

07-20-09

Jeff Adams Lecture :

1:30 PM

Example, using Atlas software, of the representations of $Sp(4, \mathbb{R})$

empty : block

Lie Type: C2 sc s

enter your choice: 2

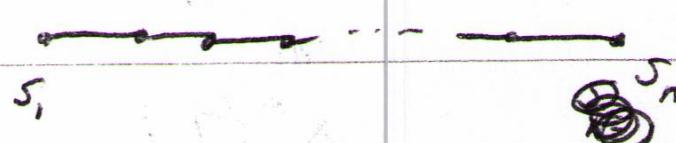
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Name an output file: ~~choice~~ just hit return



Question: Take G real reductive and a repn $(\tilde{\pi}, V)$ of G ,
How do you turn this into a finite computation?

Summary of Peter's ~~lecture~~ lecture



is a Coxeter graph. The Coxett group is

$$\langle \{s_i\} \rangle \text{ st. } s_i^2 = 1, (s_i s_j)^{m(i,j)} = 1$$

where $m(i,j) = \begin{cases} 2 & \text{no bond} \\ ? & \text{if } \exists \text{ bond between } s_i, s_j \end{cases}$

We can also write down the Hecke algebra

Let's ~~generate~~ ~~at~~ study data for real Reductive groups

Reductive Groups; Reductive Lie algebras

complex

Reductive Lie algebra is $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_n \oplus \mathfrak{f}$ where
 \mathfrak{g}_i is simple, \mathfrak{f} is abelian. The \mathfrak{g}_i are classified by
 their root systems.

Question: What are the corresponding real reductive groups?

Ex: $G = GL(n, \mathbb{C})$. $T = \text{split torus} = \left\{ \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \right\}$

Roots are $e_{ij} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = z_i/z_j$

Look by Δ the set of roots

Denote $d_{ij} := e_i - e_j$, $d_{ij}^v := e_i - e_j \in X_\alpha (+)$

$$X^*(T) \quad d_{ij}^*(z) = (1, 1, \dots, 1, z_i, \dots, z_j, \dots)$$

Lemma

1) $\Delta \longleftrightarrow \Delta^\vee$ $\alpha_{ij} \longleftrightarrow \alpha_{ij}^\vee$ bijections 2) $\langle \alpha_i, \beta^\vee \rangle \in \mathbb{Z}$

$$3) \quad \text{ ~~Δ~~ } \quad \langle \Delta, \Delta^{\vee} \rangle = 2 \quad 4) \quad s_{\Delta}(\Delta) = \Delta \quad 5) \quad s_{\Delta^{\vee}}(\Delta^{\vee}) = \Delta^{\vee}$$

6

where $s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha$.

Defn: ~~(X(T),)~~ $(X^*(T), A, X_*(T), \Delta^\vee)$ is
the root datum.

Root data classify all connected reductive groups

$$\text{Lemma: } \{ \text{Root Data} \} \underset{\cong}{\sim} \left\{ \begin{array}{l} \{ (A, B) : A, B \in M_{m \times n}(\mathbb{Z}) \text{ s.t. } \\ {}^t A^{-1} B \text{ is a Cartan matrix} \end{array} \right\}$$

where $(A, B) \sim ({}^t g^{-1} A P, g^{-1} B P)$, $g \in GL(m, \mathbb{Z})$,
 P = permutation matrix.

(A Cartan matrix is a matrix of the form $\{ \langle \alpha_i, \alpha_j^\vee \rangle \}_{ij}$ for a root system given by $\{\alpha_i\}$.)

Note: $m = \text{rank}$, $n = \text{semisimple rank}$

Exercise: If $n=1$, $m=2$, then the cardinality of
 $\{ \text{Root Data} \} \cong$ is 3.

Reductive Groups : Ex: $GL(n, \mathbb{C})$, $Sp(2n, \mathbb{C})$, $SO(n, \mathbb{C})$, ...

Examples of real reductive groups : ~~$GL(n, \mathbb{R}), U(p, q)$~~ ,

$GL(n, \mathbb{R})$

$U(p, q) = \{ g \in GL(n, \mathbb{C}) : g J_{p,q} {}^{t-} \bar{g} = J_{p,q}, \bar{g}^t = g^{-1} \}$

$SO(p, q)$

$O(p, q)$

$M_r(\mathbb{R}, \mathbb{C})$

$$J_{p,q} = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$$

(3)

Thm. 1) Given a root datum D , then $\exists G, T$
s.t. the ~~root~~ root datum of (G, T) is $\oplus D$.

2) Two groups with the same root data are isomorphic.

Jeff Adams' lecture

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Atlas: Root Datum of $GL(5, \mathbb{R})$

empty: root datum

Lie Type: A4.T1

elements of finite order... : $\frac{1}{5}, \frac{1}{5}$

Enter inner class(es): ss

The root basis matrix is the A

" " " "

co-root " " " "

in Adams' notation from page 3 of his last lecture

$\delta^t A B =$ the Cartan matrix given in the Atlas output

let ~~G~~ G = complex connected reductive

$\text{Lie}(G) = \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_n \oplus \mathbb{C}$

~~with~~ let $G_{sc} = G_1 \times \dots \times G_n \times T$ where each G_i is simply connected, simple s.t.

