then $T$ is <u>stable</u>

if $T(f) = 0$ whenever $SO_\gamma(f) = 0$ $\forall \gamma$

The idea of stabilization of trace formula is to write either side (the geometric or spectral sides) as a sum of stable distributions either on $G$ or on its "endoscopic" group $\hat{}$.

Let $k$ = local. Endoscopic groups $^H$ of $G$ are quasisplit groups defined as follows:

Let ~~$[illegible]$~~ $s$ be a semisimple element of ~~$^L H^\circ$~~.

It is a quadruple $(H, {}^L H, s, \xi)$

such that $s \in {}^L G^\circ$ s.t. ${}^L H^\circ = \text{Cent}_{{}^L G^\circ}(s)^\circ$

and $\xi : {}^L H \hookrightarrow {}^L G$ is ~~a~~ a particular embedding.

Then suppose $f \in C_c^\infty(G(k))$. Let $\gamma$ be a strongly regular semisimple element in $G(k)$. There is a norm map from stable conjugacy classes in $G(k)$ to stable conjugacy classes in $H(k)$ as follows:

${}^L G^\circ$ and ${}^L H^\circ$ share a maximal torus and then the map is just "transfer through this torus".

Conjecture: Given $f$, $\exists\, f^H \in C_c^\infty(H(k))$ such that

$$SO_\delta(f^H) = \sum_{\gamma' \text{ such that } \gamma' \text{ is stably conjugate to } \gamma} \Delta(\delta, \gamma')\, O_{\gamma'}(f)$$

This conjecture has now been proven by Ngo,

Note: The map $f \longmapsto f^H$ is at the core of the stabilization of the trace formula. Dual to this map, on the tempered range, should be those ~~distributions~~ stable distributions which were conjectured to exist through LLC. (i.e. the stable distributions comes from the tempered dual).

Let $G/k$, $H = H_s$ = endoscopic group, $s \in {}^L G^\circ$.

Recall our map $f \longmapsto f^H$.

Arthur Parameters: These are defined as

$\psi(G) = {}^L G^\circ$-orbits of maps $\psi : W_k' \times SL_2(\mathbb{C}) \longrightarrow {}^L G^{1/2}$ such that $\psi|_{W_k'} \in \Phi_{temp}(G^*)$

${}^L G^* \hookleftarrow G^*$ = quasi-split inner form

Define $\phi_\psi : W_k' \to {}^L G$ by $\phi_\psi(w) = \psi\left(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right)$

Then $\phi_\psi$ is ~~or~~ a non-tempered L-parameter.

The map $\psi \longmapsto \phi_\psi$ is an

injection $\psi(G) \hookrightarrow \Phi(G^*)$

Let $S_\psi := \mathrm{Cent}_{{}^L G^\circ}\left(\psi\left(W_k' \times SL_2(\mathbb{C})\right)\right)$

Define $\mathcal{S}_\psi := S_\psi / S_\psi^\circ$ = finite group (from ABV)

Conjecture: representations of $\mathcal{S}_\psi$ come into the dual expression of $f \longmapsto f^H$.

Well, $S_{\phi_\psi} \supset S_\psi$, so $\mathcal{S}_\psi \longrightarrow \mathcal{S}_{\phi_\psi} \longrightarrow 1$.

Irred. rep'ns of $\mathcal{S}_{\phi_\psi} = \Pi(S_{\phi_\psi}) \subset \Pi(S_\psi)$

Define $s_\psi := \psi\left(1, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right)$ and

define $\overline{s}_\psi :=$ image of $s_\psi$ in $\mathcal{S}_\psi$.

means the finite dim'l reps of $\mathcal{S}_{\phi_\psi}$, not direct product

Conjectures: (Arthur)

1) For each quasisplit group $G_1$ and for every $\psi_1 \in \Psi(G_1)$, there exists a stable distribution $f_1 \longmapsto f_1^{G_1}(\psi_1)$

which is a finite linear combination of irreducible characters of $G_1(k)$.

