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LIFTING INVOLUTIONS IN A WEYL GROUP TO THE NORMALIZER OF THE TORUS

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ABSTRACT. Let N be the normalizer of a maximal torus T in a split reductive group over $\mathbb{F}_q$ and let w be an involution in the Weyl group N/T. We construct a section of W satisfying the braid relations, such that the image of the lift n of w under the Frobenius map is equal to the inverse of n.

1. INTRODUCTION

Let G be a connected reductive algebraic group over an algebraically closed field F. Let T be a maximal torus in G, and let $W = N/T$ denote the associated Weyl group, where N denotes the normalizer of T in G, and let $X_*(T)$ denote the cocharacter lattice of T. Fix a *realization* of the root system Φ in G (see §2), a set of positive roots Φ^+, and let Δ be the associated set of simple roots. We obtain the Tits section $w \mapsto \dot{w}$ of the natural map $N \to W$ [Tit66].

Let us recall some setup from a recent work of Lusztig [Lus18i]. If F is an algebraic closure of $\mathbb{F}_q$, we define $\phi : F \to F$ by $\phi(c) = c^q$, and if $F = \mathbb{C}$, we define $\phi : F \to F$ by $\phi(c) = \overline{c}$ (complex conjugation). In the first case, we assume that G has a fixed $\mathbb{F}_q$-rational structure with Frobenius map $\phi : G \to G$ such that $\phi(t) = t^q$ for all $t \in T$. In the second case, we assume that G has a fixed $\mathbb{R}$-structure so that $G(\mathbb{R})$ is the fixed point set of an antiholomorphic involution $\phi : G \to G$ such that $\phi(y(c)) = y(\phi(c))$ for all $y \in X_*(T), c \in F^\times$. We may also assume that $\phi(\dot{w}) = \dot{w}$ for any $w \in W$.

Now let w be an involution in W. In [Lus18i], a lift n of w was constructed such that $\phi(n) = n^{-1}$. The construction was quite complicated, and involved reduction arguments and case by case computations. In this paper, we construct a natural section $\mathcal{S}$ of the entire Weyl group W that satisfies the braid relations, which accomplishes the same result (namely, that $\phi(\mathcal{S}(w)) = \mathcal{S}(w)^{-1}$ for any involution w in W). Our methodology in one sense illustrates the power of the braid relations, allowing us to prove the main result quickly. We moreover note that in a recent work [Adr22], all sections of the Weyl group that satisfy the braid relations were computed, for an almost-simple connected reductive group over an algebraically closed field.

We explain our method. Let $\mathcal{S}$ be any section of W (by a *section* of W, we mean a section of the map $N \to W$). For $\alpha \in \Delta$, we may write $\mathcal{S}(s_\alpha) = t_\alpha \dot{s}_\alpha$ for some $t_\alpha \in T$, where $s_\alpha \in W$ is the simple reflection associated to α. Let $z_\alpha \in F^\times$ be arbitrary, with $\alpha \in \Delta$, and consider now the specific torus elements $t_\alpha = \alpha^\vee(z_\alpha)$.

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We show that the map $s_\alpha \mapsto t_\alpha \dot{s}_\alpha$ extends to a section, denoted $\mathcal{S}$, of W that satisfies the braid relations. We then prove that the condition $\phi(t_\alpha) = \alpha^\vee(-1)t_\alpha$ for all $\alpha \in \Delta$ implies that $\phi(\mathcal{S}(w)) = \mathcal{S}(w)^{-1}$ for any involution w in W. We conclude that if we define a section of W by the property $s_\alpha \mapsto \alpha^\vee(z_\alpha)\dot{s}_\alpha$ where $z_\alpha \in F^\times$ for all $\alpha \in \Delta$, with the property $\alpha^\vee(\phi(z_\alpha)z_\alpha^{-1}) = \alpha^\vee(-1)$, then we accomplish the goal of the paper. In the finite field case, this equality can be accomplished by setting z_α to be a $q-1$ root of -1 for every $\alpha \in \Delta$, and in the real case the equality can be accomplished by setting z_α to be a primitive fourth root of unity for every $\alpha \in \Delta$. Our main result, therefore, is:

Theorem 1.1. *If F is an algebraic closure of $\mathbb{F}_q$, set ζ to be a $q-1$ root of -1. If $F = \mathbb{C}$, set ζ to be a primitive fourth root of unity.*

For each $\alpha \in \Delta$, define the map $s_\alpha \mapsto \alpha^\vee(\zeta)\dot{s}_\alpha$. This maps extends to a section $\mathcal{S} : W \to N$ that satisfies the braid relations. Moreover, for any involution w in W, $\phi(\mathcal{S}(w)) = \mathcal{S}(w)^{-1}$.