$\text{Lie}(G_i) = \mathfrak{g}_i$, $T = (\mathbb{C}^\times)^m$

Ex: $\mathfrak{g}_i = SO(n, \mathbb{C})$, then $G_i = Spin(n, \mathbb{C})$

①

Then $\text{Lie}(G_{sc}) = \mathfrak{o}$

Thm: If G is connected reductive, then \exists simple, simply connected G_1, G_2, \dots, G_n and a $T = (\mathbb{C}^\times)^m$

and a finite subgroup A of the center of
 $G_1 \times \dots \times G_n \times T$ s.t.

$$G \cong (G_1 \times \dots \times G_n \times T) / A$$

Pf: Key fact is that $G = G_d \cdot Z$ where
 G_d is the derived group and Z is the central torus

So to define G , we need to specify the types of
 G_1, \dots, G_n , the rank of T , and A .

Ex: $GL(5, \mathbb{C}) = \frac{\{SL(5, \mathbb{C}) \times \mathbb{C}^\times\}}{\{\{(\xi I, \xi^{-1})\}}$

where ξ is a primitive 5th root of unity.

In the Atlas software, this group is

$$A4.T1, (1/5, 4/5)$$

Ex: $\text{Spin}(16, \mathbb{C}) . \quad Z(\text{Spin}(16, \mathbb{C})) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

$$SO(16, \mathbb{C}) = \frac{\text{Spin}(16, \mathbb{C})}{\langle (-1, -1) \rangle}$$

$\{\text{root data}\} \not\cong$

Compact Groups :

Thm: $\{\text{connected compact real Lie groups}\} \not\cong \{\text{connected complex reductive groups}\}$

~~the~~ the
compact
real form of G

$G \rightarrow$ The algebraic envelope of
 G

Thm: Let G be a compact connected Lie group. Then $\exists G_1, \dots, G_n$ simple simply connected compact Lie groups,
a torus $T = (\mathbb{S}^1)^m$, and a finite subgroup
A of $G_1 \times \dots \times G_n \times T$ s.t.

$$G \cong G_1 \times \dots \times G_n \times T$$

Real reductive groups : Real Forms :

Ex: $GL(n, \mathbb{R})$ is a real form of $GL(n, \mathbb{C})$ because

$$GL(n, \mathbb{R}) = GL(n, \mathbb{C})^P \text{ where } P = G \times (\mathbb{C}/\mathbb{R})$$

and $\forall \sigma \in P$ acts as $\sigma(g) = \bar{g}$. We will write $GL(n, \mathbb{C})^P$

(3)

Defn:

$$\text{Ex: } \text{GL}(n, \mathbb{C}), \quad \sigma_{p,q}(g) := J_{p,q}^{-t} \bar{g}^{-1} J_{p,q}$$

$$J_{p,q} = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix} \quad \text{Then } \text{GL}(n, \mathbb{C})^{\sigma_{p,q}} = U(p, q).$$

$$= \{ g \in \text{GL}(n, \mathbb{C}) : (gv, gw) = (v, w) \text{ where }$$

$$(v, w) := v^t J_{p,q} \bar{w} \quad \text{--- Hermitian form} \\ \text{of } \text{GL}(n, \mathbb{C}) \quad \text{signature } p, q \}$$

$$\sigma_{n,0} = \text{get } U(n, 0) = U(n) = \{ g : g^t \bar{g} = \text{Id} \}$$

conjugacy class of

Defn: A real form of a complex G is a subgroup G^σ of G which is the fixed points of an anti-holomorphic involution σ of G .

Ex: We want to identify $\text{GL}(n, \mathbb{R})$ with its conjugates $g \text{GL}(n, \mathbb{C}) g^{-1}$, which are $(\text{GL}(n, \mathbb{C}))^{\sigma'}$ where

$$\sigma' = \text{int}(g) \circ \sigma \circ \text{int}(g)^{-1}$$

Now, let's try to work with holomorphic involutions rather than anti-holomorphic involutions:

$$\text{GL}(n, \mathbb{R}) \supset K_{\mathbb{R}} = O(n) = \{ g : g^t g = \text{Id} \}$$

$$K_{\mathbb{R}} = \text{GL}(n, \mathbb{R})^{\theta} \quad \text{where } \theta(g) = {}^t g^{-1}.$$

Extend θ to $\text{GL}(n, \mathbb{C})$:

$$\text{Defn: } K = \text{GL}(n, \mathbb{C})^{\theta} = O(n, \mathbb{C})$$

Defn: A symmetric subgroup of G is a group of the form G^{θ} where θ is an involution.