2) Let $\psi_H \in \Psi(H)$, $H = H_s$ = endoscopic. The way $s$ was defined, we may assume $s \in \tilde{S}_\psi$.

Recall $\xi : {}^L H \hookrightarrow {}^L G$. We have

$$W_k' \times SL_2(\mathbb{C}) \xrightarrow{\psi_H} {}^L H \xrightarrow{\xi} {}^L G$$

$$\psi$$

$\psi \in \Psi(G)$.

By the conjecture, we must have

$$f^H(\psi_H) = \sum_{\pi \in \Pi} \delta(s_\psi, s, \pi)\, \mathrm{tr}\, \pi(f)$$

where $\Pi_\psi$ := things that appear in the finite sum

~~All the above~~ This previous page is conjectural.

b) There exists $\rho : \mathcal{S}_\psi \longrightarrow \mathbb{C}^\times$ such that

$\rho(\mathcal{S}_\psi) = \pm 1$ and $\delta(s, \bar{\Pi})\,\rho(s)^{-1}$ depends

only on $\bar{s}$.

c) Let $\langle \bar{s}, \Pi/\rho \rangle := \delta(s, \bar{\Pi})\,\rho(s)^{-1}$. Then → this just means that we defined a function $\bar{s} \longmapsto \delta(s, \bar{\Pi})\rho(s)^{-1}$ that depends on $\bar{\Pi}$ and $\rho$

$\langle \bar{s}, \Pi/\rho \rangle$ is a character of a fin. dim'l

rep'n of $\mathcal{S}_\psi$. Moreover,

$$\langle \bar{s}_\psi s, \Pi/\rho \rangle = e_\psi(\bar{s}_\psi, \Pi/\rho)\langle \bar{s}, \Pi/\rho \rangle.$$

$$\delta(s_\psi, \Pi) = e_\psi[\bar{s}_\psi, \Pi(\rho)\,\rho(s_\psi)]\,d_\psi(\Pi)$$

where $d_\psi(\bar{\Pi}) := \langle 1, \Pi/\rho \rangle$ = degree of the

character $\langle \cdot, \Pi/\rho \rangle$.

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Examples of Fundamental Lemma: Examples of the map $f \mapsto f^H$

$F$ = local $\supset \mathcal{O}$

1) $G$ = quasisplit unramified group, i.e. splitting over unramified extension

$f = ch(G(\mathcal{O}))$ $\quad f^H = ch(H(\mathcal{O}))$

$f, f^H$ have matching orbital integrals.

2) $G = SL(2)$ $\quad H(F) = K^1$ = norm 1 elements for $[K:F] = 2$. $\quad \gamma \in K^1$.

Then $f^H(\gamma)$ = linear combination of orbital integrals.

(Labesse - Langlands 1976)

Recall: $\psi_H \in \Psi(H)$, $\psi \in \Psi(G)$

${}^LH \hookrightarrow {}^LG$ Had $\delta(s, \tilde{\pi})$, $\rho(s)$.

$$\psi_H(f^H) = \sum_{\tilde{\pi} \in \Pi_\psi} \delta(s_\psi s, \tilde{\pi}) \Theta_{\tilde{\pi}}(f)$$

$$S_\psi = \mathcal{S}_\psi / \mathcal{S}_\psi^0$$

Recall $\delta(s_\psi, \bar{\pi}) = e_\psi(\bar{\pi}) \langle 1, \bar{\pi}/\rho \rangle$

We wanted $\langle \bar{s}, \bar{\pi}/\rho \rangle := \delta(s, \bar{\pi})\, \rho(s)^{-1}$

$\|$

character of rep'n of $\mathcal{S}_\psi^\vee$

$d_\psi(\bar{\pi}) = \langle 1, \bar{\pi}/\rho \rangle = \deg \langle \bullet, \bar{\pi}/\rho \rangle$ where

$\bar{\pi} \longmapsto \langle \bullet, \bar{\pi}/\rho \rangle$.