One utility of finding a lifting of the involutions in the Weyl group, with the property that Frobenius acts by inversion, can be found in an ensuing paper of Lusztig [Lus18ii]. A key role is played by such a lifting in proving one of the main results (Theorem 0.4) of the paper.

2. Preliminaries

We first remind the reader of the definition of Tits' section from [Tit66], as well as some generalities about general sections. We follow [Spr98, §8.1, §9.3] closely. Let G be a connected reductive group over an algebraically closed field F, let T be a maximal torus in G, and let Φ be the associated set of roots. For each $\alpha \in \Phi$, let s_α denote the associated reflection in the Weyl group $W = N/T$.

Proposition 2.1 ([Spr98, Proposition 8.1.1])**.**
1. *For $\alpha \in \Phi$ there exists an isomorphism u_α of $\mathbf{G}_a$ onto a unique closed subgroup U_α of G such that $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ $(t \in T, x \in F)$.*
2. *T and the U_α $(\alpha \in \Phi)$ generate G.*

Tits then defines a representative σ_α, of s_α, in N:

Lemma 2.2 ([Spr98, Lemma 8.1.4])**.**
1. *The u_α may be chosen such that for all $\alpha \in \Phi$,*
$$\sigma_\alpha = u_\alpha(1)u_{-\alpha}(-1)u_\alpha(1)$$
lies in N and has image s_α in W. For $x \in F^\times$, we have
$$u_\alpha(x)u_{-\alpha}(-x^{-1})u_\alpha(x) = \alpha^\vee(x)\sigma_\alpha;$$
2. *$\sigma_\alpha^2 = \alpha^\vee(-1)$ and $\sigma_{-\alpha} = \sigma_\alpha^{-1}$;*
3. *If $u \in U_\alpha - \{1\}$ there is a unique $u' \in U_{-\alpha} - \{1\}$ such that $uu'u \in N$;*
4. *If $(u'_\alpha)_{\alpha\in\Phi}$ is a second family with the property (1) of Proposition 2.1 and property (1) of Lemma 2.2, there exist $c_\alpha \in F^\times$ such that*
$$u'_\alpha(x) = u_\alpha(c_\alpha x),\ c_\alpha c_{-\alpha} = 1\ (\alpha \in \Phi, x \in F).$$

A family $(u_\alpha)_{\alpha\in\Phi}$ with the properties (1) of Proposition 2.1 and Lemma 2.2 is called a *realization* of the root system Φ in G (see [Spr98, §8.1]).

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Fix a system of positive roots $\Phi^+ \subset \Phi$. Let Δ be the associated set of simple roots, and S the associated set of simple reflections. We have:

Proposition 2.3 ([Spr98, Proposition 8.3.3])**.** *Let μ be a map of S into a multiplicative monoid with the property: if $s, t \in S$, $s \neq t$, then*

$$\mu(s)\mu(t)\mu(s)\cdots = \mu(t)\mu(s)\mu(t)\cdots,$$

where in both sides the number of factors is $m(s,t)$. Then there exists a unique extension of μ to W such that if $s_1 \cdots s_h$ is a reduced decomposition for $w \in W$, we have

$$\mu(w) = \mu(s_1)\cdots\mu(s_h).$$

We now fix a realization $(u_\alpha)_{\alpha\in\Phi}$ of Φ in G. Let $\alpha, \beta \in \Phi$ be linearly independent. We denote $m(\alpha,\beta)$ the order of $s_\alpha s_\beta$. Then $m(\alpha,\beta)$ equals one of the integers $2, 3, 4, 6$.

Proposition 2.4 ([Spr98, Proposition 9.3.2])**.** *Assume that α and β are simple roots, relative to some system of positive roots. Then*

$$\sigma_\alpha\sigma_\beta\sigma_\alpha\cdots = \sigma_\beta\sigma_\alpha\sigma_\beta\cdots,$$

the number of factors on either side being $m(\alpha,\beta)$.

Following [Spr98, §9.3.3], let $w = s_{\alpha_1}\cdots s_{\alpha_h}$ be a reduced expression for $w \in W$, with $\alpha_1, \ldots, \alpha_h \in \Delta$. The element $\mathcal{N}_\circ(w) := \sigma_{\alpha_1}\cdots\sigma_{\alpha_h}$ is independent of the choice of reduced expression of w. We therefore obtain a section $\mathcal{N}_\circ : W \to N$ of the homomorphism $N \to W$. This is the section of Tits [Tit66].