$$\text{Ex: } U(p, q) = \left\{ g : g \begin{pmatrix} I_p & \\ - & I_q \end{pmatrix} {}^t g^{-1} = \begin{pmatrix} I_p & \\ - & I_q \end{pmatrix} \right\}$$

$$U(p) \times U(q) \quad \text{Then } U(p) \times U(q) = K_{\mathbb{R}}.$$

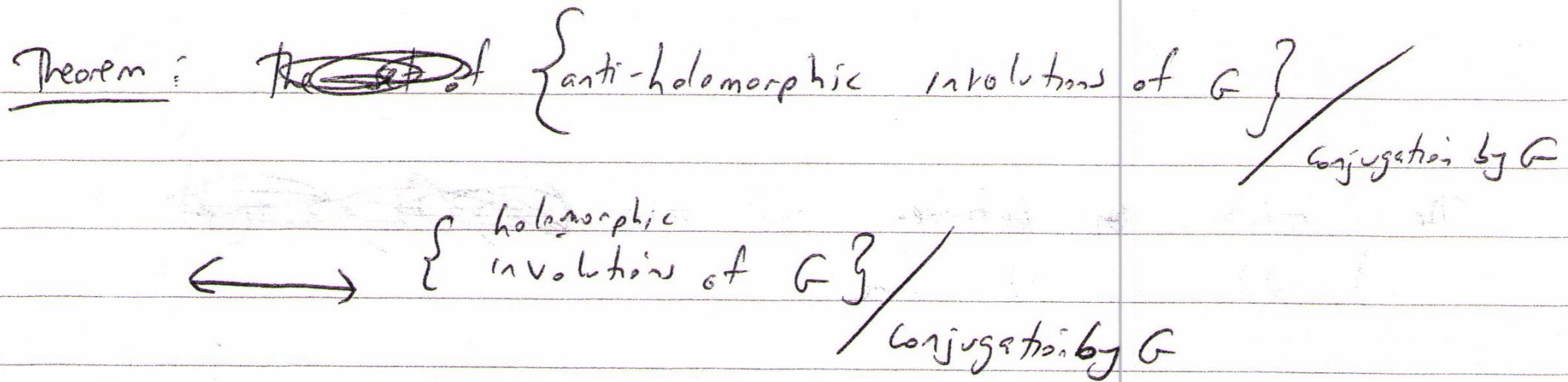
$$\theta(g) = J_{p,q} g J_{p,q}^{-1}. \quad U(p) \times U(q) = U(p, q)^{\theta}.$$

$$\text{Also } \text{GL}(p, \mathbb{C}) \times \text{GL}(q, \mathbb{C}) = \text{GL}(n, \mathbb{C})^{\theta}.$$

Table:

$$G = \text{GL}(n, \mathbb{C})$$

$G(\mathbb{R})$	$\text{GL}(n, \mathbb{R})$	$U(p, q)$
σ	$g \mapsto \bar{g}$	$g \mapsto J_{p,q} {}^t \bar{g}^{-1} J_{p,q}^{-1}$
θ	$g \mapsto {}^t g^{-1}$	$g \mapsto J_g J_g^{-1}$
$K_{\mathbb{R}}$	$O(n)$	$U(p) \times U(q)$
K	$O(n, \mathbb{C})$	$E_1(\mathbb{R}) \times \text{GL}(n, \mathbb{C})$



Pf: One direction: Given σ , $\sigma(G(\mathbb{R})) = G^\sigma$, pick maximal compact $K_{\mathbb{R}}$, pick θ such that $K_{\mathbb{R}} = G(\mathbb{R})^\theta$, ~~then~~ extend θ to G .

Since real forms = anti-holomorphic involutions, we will now work with holomorphic involutions of G instead.

Defn: The compact real form of G is the one given by the holomorphic involution \mathbf{I} .
i.e. the trivial involution $\theta(g) = g$.

Ex: $GL(n, \mathbb{C})$. How do you write holo invol of $GL(n, \mathbb{C})$? Start with the inner automorphisms (which are involutions)

First, Problems: 1) Study $\{ g \in G : g^2 = 1 \} / \text{conjugation by } G$

2) Study $\{ g \in G : \text{int}(g)^2 = 1 \} / \text{conj. by } G$

= $\{ g \in G : g^2 \in Z(G) \} / \text{conj. by } G$

3) Study $\{ \theta : \theta = \text{int}(g), \theta^2 = 1 \} / \text{conj. by } G$

The relations between the sets ~~(1), (2), and (3)~~
 (1), (2), and (3) are

$$1) \subseteq 2) \rightarrow 3).$$

Also, 3) \subseteq {real forms}

Ex: $SL(2, \mathbb{C})$. Conjugacy classes of elements in $SL(2, \mathbb{C})$
 whose square is I are $\pm Id$. This is 1)

$$2) = \{ Id, -Id, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \}$$

$$3) = \{ Id, int(t) \} \quad t$$

$$2) \rightarrow 3)$$

$$\pm Id \rightarrow Id$$

$$+ \rightarrow int(t)$$

If G is adjoint, $1) = 2) = 3)$.

Moreover, the real forms of any group are the same
 as that of the adjoint group. However, the
 sets 1), 2), 3) are important for representation theory

Ex: For $G = GL(n, \mathbb{C})$, $\mathcal{D} = \left\{ \begin{pmatrix} \epsilon_1 & & \\ & \ddots & \\ & & \epsilon_n \end{pmatrix} : \epsilon_i = \pm 1 \right\}$

Ex: $S_p(2n, \mathbb{C})$ What is \mathcal{D} ?

$$S_p(2n, \mathbb{C}) = \{ g : gJ^t g^{-1} = J \} \quad \text{for } J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

$$\cup \{ diag(z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}) \}$$

~~sd~~ ~~diag~~

Then $\mathcal{D}(2) = \{ J_{p,q} = \underbrace{(I_p, -I_q, I_p, -I_q)}_{\uparrow \quad \uparrow} \} \cup \{ (iI, -iI) \}$

$$K_{IR} = S_p(p) \times S_p(q)$$

$$K_{IR} = U(n)$$

The real forms here are $S_p(p, q)$, $S_p(2n, IR)$, respectively.