$\bar{\pi}_1 \in \Pi_{\phi_\psi} \iff d_\psi(\bar{\pi}_1) = 1.$

(This is part of the conjecture)

Assume that $\exists\, \bar{\pi}_1 \in \Pi_{\phi_\psi}$ s.t. $d_\psi(\bar{\pi}_1) = 1.$

$$\langle s, \pi/\bar{\pi}_1 \rangle = \frac{\delta(s, \bar{\pi})\, \rho(s)^{-1}}{\delta(s, \bar{\pi}_1)\, \rho(s)^{-1}} = \frac{\langle \bullet \bar{s}, \bar{\pi}/\rho \rangle}{\langle 1, \bar{\pi}_1/\rho \rangle}$$

$$\psi_H(f^H) = \delta(s_\psi, s, \bar{\pi}_1) \sum_{\bar{\pi}_\psi} \langle s, \bar{\pi} / \bar{\pi}_1 \rangle \chi_{\bar{\pi}}(f)$$

$\psi_k$ = additive character of $k^+$

Compose $\psi_k$ with the splitting of $1 \to Int(G) \to Aut(G) \to Out(G) \to 1$

We then get a generic character of $U \subset B$

Conjecture: $\exists!$ element in every tempered L-packet which is $\chi$-generic
($k = \mathbb{R}$ is done by Vogan)

Langlands quotient of $\tilde{\pi}_\chi$ := standard module with the generic representation of the Levi.

$$\bar{\pi} \longleftarrow I(\sigma \otimes \nu) \text{ tempered}$$

$$\{\sigma\} \ni \sigma_\chi$$

$$\bar{\pi}_\chi \longleftarrow I(\sigma_\chi \otimes \nu)$$

Spectral side of the trace formula:

$$\text{Spectral side}(f) = \underbrace{S_G(f)}_{\text{distribution}} + \sum \text{constant} \cdot S_H(f^H)$$

($f = \bigotimes_v f_v$)

$$f_v \longmapsto f_v^H \quad , \quad f^H = \bigotimes_v f_v^H \quad , \quad H = \text{endoscopic, elliptic}$$

$$S_H(f^H) = \sum_{\psi_H \in \Psi(H)} \psi_H(f^H)$$

You have a parameter $\psi_H$ and compares to $\psi$ — That is functoriality. That

All of the above was endoscopic transfer

Non-endoscopic transfer:

$G = GL_2 \qquad {}^L G = GL_2(\mathbb{C})$

Consider homs $GL(2,\mathbb{C}) \xrightarrow{\text{Sym}^m} GL_{m+1}(\mathbb{C})$

~~Let~~

You get these by taking a polynomial $p(x,y)$ of degree $m$, and change variables

~~by the~~ in the following way: Let $g \in GL(2,\mathbb{C})$, then consider ~~p~~ $p((x,y)g)$.

$\mathrm{Sym}^m g \in GL_{m+1}(\mathbb{C})$.

Let $\Pi$ = cuspidal rep'n on $GL_2(\mathbb{A}_k)$

$\shortparallel$

$\otimes \Pi_v$.

$$\begin{tikzcd} \Pi_v \arrow[r, leftrightarrow] & \phi_v : W_k' \arrow[r] & GL_2(\mathbb{C}) \arrow[d, "\mathrm{Sym}^m"] \\ & & GL_{m+1}(\mathbb{C}) \end{tikzcd}$$

Thus from $\mathrm{Sym}^m(\phi_v)$ you get ~~by~~

(by Langlands) $\mathrm{Sym}^m \Pi_v$ = Rep'n of $GL_{m+1}(k_v)$.

Then define $\mathrm{Sym}^m(\Pi) := \bigotimes_v \mathrm{Sym}^m \Pi_v$

= rep'n of $GL_{m+1}(\mathbb{A}_k)$.

~~By Functoriality, this rep'n appears in $L^2$.~~

For $m = 2, 3, 4,$ we know that $Sym^m \pi$ is automorphic. (we don't know others)

<u>We need Converse Theorems</u>

Let $\pi$ = irred. repn of $GL_n(\mathbb{A}_k)$.