3. The section $\mathcal{S}$

By Proposition 2.3, any section $\mathcal{S}$ of W satisfying the braid relations is determined by its values on a set of simple reflections. Let us write $\mathcal{S}(s_\alpha) = t_\alpha\sigma_\alpha$ for some $t_\alpha \in T$.

Let $\alpha, \beta \in \Delta$. In order that $t_\alpha\sigma_\alpha$ and $t_\beta\sigma_\beta$ satisfy the braid relations, it is necessary and sufficient that

$$t_\alpha\sigma_\alpha t_\beta\sigma_\beta t_\alpha\sigma_\alpha\cdots = t_\beta\sigma_\beta t_\alpha\sigma_\alpha t_\beta\sigma_\beta\cdots, \tag{1}$$

where in both sides the number of factors is $m(\alpha,\beta)$. As $\sigma_\alpha t_\beta \sigma_\alpha^{-1} = s_\alpha(t_\beta)$, and since the σ satisfy the braid relations, (1) is equivalent to

$$t_\alpha s_\alpha(t_\beta) s_\alpha s_\beta(t_\alpha)\cdots = t_\beta s_\beta(t_\alpha) s_\beta s_\alpha(t_\beta)\cdots. \tag{2}$$

Proposition 3.1. *Choose any $z_\alpha \in F^\times$, for $\alpha \in \Delta$. Then the map $s_\alpha \mapsto \alpha^\vee(z_\alpha)\sigma_\alpha$ extends to a section $\mathcal{S} : W \to N$ which satisfies the braid relations.*

Proof. As in the proof of [Spr98, Proposition 9.3.2], we need only check the cases $A_1 \times A_1, A_2, B_2$, and G_2.

$A_1 \times A_1$: We compute

$$\alpha^\vee(z_\alpha)s_\alpha(\beta^\vee(z_\beta)) = \alpha^\vee(z_\alpha)\beta^\vee(z_\beta) = \beta^\vee(z_\beta)s_\beta(\alpha^\vee(z_\alpha))$$

since s_α, s_β commute.

A_2: We compute

$$\begin{aligned} &t_\alpha s_\alpha(t_\beta)s_\alpha s_\beta(t_\alpha) \\ &\quad= \alpha^\vee(z_\alpha)s_\alpha(\beta^\vee(z_\beta))s_\alpha s_\beta(\alpha^\vee(z_\alpha)) = \alpha^\vee(z_\alpha)(\alpha+\beta)^\vee(z_\beta)s_\alpha(\alpha+\beta)^\vee(z_\alpha) \\ &\quad= \alpha^\vee(z_\alpha)(\alpha+\beta)^\vee(z_\beta)\beta^\vee(z_\alpha) = \alpha^\vee(z_\alpha)\alpha^\vee(z_\beta)\beta^\vee(z_\alpha)\beta^\vee(z_\beta) \end{aligned}$$

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since $s_\alpha(\beta^\vee) = s_\beta(\alpha^\vee) = (\alpha+\beta)^\vee = \alpha^\vee + \beta^\vee$. One computes the same result for $t_\beta s_\beta(t_\alpha) s_\beta s_\alpha(t_\beta)$.

B_2: Let α be the short root, β the long root. We have $s_\alpha(\beta^\vee) = \alpha^\vee + \beta^\vee = (2\alpha+\beta)^\vee$ and $s_\beta(\alpha^\vee) = \alpha^\vee + 2\beta^\vee = (\alpha+\beta)^\vee$. Thus,

$$\begin{aligned} & t_\alpha s_\alpha(t_\beta) s_\alpha s_\beta(t_\alpha) s_\alpha s_\beta s_\alpha(t_\beta) \\ &\quad = \alpha^\vee(z_\alpha) s_\alpha(\beta^\vee(z_\beta)) s_\alpha s_\beta(\alpha^\vee(z_\alpha)) s_\alpha s_\beta s_\alpha(\beta^\vee(z_\beta)) \\ &\quad = \alpha^\vee(z_\alpha)(2\alpha+\beta)^\vee(z_\beta) s_\alpha((\alpha+\beta)^\vee(z_\alpha)) s_\alpha s_\beta((2\alpha+\beta)^\vee(z_\beta)) \\ &\quad = \alpha^\vee(z_\alpha)(2\alpha+\beta)^\vee(z_\beta)(\alpha+\beta)^\vee(z_\alpha)\beta^\vee(z_\beta). \end{aligned}$$

One may now compute the same result for $t_\beta s_\beta(t_\alpha) s_\beta s_\alpha(t_\beta) s_\beta s_\alpha s_\beta(t_\alpha)$.