This means the matrix

$$\begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & -1 & & \\ & & & \ddots & & \\ & & & & 1 & -1 \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

with p times $(1, -1)$, q times $(-1, 1)$.

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Repn Theory of $G(\mathbb{R})$ is completely replaced by the following setting:

G is complex, θ a holomorphic involution, $K = G^\theta$,
 (G, K) . So R is completely gone from the picture.
You don't need $K(\mathbb{R})$ as well.

Given $G, \theta, K = G^\theta$.

- Q: 1) G is always connected. Is K connected? What is $\pi_0(K)$?
 2) Is K simply connected? What is $\pi_1(K)$?

Prop: 1) If G is semisimple and simply connected, then K is connected.

2) If G is simply connected and simple, then:

K is ^{not} _n simply connected \Leftrightarrow The minimal nilpotent orbit n of θ is defined over \mathbb{R}

(i.e. does the orbit intersect the real Lie algebra)

Ex: $G = SL(n, \mathbb{C})$ ^{with $n \neq 3$.} $\theta(g) = {}^t g^{-1}$. Then $K = SO(n, \mathbb{R})$

* G is simply connected, but K isn't.

Recall: Have G , want to study $\{\theta \in \text{Aut}(G) : \theta^2 = 1\}$

Ex: $G = GL(n, \mathbb{C})$, get $\{\theta_{p,q}\}$, where $p+q = n$, $p \leq q$,

$$\theta_{p,q}(g) = \text{int}(J_{p,q})(g)$$

$$J_{p,q} = \begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$$

These give $U(p, q)$.

1

What about $GL(n, \mathbb{R})$? Well, this is for $\theta(g) = {}^t g^{-1}$,
 $K = O(n)$.

Exercise: $\theta(g) = {}^t g^{-1}$ is not inner.

So study inner involutions, then the rest

Well, $1 \rightarrow \text{Int}(G) \rightarrow \text{Aut}(G) \xrightarrow{\varphi} \text{Out}(G) \rightarrow 1$

So if $\theta \in \text{Aut}(G)$ is an involution, where does it map to
under φ ? Well, $\varphi(\theta_{p,q}) = 1$ *

Ex: $\text{Out}(GL(n, \mathbb{C})) = \mathbb{Z}/2\mathbb{Z}$ which involutions go to $-1 \in \mathbb{Z}/2\mathbb{Z}$.

$\theta(g) = {}^t g^{-1}$ does. Also θ' goes to -1 where
 θ' corresponds to $GL(\mathbb{R}, \mathbb{H})$ (if n is even).

Since $\varphi(\text{involution}) = \text{involution}$, want to study the
involutions of $\text{Out}(G)$.

Ex: $G = T = (\mathbb{C}^\times)^n$. Then $\text{Out}(G) = GL(n, \mathbb{Z})$.

Dynkin Diagram

Thm: Suppose G is semisimple then $\text{Out}(G) \subset \text{Aut}(DD)$

$\text{Out}(G) \cong \text{Aut}(DD)$ if G is simply connected or adjoint.

Recall $G = G_{\text{der}} \cdot Z$, $Z = \text{torus}$.

②

Defn: θ, θ' are inner to each other if $\varphi(\theta) = \varphi(\theta')$.

~~Defn~~ An equivalence class of involutions under \sim is called an inner class.

Examples of $\{\gamma \in \text{Out}(G) : \gamma^2 = 1\}$

1) $\gamma = 1$. $\{\theta \in \text{Aut}(G) : \theta^2 = 1, \varphi(\theta) = 1\}$ is defined to be the compact inner class

This corresponds to $G(\mathbb{R})$ being compact.

In atlas, this inner class is called c ← lowercase c

2) The split inner class corresponds to the ~~non~~ inner class where $G(\mathbb{R})$ is split. In Atlas, this inner class is called s .

3) $G = G, \chi G, \quad \Theta(g, h) = (h, g), \quad K = \Delta G,$
 $G(\mathbb{R}) = G, (\mathbb{C}),$ this is called the
~~compact~~ complex inner class. Atlas denotes this C , which is a capital c.

④ There's also an unequal rank inner class.
You can look this up. Atlas calls this u .

Note: In ①, ②, and ④, G is simple or a one-dim'l torus.

Ex: $G = \text{SL}(n, \mathbb{C}), \mathbb{H}^B$.  $\hookrightarrow s \subset \leftrightarrow \text{SU}(p, q)$

 $\hookrightarrow s \leftrightarrow \text{SL}(n, \mathbb{R}), \text{SL}(\mathbb{H})$ 3

Ex: $S_p(2n, \mathbb{C})$, $\text{Out}(G) = 1$. $c = 5$



Ex: $GL(n, \mathbb{C}) = SL(n, \mathbb{C}) \times \mathbb{C}^\times$

$$\langle \mathcal{E}I, \mathcal{E}^{-1} \rangle$$

~~inner class.~~ ~~cos(Cos 5)~~

inner classes : $C \times C$, 5×5 and
(subtle)

Ex: $SO(2n, \mathbb{C})$, $n \geq 5$



Subexample 1: $SO(10, \mathbb{C})$ The two inner classes are

$\{ SO(10, 0), SO(8, 2), SO(6, 4) \}$ and

$\{ SO(9, 1), SO(7, 3), SO(5, 5) \}$.