We have Rankin-Selberg L-functions

$L(s, \pi \times \sigma)$. $\sigma$ is cuspidal repn of some $GL_m(\mathbb{A}_k)$.

If $\pi = \bigotimes_v \pi_v$

$\sigma = \bigotimes_v \sigma_v$

Thm: Assume

a) $L(s, \pi \times \sigma)$ is entire

b) $L(s, \pi \times \sigma)$ is bounded in finite vertical strips

c) $L(s, \pi \times \sigma)$ satisfies functional equation

$$L(s, \pi \times \sigma) = \varepsilon(s, \pi \times \sigma) L(1-s, \tilde{\pi} \times \tilde{\sigma})$$

Then $\pi$ appears in $L^2(GL_n(k) \backslash GL_n(\mathbb{A}_k))$

The same holds if you tensor $\sigma$ with an unramified character $\chi$

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References for all this stuff:

1) Unipotent Automorphic Representations - ~~James Arthur~~.
- Conjectures, Asterisque 171-172 (1989)

2) Unipotent Automorphic Representations -
Global Motivations (Michigan Conference Proceedings (1989))

Both by James Arthur

Let $G$ = connected reductive / $k$, $k$ = local.
Assume $G$ = quasisplit.
Fix a minimal parabolic $P_0 = M_0 N_0$
Take $P = MN \supseteq P_0$, $N \subset N_0$, $M \supseteq M_0$.
Let $\sigma$ = irred. admissible repn of $M(k)$.
Let $A_M$ = split component of $M$ = largest tori in $Z(M)^\circ$. Let $w \in W(G, A_M)$.
Can assume $A_0$ = split component of $M_0$
So thus can take $w \in W(G, A_0)$

assume
$w(M) = M$, $\wedge$ $w(\sigma) \cong \sigma$ where

$(w\sigma)(\lambda) := \sigma(w^{-1}\lambda w)$

Theory of Eisenstein series $\Rightarrow$ global intertwining operator.

They play a central role in the development of the trace formula.

Locally, the way we do this is take

$\mathrm{Ind}_{M(k)N(k)}^{G(k)} (\sigma \otimes 1) =: I_p(\sigma)$.

Then given $w$, we can define operators (locally and globally) as integrals as follows:

Let $f \in I_p(\sigma)$. Define $N_w := N_0 \cap wN^-w^{-1}$

Then define the operator $(A(\sigma,w)f)(g) := \int_{N_w} f(w^{-1}ng)\,dn$

This integral converges absolutely if the central character of $\sigma$ has absolute value lying in a cone within the positive Weyl Chamber.

For the global version, $\sigma$ = global cuspidal rep'n of $M(\mathbb{A}_k)$ etc.

Assume $k$ = local

let ~~$\mathfrak{g}_M$ := Lie algebra of~~

let ${}^L\mathfrak{n}$ be the Lie algebra of ${}^LN$

${}^LM$ = L-group of $M$

Let $\gamma_w$ := adjoint action of ${}^LM$ on ${}^L\mathfrak{n}_w$

Lie algebra of L-group of $(w^{-1}N_w w)^-$

Let $\sigma$ = irred. adm. rep'n of $M(k)$.

Then By LLC, get a $\phi: W_k' \longrightarrow {}^LM$.

Compose with $\gamma_w$ so get

$$W_k' \longrightarrow {}^LM \xrightarrow{\gamma_w} \mathrm{Aut}({}^L\mathfrak{n}_w)$$

$\phi$ ↩ $\phi$

Thus we have a rep'n of $W_k'$, so can define

L-functions $L(s, \phi)$, $\varepsilon(s, \phi, \psi_k)$

(Artin L-functions, root numbers)

and $L(s, \gamma_w \phi)$, $\varepsilon(s, \gamma_w \phi, \psi_k)$.