G_2: Let α be the short root, β the long root. We have $s_\alpha(\beta^\vee) = \alpha^\vee + \beta^\vee = (3\alpha+\beta)^\vee$ and $s_\beta(\alpha^\vee) = \alpha^\vee + 3\beta^\vee = (\alpha+\beta)^\vee$. Thus,

$$\begin{aligned} & t_\alpha s_\alpha(t_\beta) s_\alpha s_\beta(t_\alpha) s_\alpha s_\beta s_\alpha(t_\beta) s_\alpha s_\beta s_\alpha s_\beta(t_\alpha) s_\alpha s_\beta s_\alpha s_\beta s_\alpha(t_\beta) \\ &\quad = \alpha^\vee(z_\alpha) s_\alpha(\beta^\vee(z_\beta)) s_\alpha s_\beta(\alpha^\vee(z_\alpha)) s_\alpha s_\beta s_\alpha(\beta^\vee(z_\beta)) \\ &\qquad \cdot s_\alpha s_\beta s_\alpha s_\beta(\alpha^\vee(z_\alpha)) s_\alpha s_\beta s_\alpha s_\beta s_\alpha(\beta^\vee(z_\beta)) \\ &\quad = \alpha^\vee(z_\alpha)(\alpha^\vee+\beta^\vee)(z_\beta)(2\alpha^\vee+3\beta^\vee)(z_\alpha)(\alpha^\vee+2\beta^\vee)(z_\beta)(\alpha^\vee+3\beta^\vee)(z_\alpha)(\beta^\vee)(z_\beta). \end{aligned}$$

One may now compute the same result for

$$t_\beta s_\beta(t_\alpha) s_\beta s_\alpha(t_\beta) s_\beta s_\alpha s_\beta(t_\alpha) s_\beta s_\alpha s_\beta s_\alpha(t_\beta) s_\beta s_\alpha s_\beta s_\alpha s_\beta(t_\alpha). \qquad \square$$

We need the following result about involutions, see [Deo82, Theorem 5.4].

Proposition 3.2. *Any involution $w \in W$ can be obtained starting from the involution e by a sequence of length-increasing operations that are either multiplication of an involution by a simple reflection s_α with which it commutes, or conjugation by a simple reflection s_α with which it does not commute.*

Proof. By induction on the length $\ell(w)$. If $\ell(w) = 0$, then $w = e$. So suppose that $\ell(w) > 0$, and let α be a simple root such that $\ell(ws_\alpha) < \ell(w)$. Distinguish the cases on whether or not s_α commutes with w. If it commutes, then ws_α is an involution, and w is obtained from it by commuting multiplication by s_α. If they don't commute, then we can see as follows that $\ell(s_\alpha w s_\alpha) = \ell(w) - 2$. From $\ell(ws_\alpha) < \ell(w)$, there is a reduced expression for w that ends with s_α, and reversing it we obtain a reduced expression for $w^{-1} = w$ starting with s_α, say $s_\alpha s_{\alpha_2} \cdots s_{\alpha_{\ell(w)}}$. Now the exchange condition says that an expression for ws_α can be obtained by striking out one of the generators in the latter reduced expression, and it cannot be the initial s_α as that would give $s_\alpha w$ which is supposed to differ from ws_α. Now left-multiplying by s_α gives an expression for $s_\alpha w s_\alpha$ obtained by striking out a generator in $s_{\alpha_2} \cdots s_{\alpha_{\ell(w)}}$, and therefore of length $\ell(w) - 2$. This $s_\alpha w s_\alpha$ is an involution of shorter length than w, from which w can be obtained by conjugation by s_α with which it does not commute. $\square$

Proposition 3.3. *Choose any $z_\alpha \in F^\times$, for $\alpha \in \Delta$. Suppose that $t_\alpha = \alpha^\vee(z_\alpha)$, for $\alpha \in \Delta$. If $\phi(t_\alpha) = \alpha^\vee(-1)t_\alpha$ for all $\alpha \in \Delta$, then $\phi(\mathcal{S}(w)) = \mathcal{S}(w)^{-1}$ for any involution w in W.*

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Proof. We induct using Proposition 3.2. First we let $w = s_\alpha$, a simple reflection. Then $\phi(\mathcal{S}(w))\mathcal{S}(w) = \phi(t_\alpha\sigma_\alpha)t_\alpha\sigma_\alpha = \phi(t_\alpha)\sigma_\alpha t_\alpha\sigma_\alpha = \phi(t_\alpha)t_\alpha^{-1}\alpha^\vee(-1) = 1$, since $\phi(\dot{w}) = \dot{w}$ for all $w \in W$ and since $\sigma_\alpha t_\alpha \sigma_\alpha^{-1} = t_\alpha^{-1}$.