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Inner classes:

$$* \quad 1 \rightarrow \text{Int}(G) \rightarrow \text{Aut}(G) \xrightarrow{\varphi} \text{Out}(G) \rightarrow 1$$

Let $\gamma \in \text{Out}(G)$ s.t. $\gamma^2 = 1$.

$\{\theta \in \text{Aut}(G) : \varphi(\theta) = \gamma\}$ = inner class of real forms.

Basic Data: (G, γ) s.t. $\gamma \in \text{Out}(G)$, $\gamma^2 = 1$.

To this data, you get $\{$ real forms in the inner class defined by γ $\}$

$$= \{\theta \in \text{Aut}(G) : \theta^2 = 1, \text{ s.t. } \varphi(\theta) = \gamma\}.$$

Thm: * splits canonically up to conjugation by G)

~~Error~~ let s denote ~~this~~ splitting. Then if
you have some ~~*~~ $\gamma \in \text{Out}(G)$ s.t. $\gamma^2 = 1$,
then $s(\gamma) =$ "the θ for the maximally
compact" form in this inner class.

Ex: If $\gamma = 1$, then $s(\gamma) =$ Cartan involution of compact real form.

①

Defn. Given (G, δ) , define

$$G^\Gamma := G \times \Gamma \text{ where } \Gamma = \mathbb{Z}/\vartheta\mathbb{Z} = \{1, \delta\}$$

st. δ acts by $s(\delta)$, i.e.

$$\delta g \delta^{-1} = \Theta_0(g) \text{ where } \Theta_0 = \text{most compact Cartan involution}$$

Key point: Suppose Θ is in the inner class of δ , so $\psi(\Theta) = \delta$.

In particular, $\psi(\Theta_0) = \delta$. Thus, $\psi(\Theta) = \psi(\Theta_0)$.

$$\text{Thus, } \Theta = \text{int}(h) \circ \Theta_0 \quad \exists h \in G \text{ s.t. } \Theta = \text{int}(h) \circ \Theta_0.$$

$$\text{Thus, } \Theta(g) = h \Theta_0(g) h^{-1} = (h \delta) g (h \delta)^{-1}$$

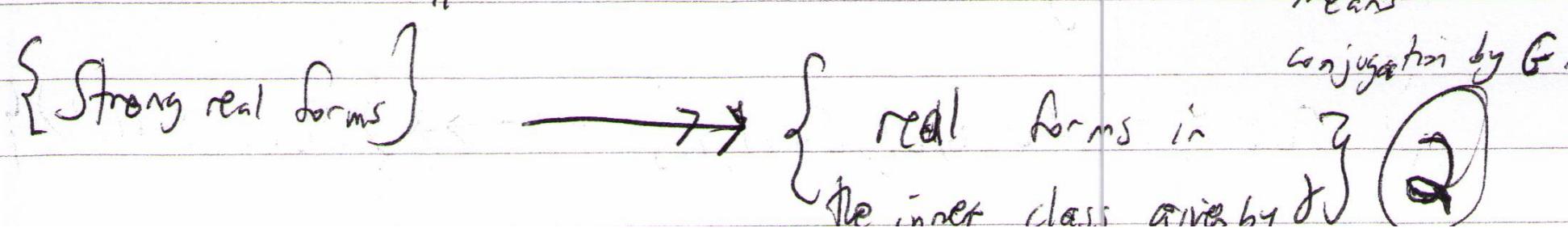
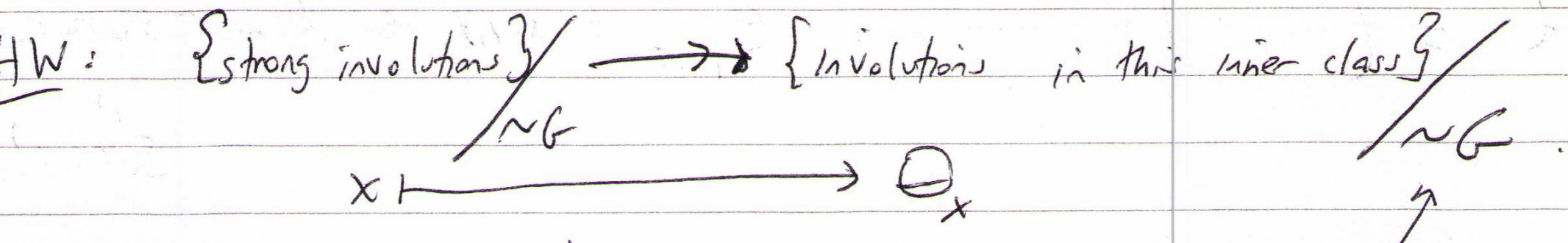
$$\text{Thus, } \Theta = \text{int}(h \delta)$$

Thus, ^{"all"} involutions in the inner class of δ are packaged in G^Γ .

Defn: A strong involution for (G, δ) is an element $x \in G^\Gamma - G$ such that $x^2 \in Z(G)$.

Define $\Theta_x = \text{int}(x)$. Then $\Theta_x^2 = 1$ and Θ_x is in the inner class of δ .