Fact: $A(\sigma, w, w_2) = A(w_2(\sigma), w_1)\, A(\sigma, w_2)$

as long as $\ell(w, w_2) = \ell(w_1) + \ell(w_2)$.

But you can normalize these operators so that you don't need the condition on $\ell(w, w_2)$.

Note: The poles of $A(\sigma, w)$ are related to L-functions

We want to now normalize the operators ~~so that~~ as follows:

Define $\tilde{A}$ by

~~Set $\tilde{A}(\sigma, w) =$~~

$$\tilde{A}(\sigma, w) := \varepsilon(0, \tilde{\gamma}_w \phi, \psi_k) \frac{L(0, \tilde{\gamma}_w \phi)}{L(1, \tilde{\gamma}_w \phi)} A(\sigma, w)$$

and $A(\sigma, w)$ is from before.

Then

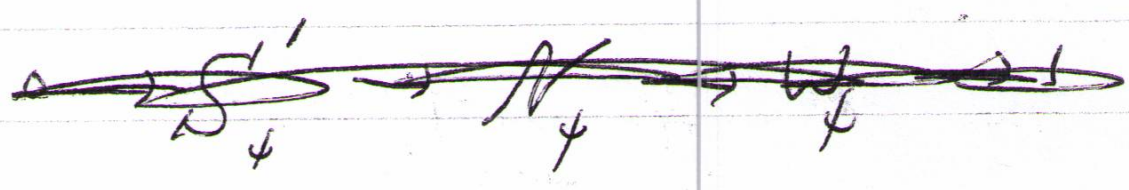
Fact: $\tilde{A}(\sigma, w, w_2) = \tilde{A}(w_2(\sigma), w_1)\, \tilde{A}(\sigma, w_2)$

(This is general. You don't have to assume $w(\sigma) \cong \sigma$. )

Let $W_\psi := $ Weyl group of $T_\psi$ in $S_\psi$.

$W_\psi^\circ := $ " " " " " $S_\psi^\circ$.

Define $S_\psi'$ and $R_\psi$ by




$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & W_\psi^\circ & = & W_\psi^\circ & & & & \\
 & & \downarrow & & & & & & \\
0 & \to & S_\psi' & \to & N_\psi & \to & W_\psi & \to & 1 \\
 & & \| & & \downarrow \alpha & & \downarrow & & \\
1 & \to & S_\psi' & \to & S_\psi & \to & R_\psi & \to & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
$$

"R-group of $\psi$"

$w S_\psi' \subset N_\psi$ is a coset in $W_\psi$.

Let $u \in N_\psi$, denote $\overline{u} := \alpha(u)$

Can talk about $\langle \bar{u}, \pi / \bar{\pi}_x \rangle$.

There is a theory of endoscopy for $M_w^+$

which leads to a character function as follows:

$$\langle \cdot, \sigma_w / \sigma_x \rangle.$$

There is a constant $c(\sigma_x, w) \in \mathbb{C}^\times$, and if

$\Lambda$ = whittaker functional, then $\Lambda_M \cdot \tilde{\sigma}_x(\bar{w}) = c(\sigma_x, \bar{w}) \Lambda_M$

where $\bar{w}$ fixes the splitting (pinning, épinglage)

$\lambda_w(\psi_h)$ = product of Langlands $\lambda$-functions.

= product of Hilbert symbols.

We should expect the following formula:

Conjecture: $c(\sigma_x, \bar{w})^{-1} \langle u, \sigma_w / \sigma_x \rangle \operatorname{trace}(A(\sigma_w) I_p(\sigma, f))$

$$= \lambda_w(\psi_h) \sum_{\bar{\pi} \in \Pi_\sigma(G)} \langle \bar{u}, \bar{\pi} / \bar{\pi}_x \rangle \chi_{\bar{\pi}}(f)$$

This is done in the Tempered case

and some generic cases (Shalidi, Keys-Shalidi, Ban,--)

Those two Arthur papers from the beginning of this talk were written to explain this last conjecture.