Now suppose $w \in W$ is an involution satisfying $\phi(\mathcal{S}(w)) = \mathcal{S}(w)^{-1}$. We need to show firstly that $\phi(\mathcal{S}(s_\alpha w)) = \mathcal{S}(s_\alpha w)^{-1}$ for any $\alpha \in \Delta$ such that s_α commutes with w and $\ell(s_\alpha w) > \ell(w)$, and secondly that $\phi(\mathcal{S}(s_\alpha w s_\alpha)) = \mathcal{S}(s_\alpha w s_\alpha)^{-1}$ for any $\alpha \in \Delta$ with $\ell(s_\alpha w s_\alpha) > \ell(w)$, such that s_α and w do not commute.

Suppose that $\alpha \in \Delta$, s_α commutes with w, and $\ell(s_\alpha w) > \ell(w)$. Then

$$\begin{aligned}\phi(\mathcal{S}(s_\alpha w))\mathcal{S}(s_\alpha w) &= \phi(t_\alpha\sigma_\alpha\mathcal{S}(w))\mathcal{S}(ws_\alpha) = \phi(t_\alpha)\sigma_\alpha\mathcal{S}(w)^{-1}\mathcal{S}(w)t_\alpha\sigma_\alpha \\ &= \phi(t_\alpha)t_\alpha^{-1}\alpha^\vee(-1) = 1,\end{aligned}$$

where in the above we are using that $\mathcal{S}$ satisfies the braid relations and that s_α commutes with w.

Moreover, for any $\alpha \in \Delta$ with $\ell(s_\alpha w s_\alpha) > \ell(w)$, we also have

$$\begin{aligned}&\phi(\mathcal{S}(s_\alpha w s_\alpha))\mathcal{S}(s_\alpha w s_\alpha) = \phi(t_\alpha\sigma_\alpha\mathcal{S}(w)t_\alpha\sigma_\alpha)t_\alpha\sigma_\alpha\mathcal{S}(w)t_\alpha\sigma_\alpha \\ &= \phi(t_\alpha)\sigma_\alpha\mathcal{S}(w)^{-1}\phi(t_\alpha)\sigma_\alpha t_\alpha\sigma_\alpha\mathcal{S}(w)t_\alpha\sigma_\alpha = \phi(t_\alpha)\sigma_\alpha\mathcal{S}(w)^{-1}\phi(t_\alpha)t_\alpha^{-1}\alpha^\vee(-1)\mathcal{S}(w)t_\alpha\sigma_\alpha \\ &= \phi(t_\alpha)\sigma_\alpha t_\alpha\sigma_\alpha = \phi(t_\alpha)t_\alpha^{-1}\alpha^\vee(-1) = 1,\end{aligned}$$

where again we are using that $\mathcal{S}$ satisfies the braid relations, and that s_α and w do not commute. □

Proposition 3.4.

(1) *Let F be an algebraic closure of $\mathbb{F}_q$. Set ζ to be a $q-1$ root of -1. If $t_\alpha = \alpha^\vee(\zeta)$ for all $\alpha \in \Delta$, then $\phi(t_\alpha) = \alpha^\vee(-1)t_\alpha$ for all $\alpha \in \Delta$.*

(2) *Let $F = \mathbb{C}$. Set ζ to be a primitive fourth root of unity. If $t_\alpha = \alpha^\vee(\zeta)$ for all $\alpha \in \Delta$, then $\phi(t_\alpha) = \alpha^\vee(-1)t_\alpha$ for all $\alpha \in \Delta$.*

Proof. For (1): Note that $\phi(t_\alpha) = t_\alpha^q$, so if we set $t_\alpha = \alpha^\vee(z_\alpha)$, then the desired equality $\phi(t_\alpha) = \alpha^\vee(-1)t_\alpha$ is equivalent to $\alpha^\vee(z_\alpha^{q-1}) = \alpha^\vee(-1)$. Setting z_α to be a $q-1$ root of -1 gives us our result.

For (2): Recall that $\phi(y(c)) = y(\phi(c))$ for all $y \in X_*(T), c \in F^\times$. Setting $t_\alpha = \alpha^\vee(z_\alpha)$, then the desired equality $\phi(t_\alpha) = \alpha^\vee(-1)t_\alpha$ is equivalent to $\alpha^\vee(\overline{z_\alpha}z_\alpha^{-1}) = \alpha^\vee(-1)$. Setting z_α to be a primitive fourth root of unity gives us our result. □

By Proposition 3.1, Proposition 3.3, and Proposition 3.4, we have now proven Theorem 1.1.

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