Defn: A strong real form is a G -conjugacy class of strong involutions.



Ex: $G = \mathrm{SL}(2, \mathbb{C})$ so $\chi = 1$. Thus

$G^P = G \times P$, so can really look in G ,
so the strong involutions are the set of

(h, δ) s.t. $h^2 \in Z(G)$, i.e.
 $\{h \in G : h^2 \in Z(G)\} = \{(1), (-1), (i, -i)\}$

up to conjugation

What are the corresponding real forms?

$\{\text{strong real form}\} \xrightarrow{\text{int}} \{\text{real forms}\}$
 $\{\text{involutions}\}$

$(1) \xrightarrow{\quad} \mathrm{Id}$

$(-1) \xrightarrow{\quad} \mathrm{Id} = \text{compact form} = \mathrm{SU}(2)$

$t := (i, -i) \xrightarrow{\quad} \Theta_{\text{split}} = \text{split real form} = \mathrm{SL}(1, \mathbb{R})$

$\mathrm{SU}(1, 1)$

So, informally:

$\{\mathrm{SU}(2, 0), \mathrm{SU}(0, 2), \mathrm{SU}(1, 1)\} \longrightarrow \{\mathrm{SU}(2), \mathrm{SU}(1, 1)\}$ ③

So if you're thinking about strong real forms, you should think of $SU(2,0)$ and $SU(0,2)$ as different.

$K \backslash G / B$ in the language of (G, γ)

Recall (Peter Trapa lecture)

$\{$ Irred. $(\mathfrak{g}, \mathfrak{k})$ -modules with trivial infinitesimal character $\}$



$\{$ pairs (Q, x) where $Q \in K \backslash G / B$ and

$x \in \begin{matrix} \text{Cent}_K(Q) \\ \diagdown \end{matrix} \quad \text{for some } q \in Q \cap \mathfrak{g}$

So: We're given (G, γ) , G^F , fixed $G \supset B \supset T$.

Define $\beta = \{(x, B'): x \text{ is a strong involution and } B' \text{ is a } \ast\text{-Borel}\}$

acts by conjugation on β . (4)

b5

1) First conjugate ~~to~~ to fixed representatives in the x variable

Want to study \mathcal{B}/\sim_G , fix $\{x_i : i \in I\}$, a set of representatives of strong real forms.

So any (x, B') is conjugate to some (x_i, B'') for some i , some B'' . From this, you can see that ~~$\mathcal{B}/\sim_G = \coprod_{i \in I} (x_i, B'')/\sim_{K_i}$~~

$$\mathcal{B}/\sim_G = \coprod_i (x_i, B'')/K_i \text{ where } K_i = G^{\theta_i} \text{ and } \theta_i := \text{int}(x_i)$$

$$\text{Thus, } \mathcal{B}/\sim_G = \coprod_i K_i \backslash G/B$$

2) Now conjugate the pairs ~~to~~ \sim_B to fixed reps in the B variable.

Well, all Borels are conjugate, so

$$(x, B') \underset{G}{\sim} (x, B)$$

Fact: (Matsuki) $\exists T' \subseteq B, \theta_x(T') = T'$

$$T' \underset{B}{\sim} T$$

Defn: $\mathcal{X} := X(G, \delta) := \{x \in \text{Norm}_{G^T \cap G}(T) : x^2 \in Z(G)\} / \sim_B$

Then, ~~(*)~~ can get $(x, B') \underset{G}{\sim} (x, B) \sim (x', B')$ ~~where~~ $x' \in \mathcal{X}$

5

Thus, $\mathcal{B} \mathcal{G}/\mathcal{B}_{\text{reg}}$ $\longleftrightarrow X$.

~~Thm~~: Thus, we have

Thm: $X \longleftrightarrow \coprod_i K_i \backslash G/B$

Example of Strong real forms : $G = T = (\mathbb{C}^*)^n$. Let $\delta = 1$.

$\{ \text{strong real forms of } T \} = \{ x \in T : x^2 \in Z(G) \} / \sim_T$

$= T$. Thus the set of representatives

of strong real forms up to conjugacy might not

be finite.

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Jeff Adams Lecture 2

$G > B > T$

Recall $(G, \gamma)_\gamma$, ~~$\gamma^2 = 1$~~ $\gamma \in \text{Out}(G)$

s.t. $\gamma^2 = 1$, and $G^\Gamma = G \times \Gamma$

$X = \{x \in \text{Norm}(T) : x^2 \in Z(G)\}$

Fix $\{x_i : i \in I\}$ a set of ~~$G^\Gamma \setminus G$~~ representatives of strong real forms

$\sim T$

Thm:

$X \longleftrightarrow \coprod_i K_i \backslash G / B$

Remark: ~~Let~~ Let $y = \{\text{strong involutions}\}$
 y (or X ?) is like the building?

since $\text{Stab}_G(x) = K_x$ = maximal compact
where $x \in y$
(G acts on ~~y~~ y)

Note:

$$x \circlearrowleft w$$

$$\text{Def: } w \cdot x := wxw^{-1}$$

$$\downarrow P$$

$$X \subset W$$

$$\mathcal{I}_W := \{ w \in W : w\theta_0(w) = 1, \theta_0 = \text{int}(\delta) \}$$

Prop: ① $X/W \longleftrightarrow \{ \text{conjugacy classes of real Cartans } H(\mathbb{R}) \subset G(\mathbb{R}) \}$

② If $x \in X/W$ corresponds to P , real Cartan $H(\mathbb{R})$, then $\text{Norm}_{G(\mathbb{R})}(H(\mathbb{R})) = \text{stab}_W(x)$

$$\text{Norm}_{G(\mathbb{R})}(H(\mathbb{R})) = \text{stab}_W(x)$$

③ Strong real forms $\longleftrightarrow X_\delta/W_\delta$

$$\text{where } X_\delta = p^{-1}(1)$$

$$W_\delta = \text{stab}_W(\delta)$$

Structure of X : Recall action of W on X

$$\text{For simple roots } \alpha, s_\alpha \cdot x = s_\alpha x s_\alpha^{-1}$$

More: Suppose you have $x, \theta_x, \theta_x(\alpha) = \alpha$

$$\theta_x(X_\alpha) = -X_\alpha, (\text{i.e. } \alpha \text{ is noncompact imaginary}) \quad (2)$$

Then $s_\alpha x \in \mathcal{X}$ (actually, there's a

representative of s_α , in G , ~~such that~~ (call it $t s_\alpha$),

such that $s_\alpha x \in \mathcal{X}$.

Punchline: You get all of $K_i \backslash G / B$ from x_i this way.

Example: $\cong \mathrm{SL}(2, \mathbb{C})$, $\mathcal{X} = \{\pm \mathrm{Id}, \pm t, w\}$,

$$t = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The x_i are $\{\mathrm{Id}, -\mathrm{Id}, t\}$

$\times \quad G(\mathbb{R}) \quad K \quad K \backslash G / B$

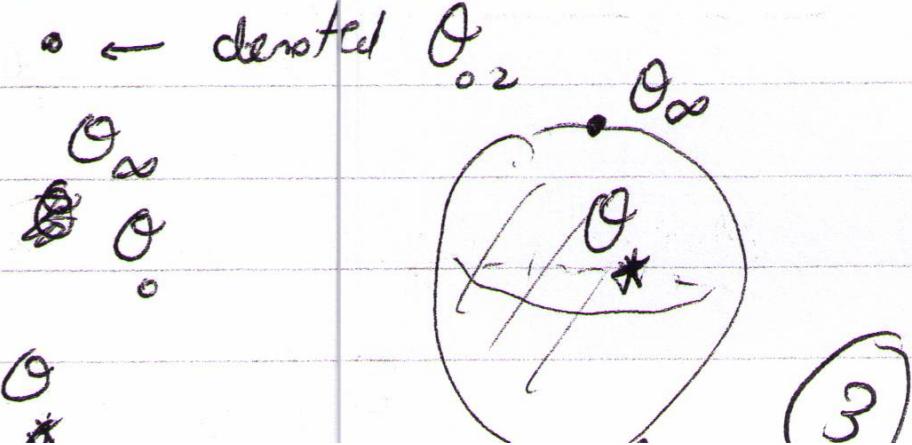
$\mathrm{Id} \quad \mathrm{SU}(2, 0) \quad G \quad \bullet \leftarrow \theta_{2,0}$

$-\mathrm{Id} \quad \mathrm{SU}(0, 2) \quad G \quad \bullet \leftarrow \text{denoted } \theta_{0,2} \quad \theta_\infty$

$t \quad \mathrm{SU}(1, 1) = \mathrm{SL}(2, \mathbb{R}) \quad \mathbb{C}^\times \quad \theta_\infty$

$-t \quad \mathrm{SU}(1, 1) \quad \mathbb{C}^\times \quad \theta_0$

$w \quad \mathrm{SU}(1, 1) \quad \mathbb{C}^\times \quad \theta_*$



(3)

Duality: Recall (G, γ) . ~~Suppose~~

Since $\gamma^2 = 1$, $\gamma \in \text{Aut}(\text{base } D)$ where

D = based root datum of G , you get

$-\gamma^t \in \text{Aut}(D^\vee)$ where D^\vee is for ${}^L G^\circ$

$\begin{matrix} ! \\ \gamma^\vee \end{matrix}$

so $\gamma^\vee \in \text{Out}({}^L G^\circ)$.

So $\gamma \rightsquigarrow \gamma^\vee$
 \uparrow \uparrow
 $\text{Out}(G)$ $\text{Out}({}^L G^\circ)$

Ex: $G = \text{SL}(n, \mathbb{C})$, $\gamma = 1$

$\rightsquigarrow {}^L G^\circ = \text{PSL}(n, \mathbb{C})$, $\gamma^\vee \neq 1$.

Thus, given (G, γ) , have $({}^L G^\circ, \gamma^\vee)$.

Can consider $(G^\vee)^\Gamma := G^\vee \times \Gamma$

where $G^\vee := {}^L G^\circ$. Perfectly symmetric
stuff for G^Γ and $(G^\vee)^\Gamma$.

Thm: $(G^\vee)^\Gamma \cong {}^L G^\circ$

(4)

Digression: Trapa talked about the trivial representation.
a lot

fact: $Z(\mathbb{V}_G) = \bigoplus_{\lambda \in X^*(T)}$ where P is weight lattice
 $= \{ \det \lambda^\vee \otimes e^\lambda \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \}$
exp($\text{ad } i\lambda$)
 $\leftarrow \lambda, \lambda \in P \subseteq h^* \cong h^*$
makes sense since 

(5)

Pick $\Lambda \subseteq \mathcal{P}$, representatives of $\mathcal{Z}(\overset{\circ}{G})$ (circled)

that are regular
int.

$$\Lambda \xrightarrow{\sim} \mathcal{Z}(\overset{\circ}{G})$$

$$\lambda \mapsto \mathcal{Z}^v(\lambda) := e^{2\pi i \lambda}$$

$$\text{Defn: } \Pi(G, \gamma, \Lambda) := \prod_{i \in I} \prod_{\lambda \in \Lambda} \text{irr}(G, K_i, \alpha)$$

(G, K_i) -modules
with irrl character

This is all representations with regular
integral infinitesimal character (up to

translation by $X^*(T)$). The Zuckerman translation principle

says that if we understand reprs with regular integral irrl char, then
we understand the reprs with irrl char

Trape explained that $\{f(Q, \tilde{\gamma}) : Q \in \mathbb{F}G/B$

$\hookrightarrow \lambda + q \quad \forall \gamma \in X^*(T)$. Thus we

only care about representations with regular integral
irrl (up to translation
by $X^*(T)$)

$$\text{and } \tilde{\gamma} \in \left(\text{Cent}_{\mathbb{F}}(q) / \text{Cent}_{\mathbb{F}}(q)^{\circ} \right)^1$$

goes to a repn $\Pi(Q, \tilde{\gamma}, \lambda)$

⑥

(not all T are allowed)

Recall Trop said $\text{Loc}_K(G/B) \hookrightarrow \text{HC}(\mathfrak{g}, K)_\text{triv}$

Some ~~bad~~ i.e. trivial inf'l character)

But you can generalize this to general regular, integral infinitesimal character. i.e.

Fix a regular ~~int~~ integral inf'l character.

Then you ~~can~~ have a map

$$\{(q, \tau) : q \in K \backslash G / B, \tau \in \text{Cent}_K(q) / \text{Cent}_K(q)^\circ\}$$



~~per~~ ~~(\mathfrak{g}, K)~~ modules

(\mathfrak{g}, K) modules with
inf'l character λ .

~~Defn:~~ Fix K_i, λ, Q .

Now: let $\varphi: W_{\mathbb{R}} \rightarrow {}^L G$. Then set

L-packet $\Pi_{\varphi}^L = \{\pi_1, \dots, \pi_n\}$.

If $\varphi(z) = z^\lambda \bar{z}^\nu$, $\lambda \in \mathfrak{h}^* \cong h^*$,

then λ is the inf'l character of all the π_i .

Thm (Vogan)

~~Defn~~ Fix K_i, λ, Q .

Define $\Pi_Q^R := \{\pi(Q, \tilde{\tau}, \lambda) : \tilde{\tau} \in \text{Cet}_K(q) / \text{Cet}(q)\}$

Then $|\Pi_Q^R \cap \Pi_{\varphi}^L| \leq 1$.

Amazing Fact: $\{\varphi: W_{\mathbb{R}} \rightarrow {}^L G \text{ with inf'l character } \lambda\} / \sim$ Care about things up to translations by $X^*(\tau)$ (and recall, we only care about things up to λ)

$$\begin{matrix} & \uparrow \\ X^* & \downarrow \end{matrix}$$

where X^* is the X for the dual data (G^*, γ^*) . (8)

Defn: Have (G, γ) .

Define $\mathcal{Z} := \{(x, y) \in X \times X^{\vee} : \theta_x^t = -\theta_y^t\}$
on h^{\vee}

(recall $\theta_x := \text{int}(x)$?)

Thm: \exists natural bijection

$$\mathcal{Z} \longleftrightarrow \coprod_{i \in I} \coprod_{\lambda \in \Lambda} \text{irr}(G, K_i, \lambda)$$

$$(x, y) \mapsto \pi_x^R \cap \pi_y^L$$

Note: What does π_y^L mean? Well, do

any $\varphi: W_{12} \rightarrow {}^L G$ you have

$$\varphi(z) = z^1 \bar{z}^{y_2}$$

$$\varphi(j) = e^{-\pi i d_j} y$$

(*) $z^v(\lambda) = y^2.$

π_x^R makes sense since $x \in K \backslash G / B$.

Note: $\mathcal{Z} \subset \left(\prod_i K_i \backslash G / B \right) \times \left(\prod_j K_j^v \backslash G^v / B^v \right)$

Symmetry is Vogan Duality This ~~should~~ should be $\prod \prod$

$$\prod \prod \text{irr}(g, k_i, \lambda) \longleftrightarrow \underbrace{\prod \prod}_{\sim} \text{irr}(g^v, k_i^v, \lambda)$$

This duality switches discrete terms

reps and principal series

(10)

Defn: A block is a subset of

$$\underbrace{\coprod_{i \in I}}_{\text{def}} \coprod_{j \in J} \text{irr}(g, K_i, d)$$

~~that~~ corresponds to

fixing an i_0^{cf} and a j_0^{cj} and running
over all the elements

$$g \in \coprod_{i \in I} K_i \backslash G / B \times \coprod_{j \in J} K_j \backslash G^{\vee} / B^{\vee}$$

that correspond to i_0 and j_0 .